1.6 PLANES IN 3-SPACE

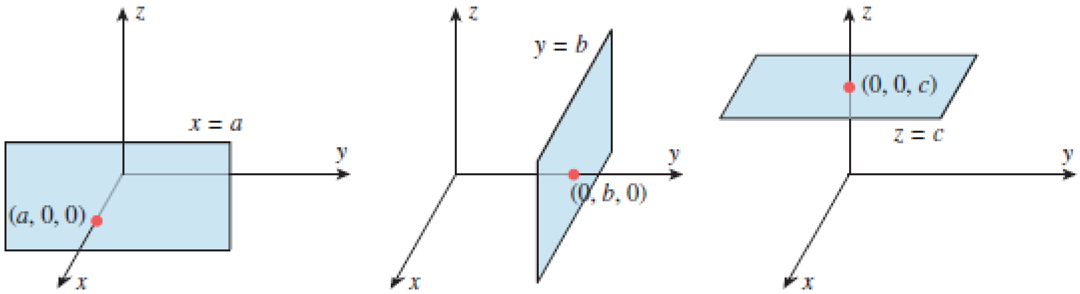
1.6.1 Planes Parallel to the Coordinate Planes

Based on below figure,

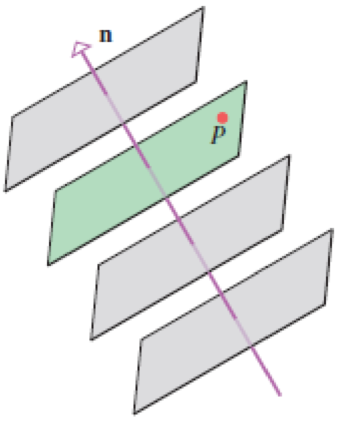
The graph of $x = a$ is the plane through $(a, 0, 0)$ that is parallel to the yz-plane,

The graph of $y = b$ is the plane through $(0, b, 0)$ that is parallel to the xz-plane,

The graph of $z = c$ is the plane through $(0, 0, c)$ that is parallel to the xy-plane.



1.6.2 Planes Determined by a Point and a Normal Vector



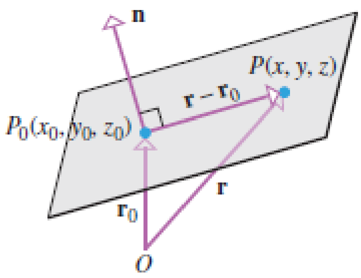
The colored plane is determined uniquely by the point P and the vector **n** perpendicular to the plane.

- A plane in 3-space can be determined uniquely by specifying a point in the plane and a vector perpendicular to the plane (see figure). A vector perpendicular to a plane is called a ***normal*** to the plane.
- Suppose that we want to find an equation of the plane passing through $P_0(x_0, y_0, z_0)$ and perpendicular to the vector $\mathbf{n} = (a, b, c)$. Define the vectors $\mathbf{r}_0$ and $\mathbf{r}$ as

$$\mathbf{r}_0 = (x_0, y_0, z_0) \text{ and } \mathbf{r} = (x, y, z)$$

- It should be evident from Figure that the plane consists precisely of those points $P(x, y, z)$ for which the vector $\mathbf{r} - \mathbf{r}_0$ is orthogonal to $\mathbf{n}$; or, expressed as an equation,

$$\mathbf{n} \,.\, (\mathbf{r} - \mathbf{r}_0) = 0$$



If preferred, we can express this vector equation in terms of components as

$$(a, b, c) \,.\, (x - x_0, y - y_0, z - z_0) = 0$$

from which we obtain

$$a\,(x - x_0) + b\,(y - y_0) + c\,(z - z_0) = 0$$

This is called the ***point-normal form*** of the equation of a plane.

Example: Find an equation of the plane passing through the point (3, −1, 7) and perpendicular to the vector **n** = (4, 2, −5).

