CHAPTER TWO

VECTOR-VALUED FUNCTIONS

2.1 INTRODUCTION TO VECTOR-VALUED FUNCTIONS

2.1.1 Parametric Curves in 3-Space

- If f and g are well-behaved functions, then the pair of parametric equations

$$x = f(t), \quad y = g(t) \tag{2-1}$$

generates a curve in 2-space that is traced in a specific direction as the parameter t increases.

- We defined this direction to be the ***orientation*** of the curve or the ***direction of increasing parameter***, and we called the curve together with its orientation the *graph* of the parametric equations or the ***parametric curve*** represented by the equations.
- Analogously, if f, g, and h are three well-behaved functions, then the parametric equations

$$x = f(t), \quad y = g(t), \quad z = h(t) \tag{2-2}$$

- Generate a curve in 3-space that is traced in a specific direction as t increases. As in 2-space, this direction is called the ***orientation*** or ***direction of increasing parameter***, and the curve together with its orientation is called the ***graph*** of the parametric equations or the ***parametric curve*** represented by the equations. If no restrictions are stated explicitly or are implied by the equations, then it will be understood that t varies over the interval $(-\infty, +\infty)$.

Example 2-1: The parametric equations $x = 1 - t, \quad y = 3t, \quad z = 2t$
represent a line in 3-space that passes through the point (1, 0, 0) and is parallel to the vector (−1, 3, 2). Since x decreases as t increases, the line has the orientation shown in Figure 2-1.

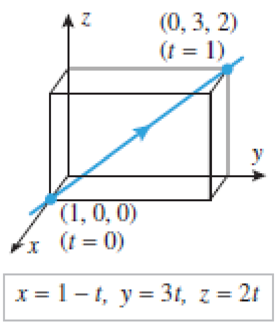


Figure 2-1

Example 2-2 Describe the parametric curve represented by the equations

$$x = a\cos t,\ y = a\sin t,\ z = ct$$

where a and c are positive constants.

Solution: As the parameter t increases, the value of $z = ct$ also increases, so the point (x, y, z) moves upward. However, as t increases, the point (x, y, z) also moves in a path directly over the circle

$$x = a\cos t,\ y = a\sin t$$

in the xy-plane. The combination of these upward and circular motions produces a corkscrew-shaped curve that wraps around a right circular cylinder of radius a centered on the z-axis (Figure 2-2). This curve is called a ***circular helix***.

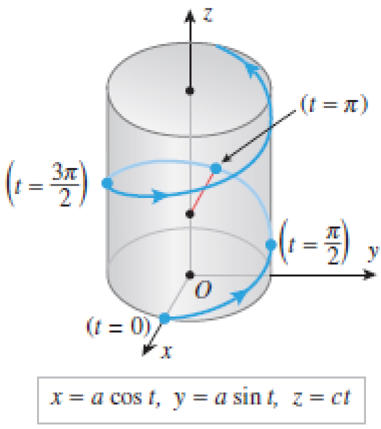


Figure 2-2

2.1.2 Parametric Curves Generated with Technology

Except in the simplest cases, parametric curves can be difficult to visualize and draw without the help of a graphing utility. For example, the ***tricuspoid*** is the graph of the parametric equations

$$x = 2\cos(t) + \cos(2t), \quad y = 2\sin(t) - \sin(2t)$$

Although it would be tedious to plot the tricuspoid by hand, a computer rendering is easy to obtain and reveals the significance of the name of the curve (Figure 2-3).

However, note that the depiction of the tricuspoid in Figure 2-3 is incomplete, since the orientation of the curve is not indicated. This is often the case for curves that are generated with a graphing utility.

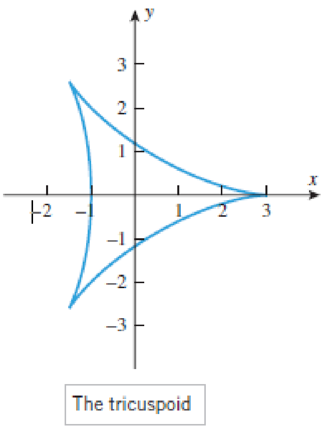


Figure 2-3

Parametric curves in 3-space can be difficult to visualize correctly even with the help of a graphing utility. For example, Figure 2-4*a* shows a parametric curve called a *torus knot* that was produced with a CAS.

Some graphing utilities provide the capability of enclosing the curve within a thin tube, as in Figure 2-4*b*. Such graphs are called ***tube plots***.

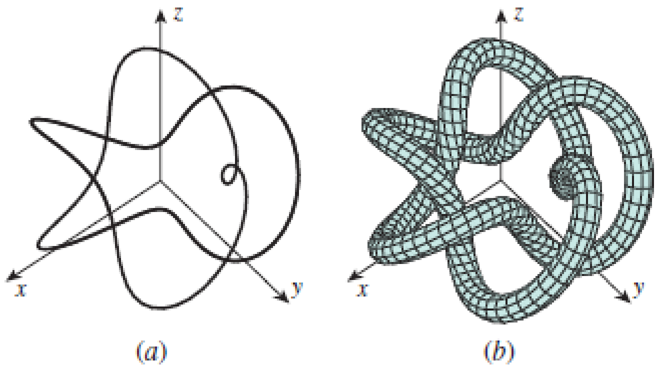


Figure 2-4

2.1.3 Parametric Equations for Intersections of Surfaces

Curves in 3-space often arise as intersections of surfaces. For example, Figure 2-5*a* shows a portion of the intersection of the cylinders $z = x^3$ and $y = x^2$.

