1.3 DOT PRODUCT; PROJECTIONS

1.3.1 Definition of the Dot Product

Definition If $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$ are vectors in 2-space, then the ***dot product*** of $\mathbf{u}$ and $\mathbf{v}$ is written as $\mathbf{u} \cdot \mathbf{v}$ and is defined as $\langle \mathbf{u} \cdot \mathbf{v} \rangle = u_1 v_1 + u_2 v_2$

Similarly, if $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$are vectors in 3-space, then their dot product is defined as $\langle \mathbf{u} \cdot \mathbf{v} \rangle = u_1 v_1 + u_2 v_2 + u_3 v_3$

- In words, the dot product of two vectors is formed by multiplying their corresponding components and adding the resulting products. Note that the dot product of two vectors is a scalar.

Example 1.12

$$\langle 3, 5 \rangle \cdot \langle -1, 2 \rangle = 3(-1) + 5(2) = 7$$
$$\langle 2, 3 \rangle \cdot \langle -3, 2 \rangle = 2(-3) + 3(2) = 0$$
$$\langle 1, -3, 4 \rangle \cdot \langle 1, 5, 2 \rangle = 1(1) + (-3)(5) + 4(2) = -6$$

Here are the same computations expressed another way:

$$(3\mathbf{i} + 5\mathbf{j}) \cdot (-\mathbf{i} + 2\mathbf{j}) = 3(-1) + 5(2) = 7$$
$$(2\mathbf{i} + 3\mathbf{j}) \cdot (-3\mathbf{i} + 2\mathbf{j}) = 2(-3) + 3(2) = 0$$
$$(\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}) \cdot (\mathbf{i} + 5\mathbf{j} + 2\mathbf{k}) = 1(1) + (-3)(5) + 4(2) = -6$$

1.3.2 Algebraic Properties of the Dot Product

Theorem *If* $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *are vectors in 2- or 3-space and k is a scalar, then:*

(a) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$

(b) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$

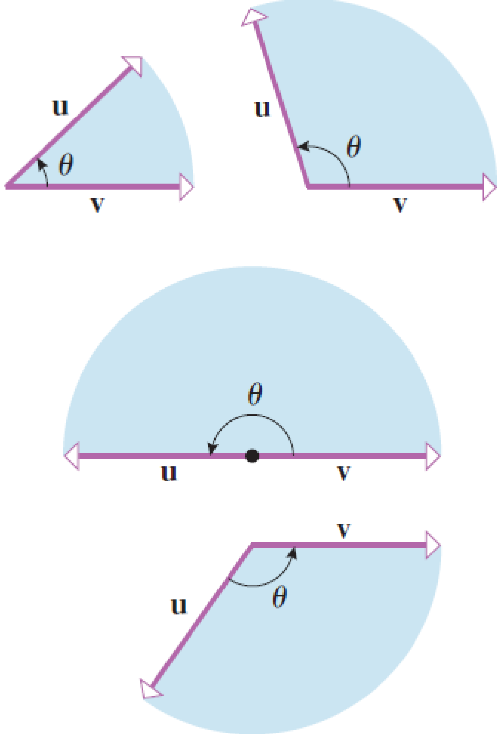
(c) $k(\mathbf{u} \cdot \mathbf{v}) = (k\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (k\mathbf{v})$

(d) $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2$

(e) $\mathbf{0} \cdot \mathbf{v} = 0$

1.3.3 Angle between Vectors

Suppose that **u** and **v** are nonzero vectors in 2-space or 3-space that are positioned so their initial points coincide. We define the ***angle between* u *and* v** to be the angle θ determined by the vectors that satisfies the condition $0 \le \theta \le \pi$ (Figure 1-21). In 2-space, θ is the smallest counter clockwise angle through which one of the vectors can be rotated until it aligns with the other.



θ is the angle between **u** and **v**.

Figure 1-21

Theorem *If* **u** *and* **v** *are nonzero vectors in 2-space or 3-space, and if θ is the angle between them, then*

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}$$

Example 1.13 Find the angle between the vector $\mathbf{u} = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$ and

(a) $\mathbf{v} = -3\mathbf{i} + 6\mathbf{j} + 2\mathbf{k}$ (b) $\mathbf{w} = 2\mathbf{i} + 7\mathbf{j} + 6\mathbf{k}$ (c) $\mathbf{z} = -3\mathbf{i} + 6\mathbf{j} - 6\mathbf{k}$

Solution (a).