Solution: a point-normal form of the equation is

$$4(x-3)+2(y+1)-5(z-7)=0$$

$$(4, 2, -5)\,.\,(x-3, y+1, z-7)=0$$

we obtain an equation of the form

$$ax+by+cz+d=0$$

$$4x+2y-5z+25=0$$

The following theorem shows that every equation represents a plane in 3-space.

Theorem

If a, b, c, and d are constants, and a, b, and c are not all zero, then the graph of the equation

$$\boldsymbol{ax+by+cz+d=0}$$

is a plane that has the vector **n** = (*a*, *b*, *c*) *as a normal.*

Example: Determine whether the planes $3x-4y+5z=0$ and $-6x+8y-10z-4=0$ are parallel.

Solution: It is clear geometrically that two planes are parallel if and only if their normals are parallel vectors. A normal to the first plane is

$$\mathbf{n}_1=(3,-4,5)$$

and a normal to the second plane is

$$\mathbf{n}_2=(-6,8,-10)$$

Since $\mathbf{n}_2$ is a scalar multiple of $\mathbf{n}_1$, the normals are parallel, and hence so are the planes.

Example: Find an equation of the plane through the points $P_1(1, 2,-1)$, $P_2(2, 3, 1)$, and $P_3(3,-1, 2)$.

Solution: Since the points P_1, P_2, and P_3 lie in the plane, the vectors $\overrightarrow{P_1P_2}=(1,1,2)$ and $\overrightarrow{P_1P_3}=(2,-3,3)$ are parallel to the plane. Therefore,

$$\overrightarrow{P_1P_2}\times\overrightarrow{P_1P_3}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 1 & 2\\ 2 & -3 & 3\end{vmatrix}$$

is normal to the plane, since it is orthogonal to both $\overrightarrow{P_1P_2}$ and $\overrightarrow{P_1P_3}$. By using this normal and the point $P_1(1, 2,-1)$ in the plane, we obtain the point-normal form

$$9(x-1)+(y-2)-5(z+1)=0$$

which can be rewritten as

$$9x+y-5z-16=0$$

Example: Determine whether the line $x = 3 + 8t$, $y = 4 + 5t$, $z = -3 - t$ is parallel to the plane $x - 3y + 5z = 12$.

Solution: The vector $\mathbf{v} = (8, 5, -1)$ is parallel to the line and the vector $\mathbf{n} = (1, -3, 5)$ is normal to the plane. For the line and plane to be parallel, the vectors **v** and **n** must be orthogonal. But this is not so, since the dot product $\mathbf{v} \cdot \mathbf{n} = (8)(1) + (5)(-3) + (-1)(5) = -12$ is nonzero. Thus, the line and plane are not parallel. ($\mathbf{v} \cdot \mathbf{n} = 0$ then right angle)

Example: Find the intersection of the line and plane in the previous example.

Solution: If we let (x_0, y_0, z_0) be the point of intersection, then the coordinates of this point satisfy both the equation of the plane and the parametric equations of the line. Thus,

$$x_0 - 3y_0 + 5z_0 = 12 \qquad (1)$$

and for some value of t, say $t = t_0$,

$$x_0 = 3 + 8t_0,\ y_0 = 4 + 5t_0,\ z_0 = -3 - t_0 \qquad (2)$$

Substituting (2) in (1) yields

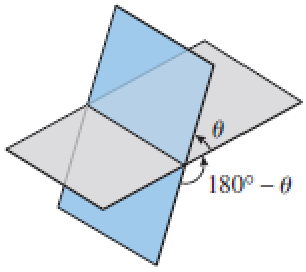
$$(3 + 8t_0) - 3(4 + 5t_0) + 5(-3 - t_0) = 12$$

Solving for t_0 yields $t_0 = -3$ and on substituting this value in (2), we obtain

