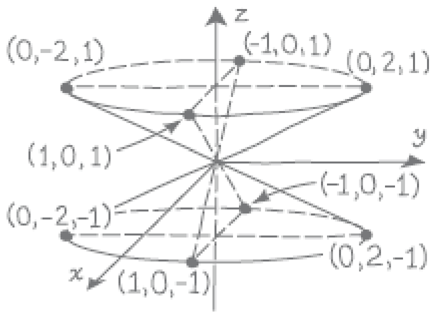
Example: Sketch the graph of the elliptic cone

$$z^2 = x^2 + \frac{y^2}{4}$$



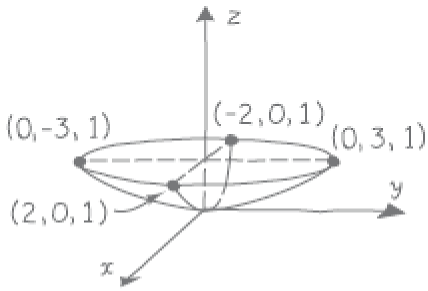
Solution: The traces in the planes z = ±1 are given by

$$x^2 + \frac{y^2}{4} = 1 \qquad (z = \pm 1)$$

Sketching these ellipses and the linear traces in the vertical coordinate planes yields the graph in the figure.

Example: Sketch the graph of the elliptic paraboloid

$$z = \frac{x^2}{4} + \frac{y^2}{9}$$



Solution: The trace in the plane $z = 1$ is

$$\frac{x^2}{4} + \frac{y^2}{9} = 1 \qquad (z = 1)$$

Sketching this ellipse and the parabolic traces in the vertical coordinate planes yields the graph in the figure.

Example: Sketch the graph of the hyperbolic paraboloid

$$z = \frac{y^2}{4} - \frac{x^2}{9} \qquad \text{(a)}$$

Solution. Setting $x = 0$ in (a) yields

$$z = \frac{y^2}{4}$$

which is a parabola in the yz-plane with vertex at the origin and opening in the positive z-direction (since $z \geq 0$), and setting $y = 0$ yields

$$z = -\frac{x^2}{9}$$

which is a parabola in the xz-plane with vertex at the origin and opening in the negative z-direction.

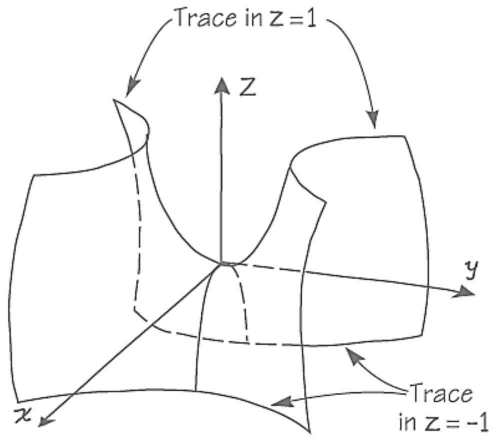
The trace in the plane $z = 1$ is

$$\frac{y^2}{4} - \frac{x^2}{9} = 1 \qquad (z = 1)$$

which is a hyperbola that opens along a line parallel to the *y*-axis, and the trace in the plane $z = -1$ is

$$\frac{x^2}{9} - \frac{y^2}{4} = 1 \qquad (z = -1)$$

which is a hyperbola that opens along a line parallel to the *x*-axis. Combining all of the above information leads to the sketch in Figure.



1.7.2 Translations of Quadric Surfaces

Example: Describe the surface $z = (x - 1)^2 + (y + 2)^2 + 3$.

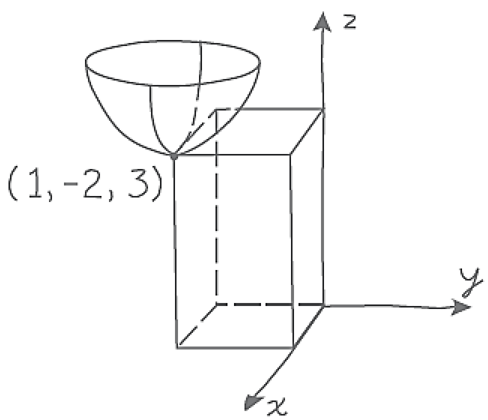
Solution. The equation can be rewritten as

$$z - 3 = (x - 1)^2 + (y + 2)^2$$

This surface is the paraboloid that results by translating the paraboloid

$$z = x^2 + y^2$$

in Figure so that the new "vertex" is at the point (1,–2, 3). A rough sketch of this paraboloid is shown in Figure.