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|} = \frac{-11}{(3)(7)} = -\frac{11}{21}$$

$$\theta = \cos^{-1}\left(-\tfrac{11}{21}\right) \approx 2.12 \text{ radians} \approx 121.6^\circ$$

Solution (b).

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{w}}{\|\mathbf{u}\|\,\|\mathbf{w}\|} = \frac{0}{\|\mathbf{u}\|\,\|\mathbf{w}\|} = 0$$

Thus, $\theta = \pi/2$, which means that the vectors are perpendicular.

Solution (c).

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{z}}{\|\mathbf{u}\|\,\|\mathbf{z}\|} = \frac{-27}{(3)(9)} = -1$$

Thus, $\theta = \pi$, which means that the vectors are oppositely directed. (In retrospect, we could have seen this without computing θ, since $\mathbf{z} = -3\mathbf{u}$.)

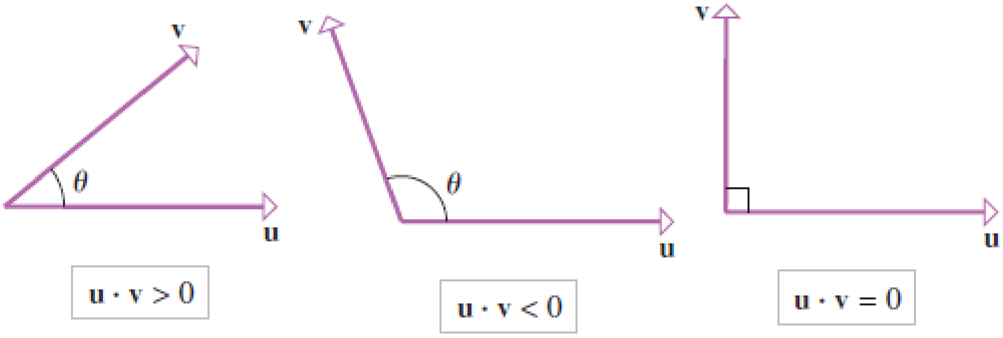


Figure 1-22

1.3.4 Interpreting the Sign of the Dot Product

$$\mathbf{u}\cdot\mathbf{v} = \|\mathbf{u}\|\,\|\mathbf{v}\|\cos\theta \qquad \text{(a)}$$

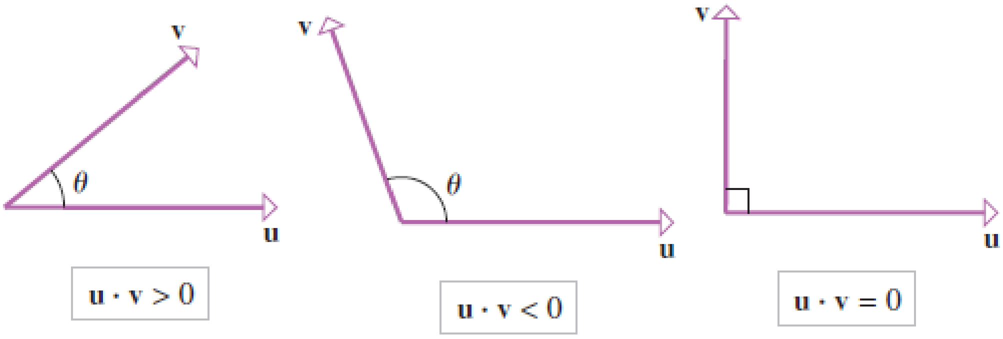


Figure 1-22

Notes:

- The terms "perpendicular," "orthogonal," and "normal" are all commonly used to describe geometric objects that meet at right angles.
- Although the zero vector does not make a well-defined angle with other vectors, we will consider 0 to be orthogonal to all vectors. This convention allows us to say that **u** and **v** are orthogonal vectors if and only if **u . v** = 0, and makes Formula (a) valid if **u** or **v** (or both) is zero.

1.3.5 Direction Angles

In an *xy*-coordinate system, the direction of a nonzero vector **v** is completely determined by the angles α and β between **v** and the unit vectors **i** and **j** (Figure 1-23), and in an *xyz*-coordinate system the direction is completely determined by the angles α, β, and γ between **v** and the unit vectors **i**, **j**, and **k** (Figure 1-23).