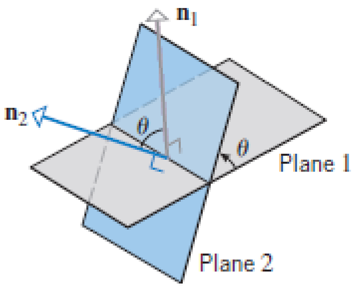
$$(x_0, y_0, z_0) = (-21, -11, 0)$$

1.6.3 Intersecting Planes

- Two distinct intersecting planes determine two positive angles of intersection—an (acute) angle θ that satisfies the condition $0 \le \theta \le \pi/2$ and the supplement of that angle (Figure *a*).
- If $\mathbf{n}_1$ and $\mathbf{n}_2$ are normals to the planes, then depending on the directions of $\mathbf{n}_1$ and $\mathbf{n}_2$, the angle θ is either the angle between $\mathbf{n}_1$ and $\mathbf{n}_2$ or the angle between $\mathbf{n}_1$ and $-\mathbf{n}_2$ (Figure *b*).
- In both cases, Theorem yields the following formula for the *acute angle θ between the planes*:



(*a*)



(*b*)

$$\cos\theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{\|\mathbf{n}_1\|\|\mathbf{n}_2\|}$$

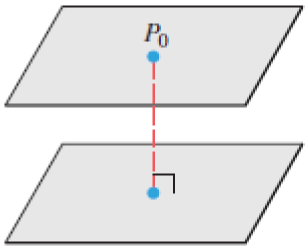
Example: Find the acute angle of intersection between the two planes $2x - 4y + 4z = 6$ and $6x + 2y - 3z = 4$

Solution: The given equations yield the normals $\mathbf{n}_1 = (2, -4, 4)$ and $\mathbf{n}_2 = (6, 2, -3)$.

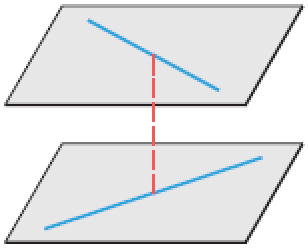
$$\cos\theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{\|\mathbf{n}_1\|\,\|\mathbf{n}_2\|} = \frac{|-8|}{\sqrt{36}\sqrt{49}} = \frac{4}{21}$$

$$\theta = \cos^{-1}\left(\frac{4}{21}\right) \approx 79^\circ$$

1.6.4 Distance Problems Involving Planes



(a)


(b)

Considering three basic distance problems in 3-space:

a. Find the distance between a point and a plane.
b. Find the distance between two parallel planes.
c. Find the distance between two skew lines.

Theorem

The distance D between a point $P_0(x_0, y_0, z_0)$ and the plane $ax + by + cz + d = 0$ is

$$\boldsymbol{D = \frac{ax_0 + by_0 + cz_0 + d}{\sqrt{a^2 + b^2 + c^2}}}$$

Example: Find the distance D between the point $(1, -4, -3)$ and the plane $2x - 3y + 6z = -1$

Solution: the plane be rewritten in the form $ax + by + cz + d = 0$.

Thus, we rewrite the equation of the given plane as

$$2x - 3y + 6z + 1 = 0$$

from which we obtain $a = 2$, $b = -3$, $c = 6$, and $d = 1$.

$$D = \frac{|(2)(1) + (-3)(-4) + 6(-3) + 1|}{\sqrt{2^2 + (-3)^2 + 6^2}} = \frac{|-3|}{7} = \frac{3}{7}$$

Example: The planes $x + 2y - 2z = 3$ and $2x + 4y - 4z = 7$ are parallel since their normals, $(1, 2, -2)$ and $(2, 4, -4)$, are parallel vectors. Find the distance between these planes.