One method for finding parametric equations for the curve of intersection is to choose one of the variables as the parameter and use the two equations to express the remaining two variables in terms of that parameter. In particular, if we choose $x = t$ as the parameter and substitute this into the equations $z = x^3$ and $y = x^2$, we obtain the parametric equations

$$x = t, \qquad y = t^2, \qquad z = t^3 \qquad (2\text{-}3)$$

This curve is called a ***twisted cubic***. The portion of the twisted cubic shown in Figure 2-5*a* corresponds to $t \geq 0$; a computer-generated graph of the twisted cubic for positive and negative values of t is shown in Figure 2-5*b*. Some other examples and techniques for finding intersections of surfaces are discussed in the exercises.

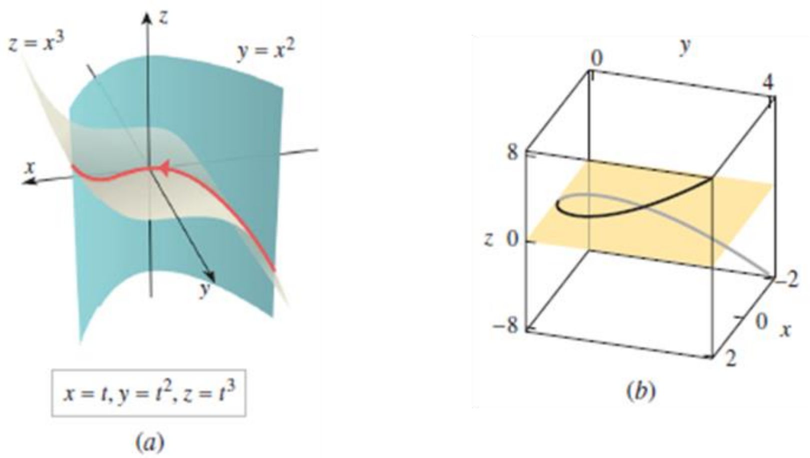


Figure 2-5

2.1.4 Vector-Valued Functions

The twisted cubic defined by the equations in (2-3) is the set of points of the form (t, t^2, t^3) for real values of t. If we view each of these points as a terminal point for a vector **r** whose initial point is at the origin,

$$\mathbf{r} = (x, y, z) = (t, t^2, t^3) = t\,\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$$

then we obtain **r** as a function of the parameter t, that is, $\mathbf{r} = \mathbf{r}(t)$. Since this function produces a *vector*, we say that $\mathbf{r} = \mathbf{r}(t)$ defines **r** as a ***vector-valued function of a real variable***, or more simply, a ***vector-valued function***. The vectors that we will consider in this text are either in 2-space or 3-space, so we will say that a vector-valued function is in 2-space or in 3-space according to the kind of vectors that it produces.

If $\mathbf{r}(t)$ is a vector-valued function in 3-space, then for each allowable value of t the vector $\mathbf{r} = \mathbf{r}(t)$ can be represented in terms of components as

$$\mathbf{r} = \mathbf{r}(t) = (x(t), y(t), z(t)) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k} \qquad (2\text{-}4)$$

The functions $x(t)$, $y(t)$, and $z(t)$ are called the ***component functions*** or the ***components*** of $\mathbf{r}(t)$.

Example 2-3: The component functions of $\mathbf{r}(t) = (t, t^2, t^3) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$

are $\quad x(t) = t,\ y(t) = t^2,\ z(t) = t^3$

The ***domain*** of a vector-valued function $\mathbf{r}(t)$ is the set of allowable values for t.

If $\mathbf{r}(t)$ is defined in terms of component functions and the domain is not specified explicitly, then it will be understood that the domain is the intersection of the natural domains of the component functions; this is called the ***natural domain*** of $\mathbf{r}(t)$.

Example 2-4: Find the natural domain of

$$\mathbf{r}(t) = \langle \ln|t-1|, e^t, \sqrt{t} \rangle = (\ln|t-1|)\mathbf{i} + e^t\mathbf{j} + \sqrt{t}\mathbf{k}$$