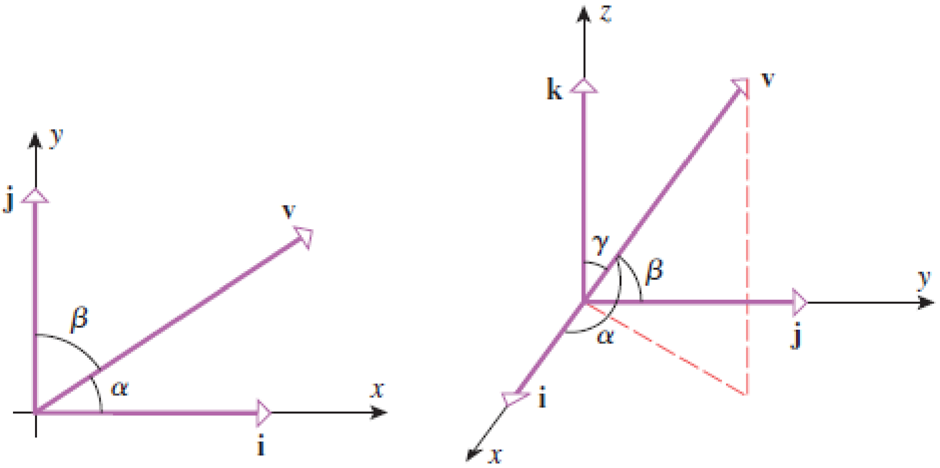


Figure 1-23

- In both 2-space and 3-space the angles between a nonzero vector **v** and the vectors **i**, **j**, and **k** are called the ***direction angles*** of **v**, and the cosines of those angles are called the ***direction cosines*** of **v**.

Theorem *The direction cosines of a nonzero vector* $\mathbf{v} = v_1\mathbf{i} + v_2\,\mathbf{j} + v_3\mathbf{k}$ *are*

$$\cos\alpha = \frac{v_1}{\|\mathbf{v}\|}, \quad \cos\beta = \frac{v_2}{\|\mathbf{v}\|}, \quad \cos\gamma = \frac{v_3}{\|\mathbf{v}\|}$$

Example 1-14 Find the direction cosines of the vector $\mathbf{v} = 2\mathbf{i} - 4\mathbf{j} + 4\mathbf{k}$, and approximate the direction angles to the nearest degree.

Solution. First we will normalize the vector **v** and then read off the components. We have

$$\|\mathbf{v}\| = \sqrt{4+16+16} = 6, \text{ so that } \mathbf{v}/\|\mathbf{v}\| = \tfrac{1}{3}\mathbf{i} - \tfrac{2}{3}\mathbf{j} + \tfrac{2}{3}\mathbf{k}. \text{ Thus,}$$

$$\cos\alpha = \tfrac{1}{3}, \quad \cos\beta = -\tfrac{2}{3}, \quad \cos\gamma = \tfrac{2}{3}$$

With the help of a calculating utility we obtain

$$\alpha = \cos^{-1}\left(\tfrac{1}{3}\right) \approx 71^\circ, \quad \beta = \cos^{-1}\left(-\tfrac{2}{3}\right) \approx 132^\circ, \quad \gamma = \cos^{-1}\left(\tfrac{2}{3}\right) \approx 48^\circ$$

Example 1-15 Find the angle between a diagonal of a cube and one of its edges.

Solution. Assume that the cube has side a, and introduce a coordinate system as shown in Figure 1-24. In this coordinate system the vector

$$\mathbf{d} = a\mathbf{i} + a\,\mathbf{j} + a\mathbf{k}$$

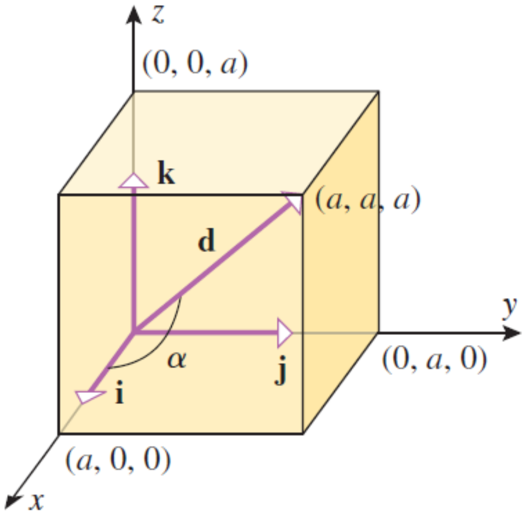


Figure 1-24

is a diagonal of the cube and the unit vectors **i**, **j**, and **k** run along the edges. By symmetry, the diagonal makes the same angle with each edge, so it is sufficient to find the angle between **d** and **i** (the direction angle α). Thus,

$$\cos\alpha = \frac{\mathbf{d}\cdot\mathbf{i}}{\|\mathbf{d}\|\,\|\mathbf{i}\|} = \frac{a}{\|\mathbf{d}\|} = \frac{a}{\sqrt{3a^2}} = \frac{1}{\sqrt{3}}$$

$$\alpha = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \approx 0.955 \text{ radian} \approx 54.7^\circ$$