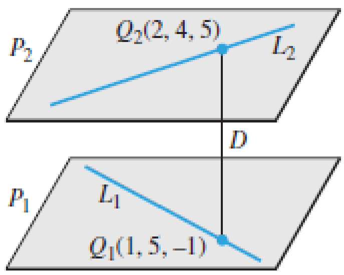
Solution: To find the distance D between the planes, we can select an *arbitrary* point in one of the planes and compute its distance to the other plane. By setting $y = z = 0$ in the equation $x + 2y - 2z = 3$, we obtain the point $P_0(3, 0, 0)$ in this plane.

The distance from P_0 to the plane $2x + 4y - 4z = 7$ is

$$D = \frac{|(2)(3) + 4(0) + (-4)(0) - 7|}{\sqrt{2^2 + 4^2 + (-4)^2}} = \frac{1}{6}$$

Example: It was shown in previous example that the lines L_1: $x = 1 + 4t$, $y = 5 - 4t$, $z = -1 + 5t$ L_2: $x = 2 + 8t$, $y = 4 - 3t$, $z = 5 + t$ are skew. Find the distance between them.

Solution: Let P_1 and P_2 denote parallel planes containing L_1 and L_2, respectively (see figure).



- To find the distance D between L_1 and L_2, we will calculate the distance from a point in $P1$ to the plane P_2.
- Since L_1 lies in plane P_1, we can find a point in P_1 by finding a point on the line L_1; we can do this by substituting any convenient value of t in the parametric equations of L_1. The simplest choice is $t = 0$, which yields the point $Q_1(1, 5, -1)$.
- The next step is to find an equation for the plane P_2. For this purpose, observe that the vector $\mathbf{u}_1 = (4, -4, 5)$ is parallel to line L_1, and therefore also parallel to planes P_1 and P_2.
- Similarly, $\mathbf{u}_2 = (8, -3, 1)$ is parallel to L_2 and hence parallel to P_1 and P_2. Therefore, the cross product

$$\mathbf{n} = \mathbf{u}_1 \times \mathbf{u}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & -4 & 5 \\ 8 & -3 & 1 \end{vmatrix} = 11\mathbf{i} + 36\mathbf{j} + 20\mathbf{k}$$

is normal to both P_1 and P_2. Using this normal and the point $Q_2(2, 4, 5)$ found by setting $t = 0$ in the equations of L_2, we obtain an equation for P_2:

$$11(x - 2) + 36(y - 4) + 20(z - 5) = 0$$

or

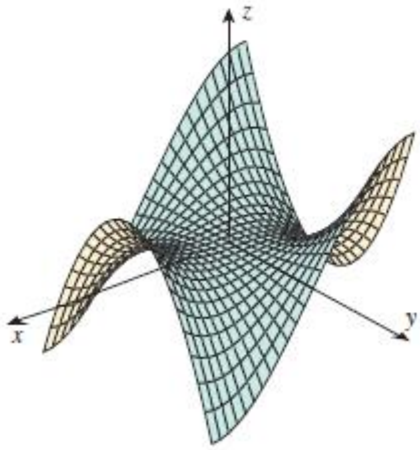
$$11x + 36y + 20z - 266 = 0$$

The distance between $Q_1(1, 5, -1)$ and this plane is

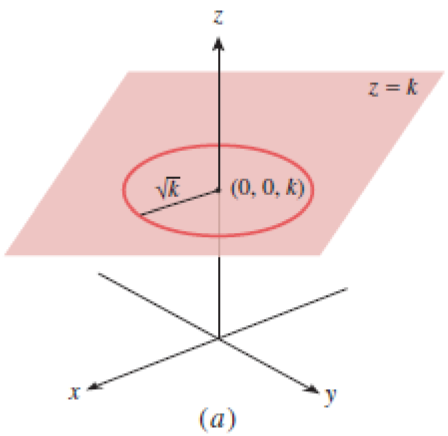
$$D = \frac{|(11)(1) + (36)(5) + (20)(-1) - 266|}{\sqrt{11^2 + 36^2 + 20^2}} = \frac{95}{\sqrt{1817}}$$

