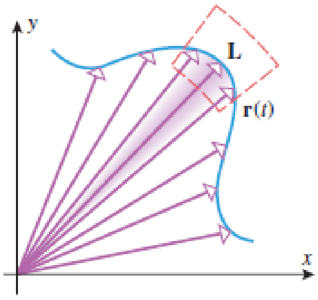
2.2 CALCULUS OF VECTOR-VALUED FUNCTIONS

2.2.1 Limits and Continuity

- A vector-valued function $\mathbf{r}(t)$ in 2-space or 3-space to approach a limiting vector $\mathbf{L}$ as t approaches a number a. That is, we want to define

$$\lim_{t \to a} \mathbf{r}(t) = \mathbf{L}$$

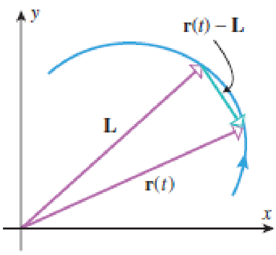
- position $\mathbf{r}(t)$ and $\mathbf{L}$ with their initial points at the origin and interpret this limit to mean that the terminal point of $\mathbf{r}(t)$ approaches the terminal point of $\mathbf{L}$ as t approaches a or, equivalently, that the vector $\mathbf{r}(t)$ approaches the vector $\mathbf{L}$ in both length and direction at t approaches a (see figure). Algebraically, this is equivalent to stating that



$\mathbf{r}(t)$ approaches $\mathbf{L}$ in length and direction if $\lim_{t \to a} \mathbf{r}(t) = \mathbf{L}$.

$$\lim_{t \to a} \|\mathbf{r}(t) - \mathbf{L}\| = \mathbf{0}$$

(the following figure). Thus, we make the following definition.



$\|\mathbf{r}(t) - \mathbf{L}\|$ is the distance between terminal points for vectors $\mathbf{r}(t)$ and $\mathbf{L}$ when positioned with the same initial points.

Definition Let $\mathbf{r}(t)$ be a vector-valued function that is defined for all t in some open interval containing the number a, except that $\mathbf{r}(t)$ need not be defined at a.

We will write

$$\lim_{t \to a} \mathbf{r}(t) = \mathbf{L}$$

if and only if

$$\lim_{t \to a} \|\mathbf{r}(t) - \mathbf{L}\| = \mathbf{0}$$

THEOREM

(a) *If* $\mathbf{r}(t) = \langle x(t), y(t) \rangle = x(t)\mathbf{i} + y(t)\mathbf{j}$, *then*

$$\lim_{t \to a} \mathbf{r}(t) = \left\langle \lim_{t \to a} x(t), \lim_{t \to a} y(t) \right\rangle = \lim_{t \to a} x(t)\mathbf{i} + \lim_{t \to a} y(t)\mathbf{j}$$

provided the limits of the component functions exist. Conversely, the limits of the component functions exist provided $\mathbf{r}(t)$ *approaches a limiting vector as* t *approaches* a.

(b) *If* $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, *then*

$$\lim_{t \to a} \mathbf{r}(t) = \left\langle \lim_{t \to a} x(t), \lim_{t \to a} y(t), \lim_{t \to a} z(t) \right\rangle = \lim_{t \to a} x(t)\mathbf{i} + \lim_{t \to a} y(t)\mathbf{j} + \lim_{t \to a} z(t)\mathbf{k}$$

provided the limits of the component functions exist. Conversely, the limits of the component functions exist provided $\mathbf{r}(t)$ *approaches a limiting vector as* t *approaches* a.