1.3.1 Decomposing Vectors into Orthogonal Components

In many applications it is desirable to "decompose" a vector into a sum of two orthogonal vectors with convenient specified directions. For example, Figure 1-25 shows a block on an inclined plane. The downward force **F** that gravity exerts on the block can be decomposed into the sum

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2$$

where the force $\mathbf{F}_1$ is parallel to the ramp and the force $\mathbf{F}_2$ is perpendicular to the ramp. The forces $\mathbf{F}_1$ and $\mathbf{F}_2$ are useful because $\mathbf{F}_1$ is the force that pulls the block *along* the ramp, and $\mathbf{F}_2$ is the force that the block exerts *against* the ramp.

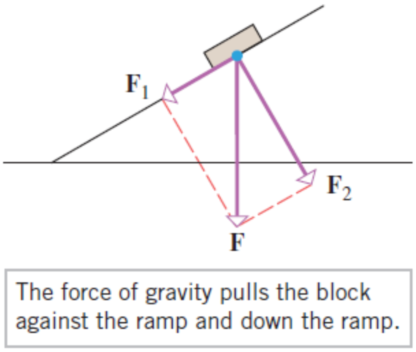


Figure 1-25

Thus, our next objective is to develop a computational procedure for decomposing a vector into a sum of orthogonal vectors. For this purpose, suppose that $\mathbf{e}_1$ and $\mathbf{e}_2$ are two orthogonal *unit* vectors in 2-space, and suppose that we want to express a given vector **v** as a sum

$$\mathbf{v} = \mathbf{w}_1 + \mathbf{w}_2$$

so that $\mathbf{w}_1$ is a scalar multiple of $\mathbf{e}_1$ and $\mathbf{w}_2$ is a scalar multiple of $\mathbf{e}_2$ (Figure 1-26*a*).

$$\mathbf{v} = (\mathbf{v} \cdot \mathbf{e}_1)\,\mathbf{e}_1 + (\mathbf{v} \cdot \mathbf{e}_2)\,\mathbf{e}_2$$

In this formula we call $(\mathbf{v} \cdot \mathbf{e}_1)\mathbf{e}_1$ and $(\mathbf{v} \cdot \mathbf{e}_2)\mathbf{e}_2$ the ***vector components*** of **v** along $\mathbf{e}_1$ and $\mathbf{e}_2$, respectively;

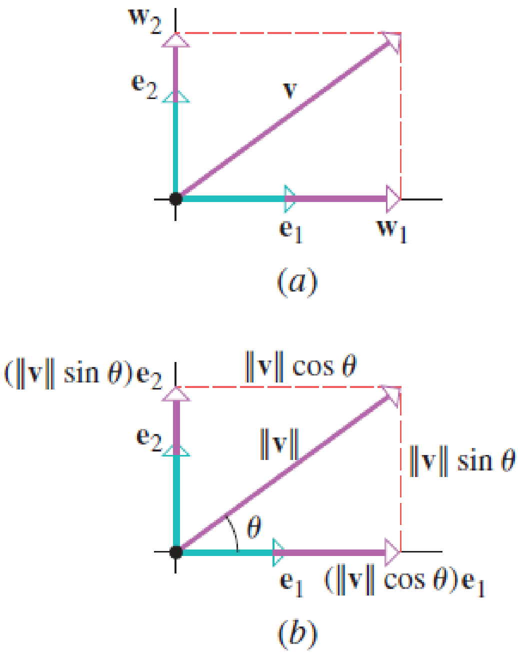


Figure 1-26

and we call $\mathbf{v} \cdot \mathbf{e}_1$ and $\mathbf{v} \cdot \mathbf{e}_2$ the ***scalar components*** of $\mathbf{v}$ along $\mathbf{e}_1$ and $\mathbf{e}_2$, respectively. If θ denotes the angle between $\mathbf{v}$ and $\mathbf{e}_1$, and the angle between $\mathbf{v}$ and $\mathbf{e}_2$ is $\pi/2$ or less, then the scalar components of $\mathbf{v}$ can be written in trigonometric form as

$$\mathbf{v} \cdot \mathbf{e}_1 = \|\mathbf{v}\| \cos\theta \qquad \text{and} \qquad \mathbf{v} \cdot \mathbf{e}_2 = \|\mathbf{v}\| \sin\theta$$