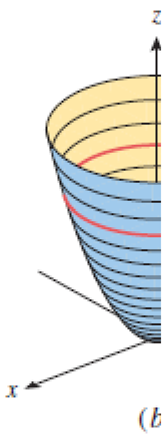
1.7 QUADRIC SURFACES

- Although the general shape of a curve in 2-space can be obtained by plotting points, this method is not usually helpful for surfaces in 3-space because too many points are required.
- It is more common to build up the shape of a surface with a network of ***mesh lines***, which are curves obtained by cutting the surface with well-chosen planes.
- For example, the figure shows the graph of $z = x^3 - 3xy^2$ rendered with a combination of mesh lines and colorization to produce the surface detail. This surface is called a "monkey saddle"]].
- The mesh line that results when a surface is cut by a plane is called the ***trace*** of the surface in the plane (see figure).



A monkey saddle





We noted that a second-degree equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

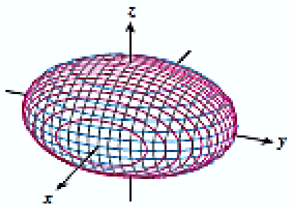
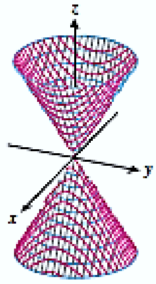
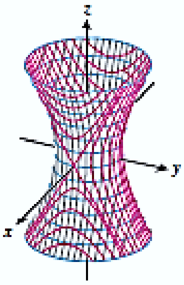
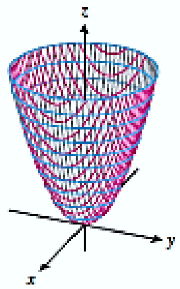
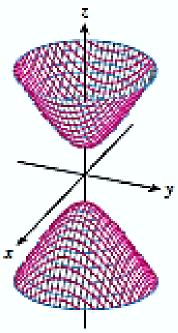
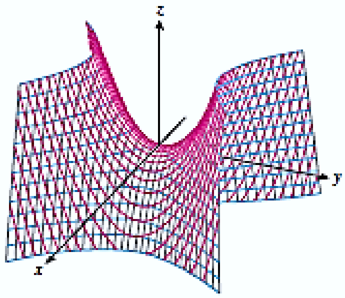
represents a conic section (possibly degenerate). The analog of this equation in an *xyz*-coordinate system is

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0$$

which is called a ***second-degree equation in x, y, and z***. The graphs of such equations are called ***quadric surfaces*** or sometimes ***quadrics***.

Six common types of quadric surfaces are shown in the following table—*ellipsoids, hyperboloids of one sheet, hyperboloids of two sheets, elliptic cones, elliptic paraboloids,* and *hyper-*

bolic paraboloids. (The constants a, b, and c that appear in the equations in the table are assumed to be positive.)

SURFACE	EQUATION	SURFACE	EQUATION
ELLIPSOID 	$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ The traces in the coordinate planes are ellipses, as are the traces in those planes that are parallel to the coordinate planes and intersect the surface in more than one point.	ELLIPTIC CONE 	$z^2=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ The trace in the xy-plane is a point (the origin), and the traces in planes parallel to the xy-plane are ellipses. The traces in the yz- and xz-planes are pairs of lines intersecting at the origin. The traces in planes parallel to these are hyperbolas.
HYPERBOLOID OF ONE SHEET 	$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ The trace in the xy-plane is an ellipse, as are the traces in planes parallel to the xy-plane. The traces in the yz-plane and xz-plane are hyperbolas, as are the traces in those planes that are parallel to these and do not pass through the x- or y-intercepts. At these intercepts the traces are pairs of intersecting lines.	ELLIPTIC PARABOLOID 	$z=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ The trace in the xy-plane is a point (the origin), and the traces in planes parallel to and above the xy-plane are ellipses. The traces in the yz- and xz-planes are parabolas, as are the traces in planes parallel to these.
HYPERBOLOID OF TWO SHEETS 	$\frac{z^2}{c^2}-\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ There is no trace in the xy-plane. In planes parallel to the xy-plane that intersect the surface in more than one point the traces are ellipses. In the yz- and xz-planes, the traces are hyperbolas, as are the traces in those planes that are parallel to these.	HYPERBOLIC PARABOLOID 	$z=\frac{y^2}{b^2}-\frac{x^2}{a^2}$ The trace in the xy-plane is a pair of lines intersecting at the origin. The traces in planes parallel to the xy-plane are hyperbolas. The hyperbolas above the xy-plane open in the y-direction, and those below in the x-direction. The traces in the yz- and xz-planes are parabolas, as are the traces in planes parallel to these.

