1.4 CROSS PRODUCT

Some of the concepts that we will develop in this section require basic ideas about ***determinants***, which are functions that assign numerical values to square arrays of numbers. For example, if a_1, a_2, b_1, and b_2 are real numbers, then we define a **2 × 2 *determinant*** by

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$$

The purpose of the arrows is to help you remember the formula—the determinant is the product of the entries on the rightward arrow minus the product of the entries on the leftward arrow. For example,

$$\begin{vmatrix} 3 & -2 \\ 4 & 5 \end{vmatrix} = (3)(5) - (4)(-2) = 15 + 8 = 23$$

A **3 × 3 *determinant*** is defined in terms of 2 × 2 determinants by

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

The right side of this formula is easily remembered by noting that a_1, a_2, and a_3 are the entries in the first "row" of the left side, and the 2 × 2 determinants on the right side arise by deleting the first row and an appropriate column from the left side. The pattern is as follows:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Example 1-19:

$$\begin{vmatrix} 3 & -2 & -5 \\ 1 & 4 & -4 \\ 0 & 3 & 2 \end{vmatrix} = 3 \begin{vmatrix} 4 & -4 \\ 3 & 2 \end{vmatrix} - (-2) \begin{vmatrix} 1 & -4 \\ 0 & 2 \end{vmatrix} + (-5) \begin{vmatrix} 1 & 4 \\ 0 & 3 \end{vmatrix}$$

$$= 3(20) + 2(2) - 5(3) = 49$$

There are also definitions of 4 × 4 determinants, 5 × 5 determinants, and higher, but we will not need them in this text. Properties of determinants are studied in a branch of mathematics called ***linear algebra***, but we will only need the two properties stated in the following theorem.

Theorem

(a) If two rows in the array of a determinant are the same, then the value of the determinant is 0.

(b) Interchanging two rows in the array of a determinant multiplies its value by −1.

Proof (*a*)

$$\begin{vmatrix} a_1 & a_2 \\ a_1 & a_2 \end{vmatrix} = a_1a_2 - a_2a_1 = 0$$

Proof (*b*)

$$\begin{vmatrix} b_1 & b_2 \\ a_1 & a_2 \end{vmatrix} = b_1a_2 - b_2a_1 = -(a_1b_2 - a_2b_1)$$

Definition

If $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ are vectors in 3-space, then the ***cross product*** $\mathbf{u} \times \mathbf{v}$ is the vector defined by

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \mathbf{k}$$

or, equivalently,

$$\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\,\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}$$

Observe that the right side of Formula has the same form as the right side of Formula, the difference being notation and the order of the factors in the three terms. Thus, we can rewrite as

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathrm{i} & \mathrm{j} & \mathrm{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

However, this is just a mnemonic device and not a true determinant since the entries in a determinant are numbers, not vectors.

Example 1-20 Let $\mathbf{u} = (1, 2, -2)$ and $\mathbf{v} = (3, 0, 1)$. Find (a) $\mathbf{u} \times \mathbf{v}$ (b) $\mathbf{v} \times \mathbf{u}$

Solution (a)

$$\mathbf{u}\times\mathbf{v}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 2 & -2\\ 3 & 0 & 1\end{vmatrix}=\begin{vmatrix}2 & -2\\ 0 & 1\end{vmatrix}\mathbf{i}-\begin{vmatrix}1 & -2\\ 3 & 1\end{vmatrix}\mathbf{j}+\begin{vmatrix}1 & 2\\ 3 & 0\end{vmatrix}\mathbf{k}=2\mathbf{i}-7\mathbf{j}-6\mathbf{k}$$

(b)

$$\mathbf{u}\times\mathbf{v}=-(\mathbf{v}\times\mathbf{u})=-2\mathbf{i}+7\mathbf{j}+6\mathbf{k}$$

1.4.1 Algebraic Properties of the Cross Product

Theorem

If **u**, **v**, and **w** are any vectors in 3-space and k is any scalar, then:

(*a*) $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$

(*b*) $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$

(*c*) $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \times \mathbf{w}) + (\mathbf{v} \times \mathbf{w})$

(*d*) $k(\mathbf{u} \times \mathbf{v}) = (k\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (k\mathbf{v})$

(*e*) $\mathbf{u} \times \mathbf{0} = \mathbf{0} \times \mathbf{u} = \mathbf{0}$

(*f*) $\mathbf{u} \times \mathbf{u} = \mathbf{0}$

The following cross products occur so frequently that it is helpful to be familiar with them:

$\mathbf{i} \times \mathbf{j} = \mathbf{k}$	$\mathbf{j} \times \mathbf{k} = \mathbf{i}$	$\mathbf{k} \times \mathbf{i} = \mathbf{j}$
$\mathbf{j} \times \mathbf{i} = -\mathbf{k}$	$\mathbf{k} \times \mathbf{j} = -\mathbf{i}$	$\mathbf{i} \times \mathbf{k} = -\mathbf{j}$

Example 1-21

$$\mathbf{i}\times\mathbf{j}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 0 & 0\\ 0 & 1 & 0\end{vmatrix}=\begin{vmatrix}0 & 0\\ 1 & 0\end{vmatrix}\mathbf{i}-\begin{vmatrix}1 & 0\\ 0 & 0\end{vmatrix}\mathbf{j}+\begin{vmatrix}1 & 0\\ 0 & 1\end{vmatrix}\mathbf{k}=\mathbf{k}$$

1.4.2 Geometric Properties of the Cross Product

Theorem

If **u** and **v** are vectors in 3-space, then:

(a) $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 0$ ($\mathbf{u} \times \mathbf{v}$ is orthogonal to **u**)

(b) $\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = 0$ ($\mathbf{u} \times \mathbf{v}$ is orthogonal to **v**)

We will prove part (a). The proof of part (b) is similar.

Proof (a)

Let $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$. Then

$\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1)$

so that $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = u_1(u_2v_3 - u_3v_2) + u_2(u_3v_1 - u_1v_3) + u_3(u_1v_2 - u_2v_1) = 0$

Example 1-22 Find a vector that is orthogonal to both of the vectors $\mathbf{u} = (2, -1, 3)$ and $\mathbf{v} = (-7, 2, -1)$.

Solution:

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 3 \\ -7 & 2 & -1 \end{vmatrix}$$

$$= \begin{vmatrix} -1 & 3 \\ 2 & -1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & 3 \\ -7 & -1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & -1 \\ -7 & 2 \end{vmatrix} \mathbf{k} = -5\mathbf{i} - 19\mathbf{j} - 3\mathbf{k}$$

It can be proved that if **u** and **v** are nonzero and nonparallel vectors, then the direction of **u** × **v** relative to **u** and **v** is determined by a right-hand rule; that is, if the fingers of the right hand are cupped so they curl from **u** toward **v** in the direction of rotation that takes **u** into **v** in less than 180◦ , then the thumb will point (roughly) in the direction of **u** × **v** (Figure 1-30).

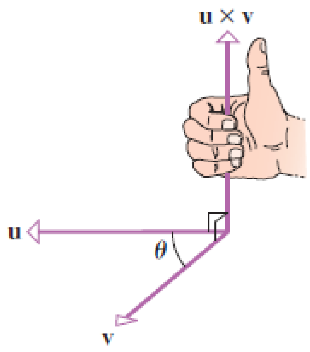


Figure 1-30

Theorem

Let **u** and **v** be nonzero vectors in 3-space, and let θ be the angle between these vectors when they are positioned so their initial points coincide.

(*a*) $\|\mathbf{u}\times\mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\|\sin\theta$

(*b*) *The area A of the parallelogram that has* **u** *and* **v** *as adjacent sides is*

$A = \|\mathbf{u}\times\mathbf{v}\|$

(*c*) $\mathbf{u}\times\mathbf{v} = \mathbf{0}$ *if and only if* **u** *and* **v** *are parallel vectors, that is, if and only if they are scalar multiples of one another.*

Proof (a)

$$\begin{aligned}
\|\mathbf{u}\|\|\mathbf{v}\|\sin\theta &= \|\mathbf{u}\|\|\mathbf{v}\|\sqrt{1-\cos^2\theta}\\
&= \|\mathbf{u}\|\|\mathbf{v}\|\sqrt{1-\frac{(\mathbf{u}\cdot\mathbf{v})^2}{\|\mathbf{u}\|^2\|\mathbf{v}\|^2}}\\
&= \sqrt{\|\mathbf{u}\|^2\|\mathbf{v}\|^2-(\mathbf{u}\cdot\mathbf{v})^2}\\
&= \sqrt{(u_1^2+u_2^2+u_3^2)(v_1^2+v_2^2+v_3^2)-(u_1v_1+u_2v_2+u_3v_3)^2}\\
&= \sqrt{(u_2v_3-u_3v_2)^2+(u_1v_3-u_3v_1)^2+(u_1v_2-u_2v_1)^2}\\
&= \|\mathbf{u}\times\mathbf{v}\|
\end{aligned}$$