(Figure 1-26*b*). Moreover, the vector components of $\mathbf{v}$ can be expressed as

$$(\mathbf{v} \cdot \mathbf{e}_1)\mathbf{e}_1 = (\|\mathbf{v}\|\cos\theta)\mathbf{e}_1 \qquad \text{and} \qquad (\mathbf{v} \cdot \mathbf{e}_2)\mathbf{e}_2 = (\|\mathbf{v}\|\sin\theta)\mathbf{e}_2$$

The decomposition can be expressed as

$$\mathbf{v} = (\|\mathbf{v}\|\cos\theta)\mathbf{e}_1 + (\|\mathbf{v}\|\sin\theta)\mathbf{e}_2$$

provided the angle between $\mathbf{v}$ and $\mathbf{e}_2$ is at most $\pi/2$.

Example 1.16 Let

$$\mathbf{v} = \langle 2, 3\rangle, \quad \mathbf{e}_1 = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle, \quad \text{and} \quad \mathbf{e}_2 = \left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$$

Find the scalar components of $\mathbf{v}$ along $\mathbf{e}_1$ and $\mathbf{e}_2$ and the vector components of $\mathbf{v}$ along $\mathbf{e}_1$ and $\mathbf{e}_2$.

Solution. The scalar components of $\mathbf{v}$ along $\mathbf{e}_1$ and $\mathbf{e}_2$ are

$$\mathbf{v} \cdot \mathbf{e}_1 = 2\left(\frac{1}{\sqrt{2}}\right) + 3\left(\frac{1}{\sqrt{2}}\right) = \frac{5}{\sqrt{2}}$$
$$\mathbf{v} \cdot \mathbf{e}_2 = 2\left(-\frac{1}{\sqrt{2}}\right) + 3\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}$$

so the vector components are

$$(\mathbf{v} \cdot \mathbf{e}_1)\mathbf{e}_1 = \frac{5}{\sqrt{2}}\left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle = \left\langle \frac{5}{2}, \frac{5}{2} \right\rangle$$
$$(\mathbf{v} \cdot \mathbf{e}_2)\mathbf{e}_2 = \frac{1}{\sqrt{2}}\left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle = \left\langle -\frac{1}{2}, \frac{1}{2} \right\rangle$$

Example 1.17 A rope is attached to a 100 lb block on a ramp that is inclined at an angle of 30◦ with the ground (Figure 1-27*a*). How much force does the block exert against the ramp, and how much force must be applied to the rope in a direction parallel to the ramp to prevent the block from sliding down the ramp? (Assume that the ramp is smooth, that is, exerts no frictional forces.)

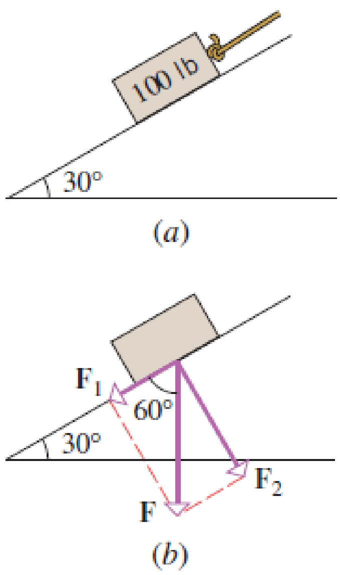


Figure 1-27

Solution. Let **F** denote the downward force of gravity on the block (so ||**F**|| = 100 lb), and let $\mathbf{F}_1$ and $\mathbf{F}_2$ be the vector components of **F** parallel and perpendicular to the ramp (as shown in Figure 1-27*b*). The lengths of $\mathbf{F}_1$ and $\mathbf{F}_2$ are

$$\|\mathbf{F}_1\| = \|\mathbf{F}\|\cos 60^\circ = 100\left(\frac{1}{2}\right) = 50 \text{ lb}$$

$$\|\mathbf{F}_2\| = \|\mathbf{F}\|\sin 60^\circ = 100\left(\frac{\sqrt{3}}{2}\right) \approx 86.6 \text{ lb}$$

Thus, the block exerts a force of approximately 86.6 lb against the ramp, and it requires a force of 50 lb to prevent the block from sliding down the ramp.