Example: Describe the surface

$$4x^2 + 4y^2 + z^2 + 8y - 4z = -4$$

Solution. Completing the squares yields

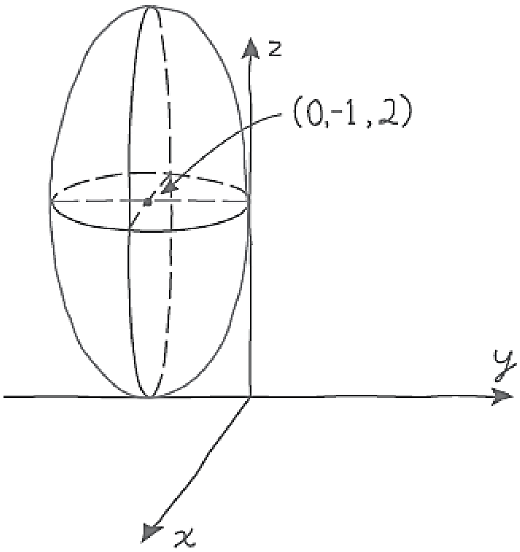
$$4x^2 + 4(y+1)^2 + (z-2)^2 = -4 + 4 + 4$$

$$x^2 + (y+1)^2 + \frac{(z-2)^2}{4} = 1$$

Thus, the surface is the ellipsoid that results when the ellipsoid

$$x^2 + y^2 + \frac{z^2}{4} = 1$$

is translated so that the new "center" is at the point (0,−1, 2). A rough sketch of this ellipsoid is shown in Figure.



1.7.3 A Technique for Identifying Quadric Surfaces

IDENTIFYING A QUADRIC SURFACE FROM THE FORM OF ITS EQUATION

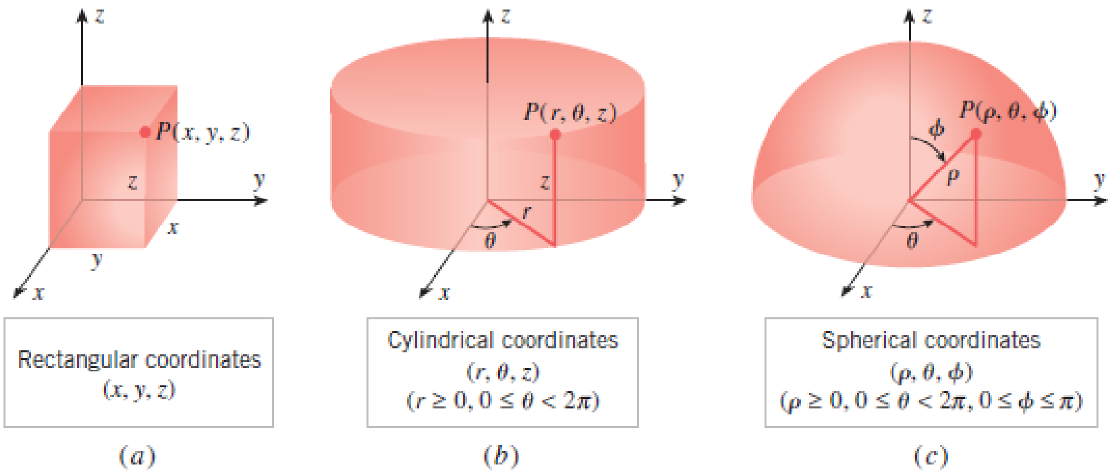
EQUATION	$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$	$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$	$\frac{z^2}{c^2}-\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	$z^2-\frac{x^2}{a^2}-\frac{y^2}{b^2}=0$	$z-\frac{x^2}{a^2}-\frac{y^2}{b^2}=0$	$z-\frac{y^2}{b^2}+\frac{x^2}{a^2}=0$
CHARACTERISTIC	No minus signs	One minus sign	Two minus signs	No linear terms	One linear term; two quadratic terms with the same sign	One linear term; two quadratic terms with opposite signs
CLASSIFICATION	Ellipsoid	Hyperboloid of one sheet	Hyperboloid of two sheets	Elliptic cone	Elliptic paraboloid	Hyperbolic paraboloid

1.8 CYLINDRICAL AND SPHERICAL COORDINATE SYSTEMS

Three coordinates are required to establish the location of a point in 3-space. We have already done this using rectangular coordinates. However, figures (a, b, c) show two other possibilities:

- part (a) of the figure shows the ***rectangular coordinates*** (x, y, z) of a point P,
- part (b) shows the ***cylindrical coordinates*** (r, θ, z) of P,
- part (c) shows the ***spherical coordinates*** (ρ, θ, ϕ) of P.