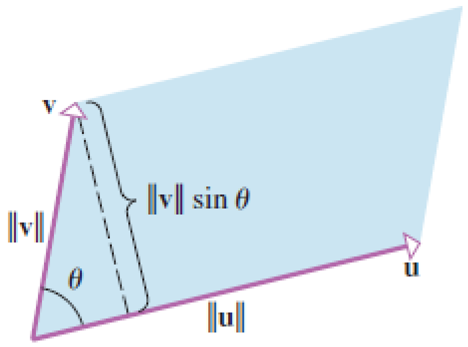


Figure 1-31

Example 1-23 Find the area of the triangle that is determined by the points $P_1(2, 2, 0)$, $P_2(-1, 0, 2)$, and $P_3(0, 4, 3)$.

Solution. The area A of the triangle is half the area of the parallelogram determined by the vectors $\overrightarrow{P_1P_2}$ and $\overrightarrow{P_1P_3}$ **Figure 1-32** But $\overrightarrow{P_1P_2} = \langle -3, -2, 2\rangle$ and $\overrightarrow{P_1P_3} = \langle -2, 2, 3\rangle$, so

$$\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} = \langle -10, 5, -10\rangle$$

(verify), and consequently

$$A = \frac{1}{2}\|\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}\| = \frac{15}{2}$$

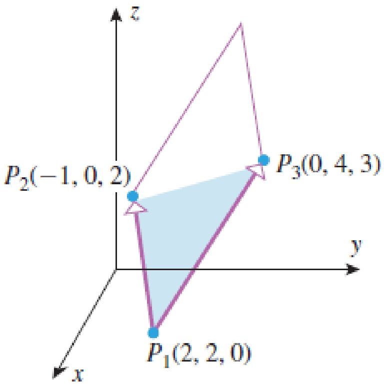


Figure 1-32

1.4.3 Scalar Triple Products

If $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3)$, and $\mathbf{w} = (w_1, w_2, w_3)$ are vectors in 3-space, then the number

$$\mathbf{u} \,.\, (\mathbf{v} \times \mathbf{w})$$

is called the ***scalar triple product*** of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$. It is not necessary to compute the dot product and cross product to evaluate a scalar triple product—the value can be obtained directly from the formula

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

The validity of which can be seen by writing

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \mathbf{u} \cdot \left(\begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} \mathbf{k} \right)$$

$$= u_1 \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} - u_2 \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} + u_3 \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}$$

$$= \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

Example 1-24 Calculate the scalar triple product **u . (v × w)** of the vectors **u** = 3**i** − 2**j** − 5**k**, **v** = **i** + 4**j** − 4**k**, **w** = 3**j** + 2**k**

Solution

$$\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = \begin{vmatrix} 3 & -2 & -5 \\ 1 & 4 & -4 \\ 0 & 3 & 2 \end{vmatrix} = 49$$

1.4.4 Geometric Properties of the Scalar Triple Product

If **u**, **v**, and **w** are nonzero vectors in 3-space that are positioned so their initial points coincide, then these vectors form the adjacent sides of a parallelepiped (see figure). The following theorem establishes a relationship between the volume of this parallelepiped and the scalar triple product of the sides.

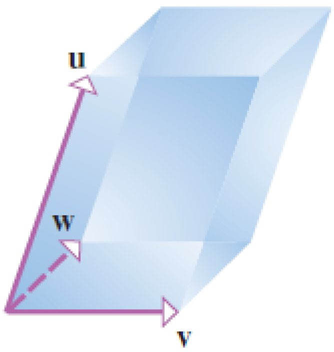
Theorem

Let **u**, **v**, *and* **w** *be nonzero vectors in 3-space.*

(a) The volume V of the parallelepiped that has **u**, **v**, *and* **w** *as adjacent edges is*

$$V = |\mathbf{u}\,.\,(\mathbf{v}\times\mathbf{w})|$$

(b) **u . (v × w)** = 0 *if and only if* **u**, **v**, *and* **w** *lie in the same plane.*



1.4.5 Algebraic Properties of the Scalar Triple Product

- The expression **u × v × w** must be avoided because it is ambiguous without parentheses. However, the expression **u . v × w** is not ambiguous—it has to mean **u . (v × w)** and not (**u . v**) × **w** because we cannot form the cross product of a scalar and a vector.
- Similarly, the expression **u × v . w** must mean (**u × v**) **. w** and not **u** × (**v . w**). Thus, when you see an expression of the form **u . v × w** or **u × v . w**, the cross product is formed first and the dot product second.
- Since interchanging two rows of a determinant multiplies its value by −1, making two row interchanges in a determinant has no effect on its value. This being the case, it follows that