Solution: The natural domains of the component functions

$$x(t) = \ln|t-1|, \quad y(t) = e^t, \quad z(t) = \sqrt{t}$$

are

$$(-\infty, 1) \cup (1, +\infty), \quad (-\infty, +\infty), \quad [0, +\infty)$$

respectively. The intersection of these sets is

$$[0, 1) \cup (1, +\infty)$$

(verify), so the natural domain of $\mathbf{r}(t)$ consists of all values of t such that

$$0 \le t < 1 \quad \text{or} \quad t > 1$$

2.1.5 Graphs of Vector-Valued Functions

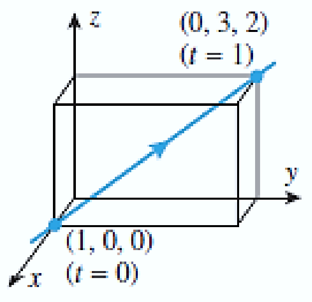
If $\mathbf{r}(t)$ is a vector-valued function in 2-space or 3-space, then we define the ***graph*** of $\mathbf{r}(t)$ to be the parametric curve described by the component functions for $\mathbf{r}(t)$. For example, if

$$\mathbf{r}(t) = (1 - t, 3t, 2t) = (1 - t)\mathbf{i} + 3t\,\mathbf{j} + 2t\mathbf{k} \qquad (2\text{-}5)$$

then the graph of $\mathbf{r} = \mathbf{r}(t)$ is the graph of the parametric equations

$$x = 1 - t,\ y = 3t,\ z = 2t$$

Thus, the graph of (2-5) is the line in Figure 2-1.



$x = 1 - t,\ y = 3t,\ z = 2t$

Example 2-5: Sketch the graph and a radius vector of

(a) $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j},\ 0 \le t \le 2\pi$

(b) $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + 2\mathbf{k},\ 0 \le t \le 2\pi$

Solution (a): The corresponding parametric equations are

$$x = \cos t,\ y = \sin t\ (0 \le t \le 2\pi)$$

So the graph is a circle of radius 1, centered at the origin, and oriented counter clockwise. The graph and a radius vector are shown in Figure 2-6.

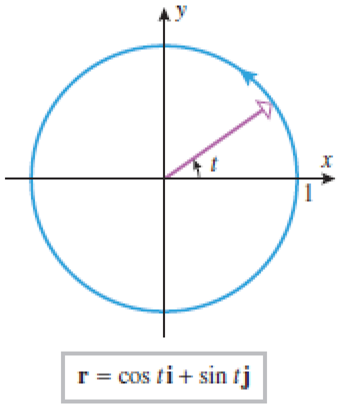


Figure 2-6

Solution (b): The corresponding parametric equations are

$$x = \cos t,\ y = \sin t,\ z = 2\ (0 \le t \le 2\pi)$$

From the third equation, the tip of the radius vector traces a curve in the plane $z = 2$, and from the first two equations, the curve is a circle of radius 1 centered at the point $(0, 0, 2)$ and traced counter clockwise looking down the z-axis. The graph and a radius vector are shown in Figure 2-7.

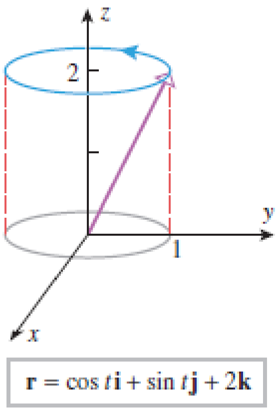


Figure 2-7

2.1.6 Vector Form of a Line Segment

If $\mathbf{r}_0$ is a vector in 2-space or 3-space with its initial point at the origin, then the line that passes through the terminal point of $\mathbf{r}_0$ and is parallel to- the vector $\mathbf{v}$ can be expressed in vector form as

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$$

In particular, if $\mathbf{r}_0$ and $\mathbf{r}_1$ are vectors in 2-space or 3-space with their initial points at the origin, then the line that passes through the terminal points of these vectors can be expressed in vector form as

$$\mathbf{r} = \mathbf{r}_0 + t(\mathbf{r}_1 - \mathbf{r}_0) \qquad (2\text{-}6) \qquad \text{or } \mathbf{r} = (1 - t)\mathbf{r}_0 + t\mathbf{r}_1 \quad (2\text{-}7)$$

as indicated in Figure 2-8.

It is common to call either (2-6) or (2-7) the ***two-point vector form of a line*** and to say, for simplicity, that the line passes through the *points* $\mathbf{r}_0$ and $\mathbf{r}_1$ (as opposed to saying that it passes through the *terminal points* of $\mathbf{r}_0$ and $\mathbf{r}_1$).

It is understood in (6) and (7) that t varies from $-\infty$ to $+\infty$. However, if we restrict t to vary over the interval $0 \le t \le 1$, then $\mathbf{r}$ will vary from $\mathbf{r}_0$ to $\mathbf{r}_1$. Thus, the equation

$$\mathbf{r} = (1 - t)\mathbf{r}_0 + t\mathbf{r}_1 \ (0 \le t \le 1) \ (2\text{-}8)$$

represents the line segment in 2-space or 3-space that is traced from $\mathbf{r}_0$ to $\mathbf{r}_1$.

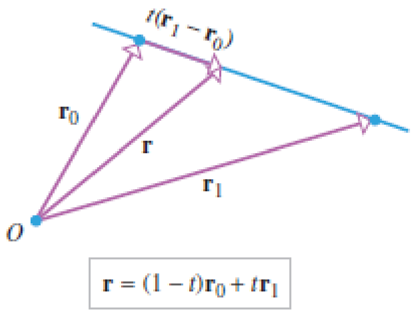


Figure 2-8