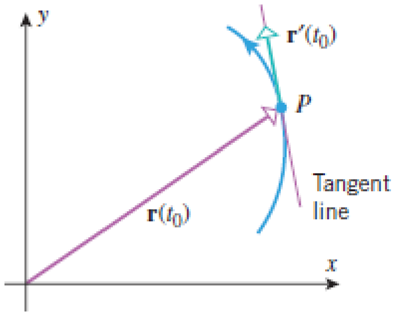
2.2.1 Tangent Lines to Graphs of Vector-Valued Functions

Definition Let P be a point on the graph of a vector-valued function $\mathbf{r}(t)$, and let $\mathbf{r}(t_0)$ be the radius vector from the origin to P (see below figure). If $\mathbf{r}'(t_0)$ exists and $\mathbf{r}'(t_0) \neq \mathbf{0}$, then we call $\mathbf{r}'(t_0)$ a ***tangent vector*** to the graph of $\mathbf{r}(t)$ at $\mathbf{r}(t_0)$, and we call the line through P that is parallel to the tangent vector the ***tangent line*** to the graph of $\mathbf{r}(t)$ at $\mathbf{r}(t_0)$.



Let $\mathbf{r}_0 = \mathbf{r}(t_0)$ and $\mathbf{v}_0 = \mathbf{r}'(t_0)$. The tangent line to the graph of $\mathbf{r}(t)$ at $\mathbf{r}_0$ is given by the vector equation

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}_0$$

Example: Find parametric equations of the tangent line to the circular helix

$$x = \cos t,\ y = \sin t,\ z = t$$

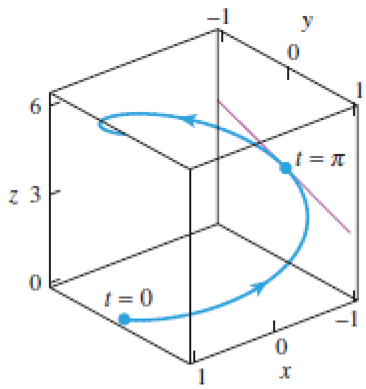
where $t = t_0$, and use that result to find parametric equations for the tangent line at the point where $t = \pi$.

Solution: The vector equation of the helix is

$$\mathbf{r}(t) = \cos t\ \mathbf{i} + \sin t\ \mathbf{j} + t\ \mathbf{k}$$

$$\mathbf{r}_0 = \mathbf{r}(t_0) = \cos t_0\mathbf{i} + \sin t_0\ \mathbf{j} + t_0\mathbf{k}$$

$$\mathbf{v}_0 = \mathbf{r}'(t_0) = (-\sin t_0)\mathbf{i} + \cos t_0\ \mathbf{j} + \mathbf{k}$$



The vector equation of the tangent line at $t = t_0$ is

$$\mathbf{r} = \cos t_0\mathbf{i} + \sin t_0\ \mathbf{j} + t_0\mathbf{k} + t\,[(-\sin t_0)\mathbf{i} + \cos t_0\ \mathbf{j} + \mathbf{k}]$$

$$= (\cos t_0 - t\sin t_0)\mathbf{i} + (\sin t_0 + t\cos t_0)\mathbf{j} + (t_0 + t)\mathbf{k}$$

Thus, the parametric equations of the tangent line at $t = t_0$ are

$$x = \cos t_0 - t\sin t_0,\ y = \sin t_0 + t\cos t_0,\ z = t_0 + t$$

In particular, the tangent line at $t = \pi$ has parametric equations

$$x = -1,\ y = -t,\ z = \pi + t$$

The graph of the helix and this tangent line are shown in figure.

Example: Let

$$\mathbf{r}_1(t) = (\tan^{-1} t)\mathbf{i} + (\sin t)\mathbf{j} + t^2\mathbf{k}$$

and

$$\mathbf{r}_2(t) = (t^2 - t)\mathbf{i} + (2t - 2)\mathbf{j} + (\ln t)\mathbf{k}$$

The graphs of $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$ intersect at the origin. Find the degree measure of the acute angle between the tangent lines to the graphs of $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$ at the origin.