In a rectangular coordinate system the coordinates can be any real numbers, but in cylindrical and spherical coordinate systems there are restrictions on the allowable values of the coordinates.



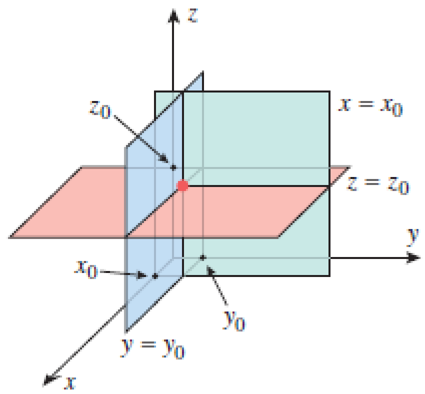
1.8.1 Constant Surfaces

In rectangular coordinates the surfaces represented by equations of the form

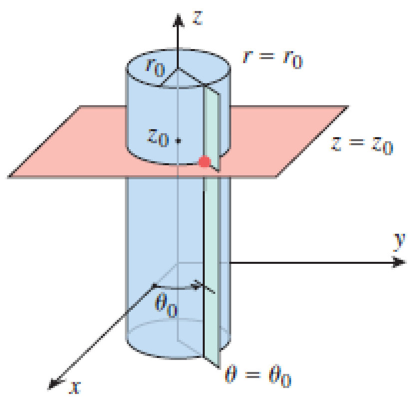
$$x = x_0,\ y = y_0, \text{ and } z = z_0$$

where x_0, y_0, and z_0 are constants, are planes parallel to the yz-plane, xz-plane, and xy-plane, respectively (see figure). In cylindrical coordinates the surfaces represented by equations of the form

$$r = r_0,\ \theta = \theta_0, \text{ and } z = z_0$$



where r_0, θ_0, and z_0 are constants, are shown in the following figure:



• The surface $r = r_0$ is a right circular cylinder of radius r_0 centered on the z-axis.

• The surface $\theta = \theta_0$ is a half-plane attached along the z-axis and making an angle θ_0 with the positive x-axis.

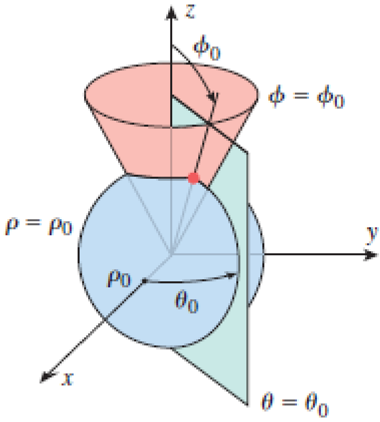
• The surface $z = z_0$ is a horizontal plane.

In spherical coordinates the surfaces represented by equations of the form

$$\rho = \rho_0,\ \theta = \theta_0, \text{ and } \varphi = \varphi_0$$

Where ρ_0, θ_0, and φ_0 are constants, are shown in the following figure:

• The surface $\rho = \rho_0$ consists of all points whose distance ρ from the origin is ρ_0. Assuming ρ_0 to be nonnegative, this is a sphere of radius ρ_0 centered at the origin.



• As in cylindrical coordinates, the surface $\theta = \theta_0$ is a half-plane attached along the z-axis, making an angle of θ_0 with the positive x-axis.

• The surface $\varphi = \varphi_0$ consists of all points from which a line segment to the origin makes an angle of φ_0 with the positive z-axis. If $0 < \varphi_0 < \pi/2$, this will be the nappe of a cone opening up, while if $\pi/2 < \varphi_0 < \pi$, this will be the nappe of a cone opening down. (If $\varphi_0 = \pi/2$, then the cone is flat, and the surface is the xy-plane.)