Example: Let $\mathbf{r}(t) = t^2\mathbf{i} + e^t\,\mathbf{j} - 2\cos(\pi t)\,\mathbf{k}$. Then

$$\lim_{t \to 0} \mathbf{r}(t) = \left(\lim_{t \to 0} t^2\right)\mathbf{i} + \left(\lim_{t \to 0} e^t\right)\mathbf{j} - \left(\lim_{t \to 0} 2\cos \pi t\right)\mathbf{k} = \mathbf{j} - 2\mathbf{k}$$

Alternatively, using the angle bracket notation for vectors,

$$\lim_{t \to 0} \mathbf{r}(t) = \lim_{t \to 0} \langle t^2, e^t, -2\cos \pi t \rangle = \left\langle \lim_{t \to 0} t^2, \lim_{t \to 0} e^t, \lim_{t \to 0} (-2\cos \pi t) \right\rangle = \langle 0, 1, -2 \rangle$$

Motivated by the definition of continuity for real-valued functions, we define a vector valued function $\mathbf{r}(t)$ to be ***continuous*** at $t = a$ if

$$\lim_{t \to a} \mathbf{r}(t) = \mathbf{r}(a)$$

That is, $\mathbf{r}(a)$ is defined, the limit of $\mathbf{r}(t)$ as $t \to a$ exists, and the two are equal. As in the case for real-valued functions, we say that $\mathbf{r}(t)$ is ***continuous on an interval*** I if it is continuous at each point of I [with the understanding that at an endpoint in I the two-sided limit in (above equation) is replaced by the appropriate one-sided limit].

A vector-valued function is continuous at $t = a$ if and only if its component functions are continuous at $t = a$.

2.2.2 Derivatives

The derivative of a vector-valued function is defined by a limit similar to that for the derivative of a real-valued function.

Definition If **r**(t) is a vector-valued function, we define the ***derivative of* r *with respect to t*** to be the vector-valued function **r** given by

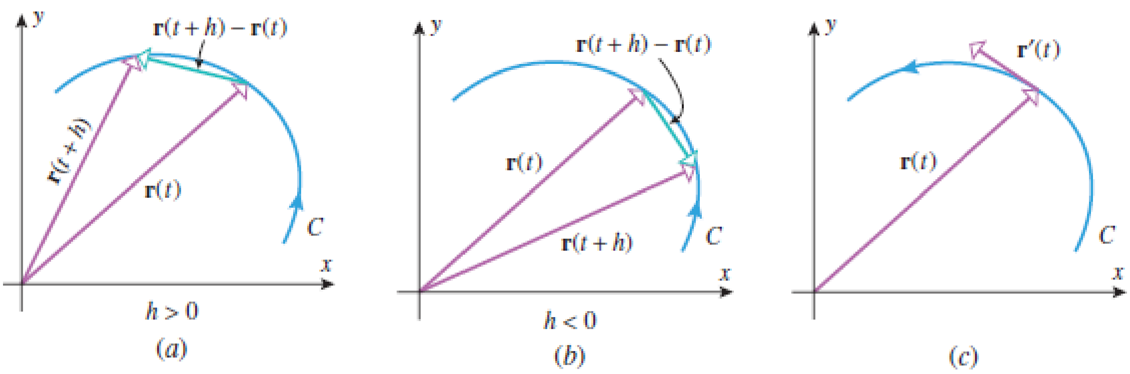
$$\mathbf{r}'(t) = \lim_{h \to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}$$

The domain of **r** consists of all values of t in the domain of **r**(t) for which the limit exists.

-The function **r**(t) is ***differentiable*** at t if the limit exists.
-The derivative of **r**(t) can be expressed as

$$\frac{d}{dt}[\mathbf{r}(t)], \quad \frac{d\mathbf{r}}{dt}, \quad \mathbf{r}'(t), \quad \text{or} \quad \mathbf{r}'$$

It is important to keep in mind that **r**(t) is a vector, not a number, and hence has a magnitude and a direction for each value of t [except if **r**(t) = **0**, in which case **r**(t) has magnitude zero but no specific direction].



These illustrations show the graph C of **r**(t) (with its orientation) and the vectors **r**(t), **r**$(t + h)$, and **r**$(t + h)$ − **r**(t) for positive h and for negative h.