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}) = \mathbf{v} \cdot (\mathbf{w} \times \mathbf{u})$$

- Since the 3 × 3 determinants that are used to compute these scalar triple products can be obtained from one another by two row interchanges.
- Another useful formula can be obtained by rewriting the first equality as

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$$

and then omitting the superfluous parentheses to obtain

$$\mathbf{u} \cdot \mathbf{v} \times \mathbf{w} = \mathbf{u} \times \mathbf{v} \cdot \mathbf{w}$$

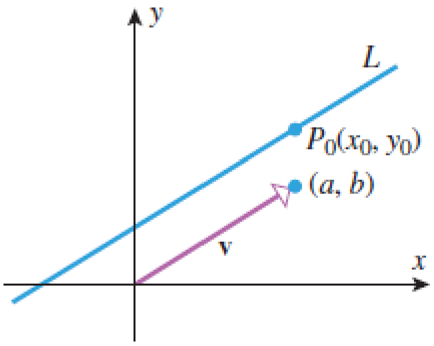
1.5 PARAMETRIC EQUATIONS OF LINES

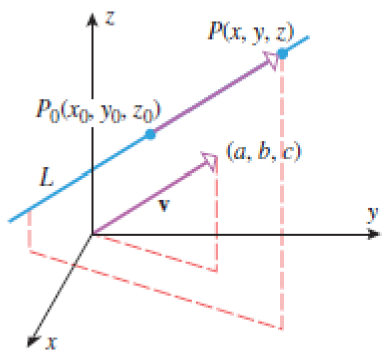
1.5.1 Lines Determined By a Point and a Vector

A line in 2-space or 3-space can be determined uniquely by specifying a point on the line and a nonzero vector parallel to the line (see Figure).

For example, consider a line L in 3-space that passes through the point $P_0(x_0, y_0, z_0)$ and is parallel to the nonzero vector $\mathbf{v} = (a, b, c)$. Then L consists precisely of those points $P(x, y, z)$ for which the vector $\overrightarrow{P_0P}$ is parallel to $\mathbf{v}$ (see figure). In other words, the point $P(x, y, z)$ is on L if and only if $\overrightarrow{P_0P}$ is a scalar multiple of $\mathbf{v}$, say

$$\overrightarrow{P_0P} = t\mathbf{v}$$





Theorem

(a) The line in 2-space that passes through the point $P_0(x_0, y_0)$ and is parallel to the nonzero vector $\mathbf{v} = (a, b) = a\mathbf{i} + b\mathbf{j}$ *has parametric equations*

$$x = x_0 + at, \qquad y = y_0 + bt$$

(b) The line in 3-space that passes through the point $P_0(x_0, y_0, z_0)$ and is parallel to the non-zero vector $\mathbf{v} = (a, b, c) = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ *has parametric equations*

$$x = x_0 + at, \qquad y = y_0 + bt, \qquad z = z_0 + ct$$

Example: Find parametric equations of the line (a) passing through (4, 2) and parallel to **v** = (−1, 5); (b) passing through (1, 2, −3) and parallel to **v** = 4**i** + 5**j** − 7**k**; (c) passing through the origin in 3-space and parallel to **v** = (1, 1, 1).

Solution: (a). $x_0 = 4$, $y_0 = 2$, $a = -1$, and $b = 5$

$x = 4 - t,\ y = 2 + 5t$

Solution: (b). $x = 1 + 4t, \quad y = 2 + 5t, \quad z = -3 - 7t$

Solution: (c). $x_0 = 0, \quad y_0 = 0, \quad z_0 = 0, \quad a = 1, \quad b = 1$, and $c = 1$

$$x = t,\ y = t,\ z = t$$

Example: (a) Find parametric equations of the line L passing through the points $P_1(2, 4, -1)$ and $P_2(5, 0, 7)$. (b) Where does the line intersect the xy-plane?

Solution: (a). The vector $\overrightarrow{P_1P_2} = (3, -4, 8)$ is parallel to L and the point $P_1(2, 4, -1)$ lies on L, so it follows from equation that L has parametric equations

$$x = 2 + 3t, \quad y = 4 - 4t, \quad z = -1 + 8t$$

Had we used P_2 as the point on L rather than P_1, we would have obtained the equations

$$x = 5 + 3t, \qquad y = -4t, \qquad z = 7 + 8t$$

Solution: (b). the line intersects the xy-plane at the point where $z = -1 + 8t = 0$, that is, when $t = \frac{1}{8}$. Substituting this value of t in equation yields the point of intersection $(x, y, z) = \left(\frac{19}{8}, \frac{7}{2}, 0\right)$.