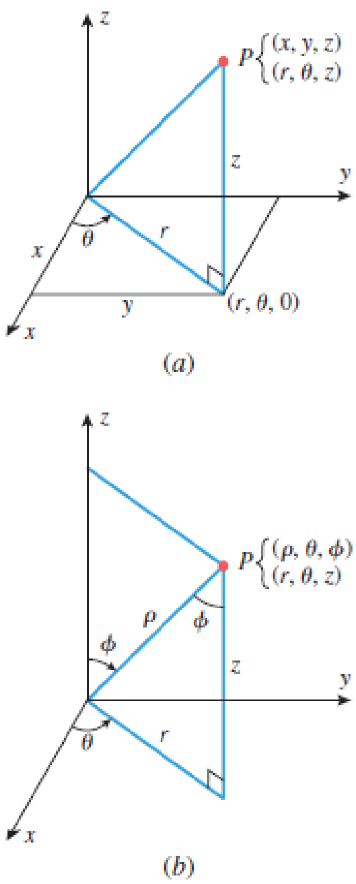
1.8.2 Converting Coordinates

Just as we needed to convert between rectangular and polar coordinates in 2-space, so we will need to be able to convert between rectangular, cylindrical, and spherical coordinates in 3-space. The following table provides formulas for making these conversions.

CONVERSION FORMULAS FOR COORDINATE SYSTEMS

CONVERSION		FORMULAS	RESTRICTIONS
Cylindrical to rectangular	$(r, \theta, z) \to (x, y, z)$	$x = r\cos\theta,\quad y = r\sin\theta,\quad z = z$	$r \ge 0, \rho \ge 0$ $0 \le \theta < 2\pi$ $0 \le \phi \le \pi$
Rectangular to cylindrical	$(x, y, z) \to (r, \theta, z)$	$r = \sqrt{x^2+y^2},\quad \tan\theta = y/x,\quad z = z$	
Spherical to cylindrical	$(\rho, \theta, \phi) \to (r, \theta, z)$	$r = \rho\sin\phi,\quad \theta = \theta,\quad z = \rho\cos\phi$	
Cylindrical to spherical	$(r, \theta, z) \to (\rho, \theta, \phi)$	$\rho = \sqrt{r^2+z^2},\quad \theta = \theta,\quad \tan\phi = r/z$	
Spherical to rectangular	$(\rho, \theta, \phi) \to (x, y, z)$	$x = \rho\sin\phi\cos\theta,\quad y = \rho\sin\phi\sin\theta,\quad z = \rho\cos\phi$	
Rectangular to spherical	$(x, y, z) \to (\rho, \theta, \phi)$	$\rho = \sqrt{x^2+y^2+z^2},\quad \tan\theta = y/x,\quad \cos\phi = z/\sqrt{x^2+y^2+z^2}$	

The diagrams in the following figure will help you to understand how the formulas in the table are derived.



Comparison of coordinate systems

For example, part (*a*) of the figure shows that in converting between rectangular coordinates (x, y, z) and cylindrical coordinates (r, θ, z), we can interpret (r, θ) as polar coordinates of (x, y). Thus, the polar-to-rectangular and rectangular-to-polar conversion formulas (1) and (2) provide the conversion formulas between rectangular and cylindrical coordinates in the table.

Part (*b*) of Figure suggests that the spherical coordinates (ρ, θ, φ) of a point P can be converted to cylindrical coordinates (r, θ, z) by the conversion formulas

$$r = \rho \sin \varphi,\ \theta = \theta,\ z = \rho \cos \varphi \qquad (1)$$

Moreover, since the cylindrical coordinates (r, θ, z) of P can be converted to rectangular coordinates (x, y, z) by the conversion formulas

$$x = r \cos \theta,\ y = r \sin \theta,\ z = z \qquad (2)$$

We can obtain direct conversion formulas from spherical coordinates to rectangular coordinates by substituting (1) in (2). This yields

$$x = \rho \sin \varphi \cos \theta,\ y = \rho \sin \varphi \sin \theta,\ z = \rho \cos \varphi \qquad (3)$$

Example:

(a) Find the rectangular coordinates of the point with cylindrical coordinates

$$(r, \theta, z) = (4, \pi/3, -3)$$

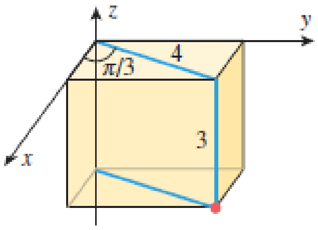
(b) Find the rectangular coordinates of the point with spherical coordinates

$$(\rho, \theta, \varphi) = (4, \pi/3, \pi/4)$$

Solution (a): Applying the cylindrical-to-rectangular conversion formulas in the table yields

$$x = r\cos\theta = 4\cos\frac{\pi}{3} = 2, \quad y = r\sin\theta = 4\sin\frac{\pi}{3} = 2\sqrt{3}, \quad z = -3$$

Thus, the rectangular coordinates of the point are $(x, y, z) = (2, 2\sqrt{3}, -3)$ (see figure).