In both cases, the vector **r**$(t + h)$ − **r**(t) runs along the secant line joining the terminal points of **r**$(t + h)$ and **r**(t), but with opposite directions in the two cases. In the case where h is positive the vector **r**$(t + h)$ − **r**(t) points in the direction of increasing parameter, and in the case where h is negative it points in the opposite direction. However, in the case where h is negative the direction gets reversed when we multiply by $1/h$, so in both cases the vector

$$\frac{1}{h}[\mathbf{r}(t+h) - \mathbf{r}(t)] = \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}$$

points in the direction of increasing parameter and runs along the secant line. As $h \to 0$, the secant line approaches the tangent line at the terminal point of **r**(t), so we can conclude that the limit

$$\mathbf{r}'(t) = \lim_{h \to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}$$

(if it exists and is nonzero) is a vector that is tangent to the curve C at the tip of **r**(t) and points in the direction of increasing parameter (Figure *c*). We can summarize all of this as follows.

Geometric interpretation of the derivative

Suppose that C is the graph of a vector-valued function **r**(t) in 2-space or 3-space and that **r**´ (t) exists and is nonzero for a given value of t . If the vector **r**´ (t) is positioned with its initial point at the terminal point of the radius vector **r**(t), then **r**´(t) is tangent to C and points in the direction of increasing parameter.

Theorem

If **r**(t) *is a vector-valued function, then* **r** *is differentiable at t if and only if each of its component functions is differentiable at t, in which case the component functions of* **r**´ (t) *are the derivatives of the corresponding component functions of* **r**(t).

Proof

For simplicity, we give the proof in 2-space; the proof in 3-space is identical, except for the additional component. Assume that $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$. Then

$$\begin{aligned}
\mathbf{r}'(t) &= \lim_{h \to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h} \\
&= \lim_{h \to 0} \frac{[x(t+h)\mathbf{i} + y(t+h)\mathbf{j}] - [x(t)\mathbf{i} + y(t)\mathbf{j}]}{h} \\
&= \left(\lim_{h \to 0} \frac{x(t+h) - x(t)}{h}\right)\mathbf{i} + \left(\lim_{h \to 0} \frac{y(t+h) - y(t)}{h}\right)\mathbf{j} \\
&= x'(t)\mathbf{i} + y'(t)\mathbf{j}
\end{aligned}$$

Example: Let $\mathbf{r}(t) = t^2\,\mathbf{i} + e^t\,\mathbf{j} - 2\cos(\pi t)\mathbf{k}$. Then

Solution:

$$\mathbf{r}'(t) = \frac{d}{dt}(t^2)\mathbf{i} + \frac{d}{dt}(e^t)\mathbf{j} - \frac{d}{dt}(2\cos \pi t)\mathbf{k}$$
$$= 2t\mathbf{i} + e^t\mathbf{j} + (2\pi \sin \pi t)\mathbf{k}$$

2.2.3 Derivative Rules

Theorem

(*Rules of Differentiation*)

Let $\mathbf{r}(t)$, $\mathbf{r}_1(t)$, *and* $\mathbf{r}_2(t)$ *be differentiable vector-valued functions that are all in 2-space or all in 3-space, and let f(t) be a differentiable real-valued function, k a scalar, and* $\mathbf{c}$ *a constant vector (that is, a vector whose value does not depend on t). Then the following rules of differentiation hold:*

(a) $\frac{d}{dt}[\mathbf{c}] = \mathbf{0}$

(b) $\frac{d}{dt}[k\mathbf{r}(t)] = k\frac{d}{dt}[\mathbf{r}(t)]$

(c) $\frac{d}{dt}[\mathbf{r}_1(t) + \mathbf{r}_2(t)] = \frac{d}{dt}[\mathbf{r}_1(t)] + \frac{d}{dt}[\mathbf{r}_2(t)]$

(d) $\frac{d}{dt}[\mathbf{r}_1(t) - \mathbf{r}_2(t)] = \frac{d}{dt}[\mathbf{r}_1(t)] - \frac{d}{dt}[\mathbf{r}_2(t)]$

(e) $\frac{d}{dt}[f(t)\mathbf{r}(t)] = f(t)\frac{d}{dt}[\mathbf{r}(t)] + \frac{d}{dt}[f(t)]\mathbf{r}(t)$