1.5.2 Line Segments

Sometimes one is not interested in an entire line, but rather some *segment* of a line. Parametric equations of a line segment can be obtained by finding parametric equations for the entire line, and then restricting the parameter appropriately so that only the desired segment is generated.

Example: Find parametric equations describing the line segment joining the points $P_1(2, 4, -1)$ and $P_2(5, 0, 7)$.

Solution: From Example 2, the line through the points P_1 and P_2 has parametric equations $x = 2 + 3t$, $y = 4 - 4t$, $z = -1 + 8t$. With these equations, the point P_1 corresponds to $t = 0$ and P_2 to $t = 1$. Thus, the line segment that joins P_1 and P_2 is given by

$$x = 2 + 3t, \qquad y = 4 - 4t, \qquad z = -1 + 8t \qquad\qquad (0 \leq t \leq 1)$$

1.5.3 Vector Equations of Lines

We will now show how vector notation can be used to express the parametric equations of a line more compactly. Because two vectors are equal if and only if their components are equal, can be written in vector form as

$$(x, y) = (x_0 + at, y_0 + bt)$$

$$(x, y, z) = (x_0 + at, y_0 + bt, z_0 + ct)$$

or, equivalently, as

$$(x, y) = (x_0, y_0) + t(a, b) \qquad (1)$$

$$(x, y, z) = (x_0, y_0, z_0) + t(a, b, c) \qquad (2)$$

For the equation in 2-space we define the vectors $\mathbf{r}$, $\mathbf{r}_0$, and $\mathbf{v}$ as

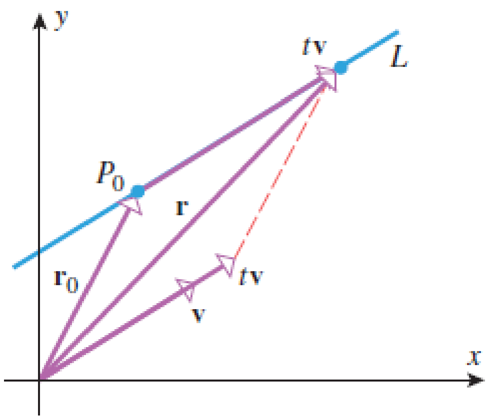
$$\mathbf{r} = (x, y), \quad \mathbf{r}_0 = (x_0, y_0), \quad \mathbf{v} = (a, b) \qquad (3)$$

and for the equation in 3-space we define them as

$$\mathbf{r} = (x, y, z), \quad \mathbf{r}_0 = (x_0, y_0, z_0), \quad \mathbf{v} = (a, b, c) \qquad (4)$$

Substituting (3) and (4) in (1) and (2), respectively, yields the equation

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$$



$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$

In this equation, $\mathbf{v}$ is a nonzero vector parallel to the line, and $\mathbf{r_0}$ is a vector whose components are the coordinates of a point on the line.

We can interpret Equation (4) geometrically by positioning the vectors $\mathbf{r_0}$ and $\mathbf{v}$ with their initial points at the origin and the vector $t\mathbf{v}$ with its initial point at P_0 (see figure). The vector $t\mathbf{v}$ is a scalar multiple of v and hence is parallel to $\mathbf{v}$ and L. Moreover, since

Example: The equation

$$(x, y, z) = (-1, 0, 2) + t\,(1, 5, -4)$$

is of form (4) with

$$\mathbf{r_0} = (-1, 0, 2) \text{ and } \mathbf{v} = (1, 5, -4)$$

Thus, the equation represents the line in 3-space that passes through the point (−1, 0, 2) and is parallel to the vector (1, 5, −4).

Example: Find an equation of the line in 3-space that passes through the points $P_1(2, 4, -1)$ and $P_2(5, 0, 7)$.

Solution: The vector $\overrightarrow{P_1P_2} = (3, -4, 8)$ is parallel to the line, so it can be used as **v** in (4). For $\mathbf{r_0}$ we can use either the vector from the origin to P_1 or the vector from the origin to P_2. Using the former yields $\mathbf{r_0} = (2, 4, -1)$

Thus, a vector equation of the line through P_1 and P_2 is

$(x, y, z) = (2, 4, -1) + t\,(3, -4, 8)$

If needed, we can express the line parametrically by equating corresponding components on the two sides of this vector equation, in which case we obtain the parametric equations in Example 2 (verify).