cylindrical: $(4, \pi/3, -3)$
rectangular: $(2, 2\sqrt{3}, -3)$

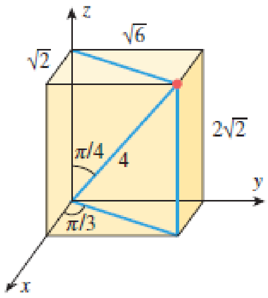
Solution (b): Applying the spherical-to-rectangular conversion formulas in the table yields

$$x = \rho\sin\phi\cos\theta = 4\sin\frac{\pi}{4}\cos\frac{\pi}{3} = \sqrt{2}$$

$$y = \rho\sin\phi\sin\theta = 4\sin\frac{\pi}{4}\sin\frac{\pi}{3} = \sqrt{6}$$

$$z = \rho\cos\phi = 4\cos\frac{\pi}{4} = 2\sqrt{2}$$

The rectangular coordinates of the point are $(x, y, z) = (\sqrt{2}, \sqrt{6}, 2\sqrt{2})$



spherical: $(4, \pi/3, \pi/4)$
rectangular: $(\sqrt{2}, \sqrt{6}, 2\sqrt{2})$

Example: Find the spherical coordinates of the point that has rectangular coordinates

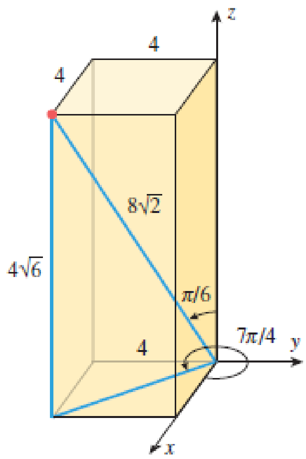
$$(x, y, z) = (4, -4, 4\sqrt{6})$$

Solution: From the rectangular-to-spherical conversion formulas in the table we obtain

$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{16 + 16 + 96} = \sqrt{128} = 8\sqrt{2}$$

$$\tan\theta = \frac{y}{x} = -1$$

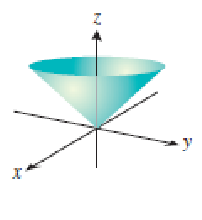
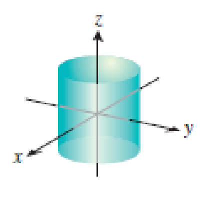
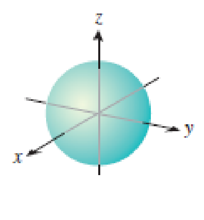
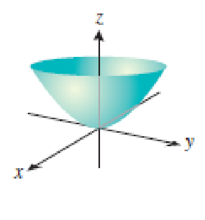
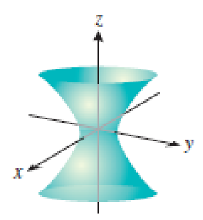
$$\cos\phi = \frac{z}{\sqrt{x^2 + y^2 + z^2}} = \frac{4\sqrt{6}}{8\sqrt{2}} = \frac{\sqrt{3}}{2}$$



rectangular: $(4, -4, 4\sqrt{6})$
spherical: $(8\sqrt{2}, 7\pi/4, \pi/6)$

From the restriction $0 \le \theta < 2\pi$ and the computed value of $\tan\theta$, the possibilities for θ are $\theta = 3\pi/4$ and $\theta = 7\pi/4$. However, the given point has a negative y-coordinate, so we must have $\theta = 7\pi/4$. Moreover, from the restriction $0 \le \varphi \le \pi$ and the computed value of $\cos\varphi$, the only possibility for φ is $\varphi = \pi/6$. Thus, the spherical coordinates of the point are $(\rho, \theta, \varphi) = (8\sqrt{2}, 7\pi/4, \pi/6)$.

1.8.3 Equations of Surfaces in Cylindrical and Spherical Coordinates

	CONE	CYLINDER	SPHERE	PARABOLOID	HYPERBOLOID
					
RECTANGULAR	$z = \sqrt{x^2 + y^2}$	$x^2 + y^2 = 1$	$x^2 + y^2 + z^2 = 1$	$z = x^2 + y^2$	$x^2 + y^2 - z^2 = 1$
CYLINDRICAL	$z = r$	$r = 1$	$z^2 = 1 - r^2$	$z = r^2$	$z^2 = r^2 - 1$
SPHERICAL	$\phi = \pi/4$	$\rho = \csc\phi$	$\rho = 1$	$\rho = \cos\phi \csc^2\phi$	$\rho^2 = -\sec 2\phi$