1.7.1 Techniques for Graphing Quadric Surfaces

A rough sketch of an ellipsoid

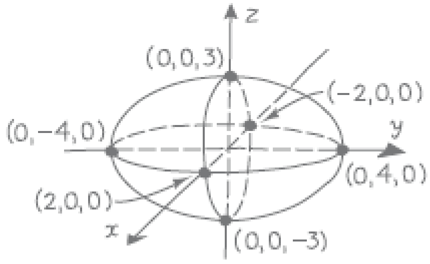
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \qquad (a>0, b>0, c>0)$$

can be obtained by first plotting the intersections with the coordinate axes, and then sketching the elliptical traces in the coordinate planes.

Example: Sketch the ellipsoid

$$\frac{x^2}{4} + \frac{y^2}{16} + \frac{z^2}{9} = 1$$

Solution: The x-intercepts can be obtained by setting $y = 0$ and $z = 0$ in. This yields $x = \pm 2$. Similarly, the y-intercepts are $y = \pm 4$, and the z-intercepts are $z = \pm 3$. Sketching the elliptical traces in the coordinate planes yields the graph in the figure.



Example: Sketch the graph of the hyperboloid of one sheet

$$x^2 + y^2 - \frac{z^2}{4} = 1$$

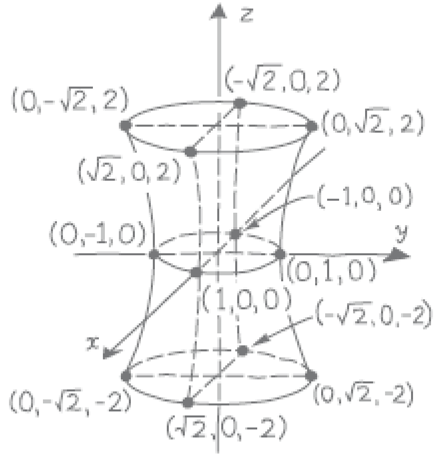
Solution: The trace in the xy-plane, obtained by setting $z = 0$, is

$$x^2 + y^2 = 1 \ (z = 0)$$

which is a circle of radius 1 centered on the z-axis. The traces in the planes $z = 2$ and $z = -2$, obtained by setting $z = \pm 2$, are given by

$$x^2 + y^2 = 2 \ (z = \pm 2)$$

which are circles of radius $\sqrt{2}$ centered on the z-axis. Joining these circles by the hyperbolic traces in the vertical coordinate planes yields the graph in the following figure.



Example: Sketch the graph of the hyperboloid of two sheets

$$z^2 - x^2 - \frac{y^2}{4} = 1$$

Solution: The z-intercepts, obtained by setting $x = 0$ and $y = 0$, are $z = \pm 1$. The traces in the planes $z = 2$ and $z = -2$, obtained by setting $z = \pm 2$ in (10), are given by

$$\frac{x^2}{3} + \frac{y^2}{12} = 1 \qquad (z = \pm 2)$$

Sketching these ellipses and the hyperbolic traces in the vertical coordinate planes yields the following figure.