Solution: The graph of $\mathbf{r}_1(t)$ passes through the origin at $t = 0$, where its tangent vector is

$$\acute{r}_1(0) = \langle \frac{1}{1+t^2}, \cos t, 2t\rangle\Big|_{t=0} = \langle 1, 1, 0\rangle$$

The graph of $\mathbf{r}_2(t)$ passes through the origin at $t = 1$ (verify), where its tangent vector is

$$\acute{r}_2(1) = \langle 2t - 1, 2, \frac{1}{t}\rangle\Big|_{t=1} = \langle 1, 2, 1\rangle$$

the angle θ between these two tangent vectors satisfies

$$\cos\theta = \frac{\langle 1,1,0\rangle \cdot \langle 1,2,1\rangle}{\|\langle 1,1,0\rangle\|\|\langle 1,2,1\rangle\|} = \frac{1+2+0}{\sqrt{2}\sqrt{6}} = \frac{3}{\sqrt{12}} = \frac{\sqrt{3}}{2}$$

It follows that $\theta = \pi/6$ radians, or 30°.

2.2.2 Derivatives of Dot and Cross Products

The following rules, which are derived in the exercises, provide a method for differentiating dot products in 2-space and 3-space and cross products in 3-space.

$$\frac{d}{dt}[\mathbf{r}_1(t)\cdot\mathbf{r}_2(t)] = \mathbf{r}_1(t)\cdot\frac{d\mathbf{r}_2}{dt} + \frac{d\mathbf{r}_1}{dt}\cdot\mathbf{r}_2(t) \qquad \text{(a)}$$

$$\frac{d}{dt}[\mathbf{r}_1(t)\times\mathbf{r}_2(t)] = \mathbf{r}_1(t)\times\frac{d\mathbf{r}_2}{dt} + \frac{d\mathbf{r}_1}{dt}\times\mathbf{r}_2(t) \qquad \text{(b)}$$

Theorem

If **r**(*t*) *is a differentiable vector-valued function in 2-space or 3-space and* ||**r**(*t*)|| *is constant for all t, then*

$$\mathbf{r}(t)\,.\,\mathbf{r}'(t) = 0$$

that is, **r**(*t*) *and* **r**′(*t*) *are orthogonal vectors for all t.*

Proof:

It follows from (a) with $\mathbf{r}_1(t) = \mathbf{r}_2(t) = \mathbf{r}(t)$ that

$$\frac{d}{dt}[\mathbf{r}(t)\cdot\mathbf{r}(t)] = \mathbf{r}(t)\cdot\frac{d\mathbf{r}}{dt} + \frac{d\mathbf{r}}{dt}\cdot\mathbf{r}(t)$$

or, equivalently,

$$\frac{d}{dt}[\|\mathbf{r}(t)\|^2] = 2\mathbf{r}(t)\cdot\frac{d\mathbf{r}}{dt}$$

But $\|\mathbf{r}(t)\|^2$ is constant, so its derivative is zero. Thus

$$2\mathbf{r}(t)\cdot\frac{d\mathbf{r}}{dt} = 0$$

2.2.3 Definite Integrals of Vector-Valued Functions

If **r**(*t*) is a vector-valued function that is continuous on the interval $a \le t \le b$, then we define the ***definite integral*** of **r**(*t*) over this interval as a limit of Riemann sums. Specifically, we define

$$\int_a^b \mathbf{r}(t)\,dt = \lim_{\max \Delta t_k \to 0} \sum_{k=1}^{n} \mathbf{r}(t_k^*)\Delta t_k$$

The definite integral of $\mathbf{r}(t)$ over the interval $a \le t \le b$ can be expressed as a vector whose components are the definite integrals of the component functions of $\mathbf{r}(t)$. For example, if $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$, then

$$\begin{aligned}\int_a^b \mathbf{r}(t)\,dt &= \lim_{\max \Delta t_k \to 0} \sum_{k=1}^{n} \mathbf{r}(t_k^*)\Delta t_k \\ &= \lim_{\max \Delta t_k \to 0} \left[\left(\sum_{k=1}^{n} x(t_k^*)\Delta t_k\right)\mathbf{i} + \left(\sum_{k=1}^{n} y(t_k^*)\Delta t_k\right)\mathbf{j}\right] \\ &= \left(\lim_{\max \Delta t_k \to 0} \sum_{k=1}^{n} x(t_k^*)\Delta t_k\right)\mathbf{i} + \left(\lim_{\max \Delta t_k \to 0} \sum_{k=1}^{n} y(t_k^*)\Delta t_k\right)\mathbf{j} \\ &= \left(\int_a^b x(t)\,dt\right)\mathbf{i} + \left(\int_a^b y(t)\,dt\right)\mathbf{j}\end{aligned}$$

In general, we have

$$\int_a^b \mathbf{r}(t)\,dt = \left(\int_a^b x(t)\,dt\right)\mathbf{i} + \left(\int_a^b y(t)\,dt\right)\mathbf{j} \qquad \text{2-space}$$

$$\int_a^b \mathbf{r}(t)\,dt = \left(\int_a^b x(t)\,dt\right)\mathbf{i} + \left(\int_a^b y(t)\,dt\right)\mathbf{j} + \left(\int_a^b z(t)\,dt\right)\mathbf{k} \qquad \text{3-space}$$