1.3.2 Orthogonal Projections

The vector components of **v** along $\mathbf{e}_1$ and $\mathbf{e}_2$ in previous equation are also called the *orthogonal projections* of **v** on $\mathbf{e}_1$ and $\mathbf{e}_2$ and are commonly denoted by

$$\text{proj}_{\mathbf{e}_1}\mathbf{v} = (\mathbf{v}\cdot\mathbf{e}_1)\mathbf{e}_1 \quad \text{and} \quad \text{proj}_{\mathbf{e}_2}\mathbf{v} = (\mathbf{v}\cdot\mathbf{e}_2)\mathbf{e}_2$$

In general, if **e** is a unit vector, then we define the ***orthogonal projection of* v *on* e** to be

$$\text{proj}_{\mathbf{e}}\mathbf{v} = (\mathbf{v}\cdot\mathbf{e})\mathbf{e}$$

$$\text{proj}_{\mathbf{b}}\mathbf{v} = \frac{\mathbf{v}\cdot\mathbf{b}}{\|\mathbf{b}\|^2}\mathbf{b}$$

Geometrically, if **b** and **v** have a common initial point, then $\text{proj}_{\mathbf{b}}\mathbf{v}$ is the vector that is determined when a perpendicular is dropped from the terminal point of **v** to the line through **b** (illustrated in Figure 1-28 in two cases).

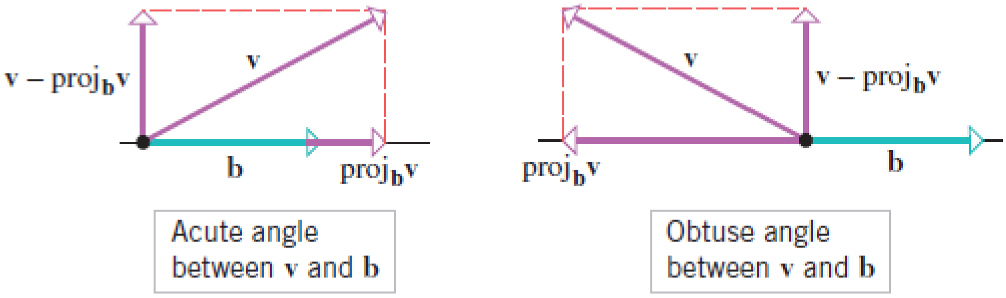


Figure 1-28

Example 1-18 Find the orthogonal projection of **v** = **i** + **j** + **k** on **b** = 2**i** + 2**j**, and then find the vector component of **v** orthogonal to **b**.

Solution. We have

$$\mathbf{v}\cdot\mathbf{b} = (\mathbf{i}+\mathbf{j}+\mathbf{k})\cdot(2\mathbf{i}+2\mathbf{j}) = 2+2+0 = 4$$
$$\|\mathbf{b}\|^2 = 2^2+2^2 = 8$$

Thus, the orthogonal projection of **v** on **b** is

$$\text{proj}_{\mathbf{b}}\mathbf{v} = \frac{\mathbf{v}\cdot\mathbf{b}}{\|\mathbf{b}\|^2}\mathbf{b} = \frac{4}{8}(2\mathbf{i}+2\mathbf{j}) = \mathbf{i}+\mathbf{j}$$

and the vector component of **v** orthogonal to **b** is

$$\mathbf{v} - \text{proj}_{\mathbf{b}}\mathbf{v} = (\mathbf{i}+\mathbf{j}+\mathbf{k}) - (\mathbf{i}+\mathbf{j}) = \mathbf{k}$$

These results are consistent with Figure 1-29.

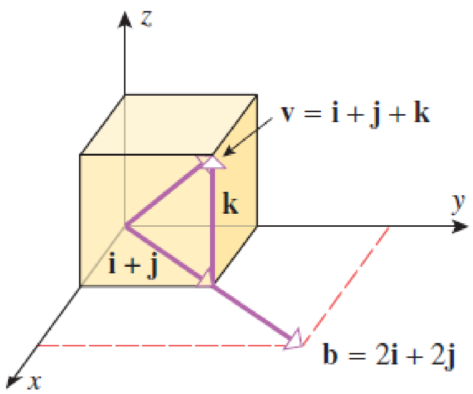


Figure 1-29