Example: Let $\mathbf{r}(t) = t^2\mathbf{i} + e^t\,\mathbf{j} - (2\cos \pi t)\mathbf{k}$. Then

$$\begin{aligned}\int_0^1 \mathbf{r}(t)\,dt &= \left(\int_0^1 t^2\,dt\right)\mathbf{i} + \left(\int_0^1 e^t\,dt\right)\mathbf{j} - \left(\int_0^1 2\cos \pi t\,dt\right)\mathbf{k} \\ &= \frac{t^3}{3}\bigg]_0^1 \mathbf{i} + e^t\bigg]_0^1 \mathbf{j} - \frac{2}{\pi}\sin \pi t\bigg]_0^1 \mathbf{k} = \frac{1}{3}\mathbf{i} + (e-1)\mathbf{j}\end{aligned}$$

2.2.4 Rules of Integration

Theorem:

(*Rules of Integration*) *Let* $\mathbf{r}(t)$, $\mathbf{r}_1(t)$, *and* $\mathbf{r}_2(t)$ *be vector-valued functions in 2-space or 3-space that are continuous on the interval* $a \le t \le b$, *and let k be a scalar. Then the following rules of integration hold:*

$$(a)\quad \int_a^b k\mathbf{r}(t)\,dt = k\int_a^b \mathbf{r}(t)\,dt$$

$$(b)\quad \int_a^b [\mathbf{r}_1(t) + \mathbf{r}_2(t)]\,dt = \int_a^b \mathbf{r}_1(t)\,dt + \int_a^b \mathbf{r}_2(t)\,dt$$

$$(c)\quad \int_a^b [\mathbf{r}_1(t) - \mathbf{r}_2(t)]\,dt = \int_a^b \mathbf{r}_1(t)\,dt - \int_a^b \mathbf{r}_2(t)\,dt$$

2.2.5 Antiderivatives of Vector-Valued Functions

An ***antiderivative*** for a vector-valued function **r***(t)* is a vector-valued function **R***(t)* such that

$$\mathbf{R}'(t) = \mathbf{r}(t)$$

we express Equation using integral notation as

$$\int \mathbf{r}(t)\,dt = \mathbf{R}(t) + \mathbf{C}$$

where **C** represents an arbitrary constant *vector*.

Since differentiation of vector-valued functions can be performed componentwise, it follows that anti-differentiation can be done this way as well.

Example:

$$\begin{aligned}\int (2t\mathbf{i} + 3t^2\mathbf{j})\,dt &= \left(\int 2t\,dt\right)\mathbf{i} + \left(\int 3t^2\,dt\right)\mathbf{j}\\ &= (t^2 + C_1)\mathbf{i} + (t^3 + C_2)\mathbf{j}\\ &= (t^2\mathbf{i} + t^3\mathbf{j}) + (C_1\mathbf{i} + C_2\mathbf{j}) = (t^2\mathbf{i} + t^3\mathbf{j}) + \mathbf{C}\end{aligned}$$

where $\mathbf{C} = C_1\mathbf{i} + C_2\mathbf{j}$ is an arbitrary vector constant of integration.

Most of the familiar integration properties have vector counterparts. For example, vector differentiation and integration are inverse operations in the sense that

$$\frac{d}{dt}\left[\int \mathbf{r}(t)\,dt\right] = \mathbf{r}(t) \qquad \text{and} \qquad \int \mathbf{r}'(t)\,dt = \mathbf{r}(t) + \mathbf{C}$$

Moreover, if **R***(t)* is an antiderivative of **r***(t)* on an interval containing $t = a$ and $t = b$, then we have the following vector form of the Fundamental Theorem of Calculus:

$$\int_a^b \mathbf{r}(t)\,dt = \mathbf{R}(t)\Big]_a^b = \mathbf{R}(b) - \mathbf{R}(a)$$

Example: Evaluate the definite integral $\int_0^2 (2t\mathbf{i} + 3t^2\mathbf{j})\,dt.$

Solution: Integrating the components yields

$$\int_0^2 (2t\mathbf{i} + 3t^2\mathbf{j})\,dt = t^2\Big]_0^2 \mathbf{i} + t^3\Big]_0^2 \mathbf{j} = 4\mathbf{i} + 8\mathbf{j}$$

Alternative Solution: The function $\mathbf{R}(t) = t^2\mathbf{i} + t^3\mathbf{j}$ is an antiderivative of the integrand since $\mathbf{R}'(t) = 2t\,\mathbf{i} + 3t^2\mathbf{j}$. Thus,

$$\int_0^2 (2t\mathbf{i} + 3t^2\mathbf{j})\,dt = \mathbf{R}(t)\Big]_0^2 = t^2\mathbf{i} + t^3\mathbf{j}\Big]_0^2 = (4\mathbf{i} + 8\mathbf{j}) - (0\mathbf{i} + 0\mathbf{j}) = 4\mathbf{i} + 8\mathbf{j}$$

Example: Find $\mathbf{r}(t)$ given that $\mathbf{r}'(t) = (3, 2t)$ and $\mathbf{r}(1) = (2, 5)$.

Solution: Integrating $\mathbf{r}'(t)$ to obtain $\mathbf{r}(t)$ yields

$$\mathbf{r}(t) = \int \mathbf{r}'(t)\,dt = \int \langle 3, 2t\rangle\,dt = \langle 3t, t^2\rangle + \mathbf{C}$$

where $\mathbf{C}$ is a vector constant of integration. To find the value of $\mathbf{C}$ we substitute $t = 1$ and use the given value of $\mathbf{r}(1)$ to obtain

$$\mathbf{r}(1) = (3, 1) + \mathbf{C} = (2, 5)$$

so that $\mathbf{C} = (-1, 4)$. Thus,

$$\mathbf{r}(t) = (3t, t^2) + (-1, 4) = (3t - 1, t^2 + 4)$$

2.3 CHANGE OF PARAMETER; ARC LENGTH

2.3.1 Arc Length from the Vector Viewpoint

The arc length L of a parametric curve

$$x = x(t), \quad y = y(t) \quad (a \le t \le b)$$

is given by the formula

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Analogously, the arc length L of a parametric curve

$$x = x(t), \quad y = y(t), \quad z = z(t) \quad (a \le t \le b)$$

in 3-space is given by the formula

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

vector forms that we can obtain by letting

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} \quad \text{or} \quad \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$$

It follows that

$$\frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} \quad \text{or} \quad \frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k}$$

and hence

$$\left\|\frac{d\mathbf{r}}{dt}\right\| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \quad \text{or} \quad \left\|\frac{d\mathbf{r}}{dt}\right\| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

Theorem:

If C is the graph in 2-space or 3-space of a smooth vector-valued function $\mathbf{r}(t)$, then its arc length L from $t = a$ to $t = b$ is

$$L = \int_a^b \left\|\frac{d\mathbf{r}}{dt}\right\| dt$$

Example: Find the arc length of that portion of the circular helix $x = \cos t$, $y = \sin t$, $z = t$ from $t = 0$ to $t = \pi$.

Solution: Set $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k} = (\cos t, \sin t, t)$. Then

$$\mathbf{r}'(t) = \langle -\sin t, \cos t, 1\rangle \quad \text{and} \quad \|\mathbf{r}'(t)\| = \sqrt{(-\sin t)^2 + (\cos t)^2 + 1} = \sqrt{2}$$

From Theorem the arc length of the helix is

$$L = \int_0^{\pi} \left\| \frac{d\mathbf{r}}{dt} \right\| dt = \int_0^{\pi} \sqrt{2}\, dt = \sqrt{2}\pi$$

2.3.2 Arc Length as a Parameter

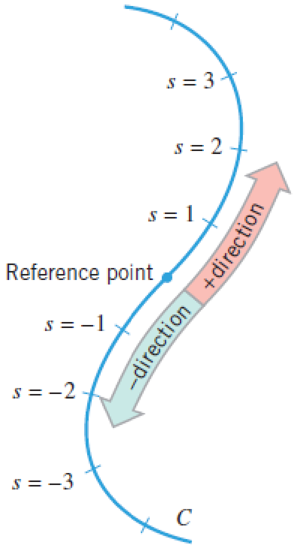
For many purposes the best parameter to use for representing a curve in 2-space or 3-space parametrically is the length of arc measured along the curve from some fixed reference point. This can be done as follows:

Using Arc Length as a Parameter

Step 1. Select an arbitrary point on the curve C to serve as a ***reference point***.

Step 2. Starting from the reference point, choose one direction along the curve to be the ***positive direction*** and the other to be the ***negative direction***.

Step 3. If P is a point on the curve, let s be the "signed" arc length along C from the reference point to P, where s is positive if P is in the positive direction from the reference point and s is negative if P is in the negative direction. The below figure illustrates this idea.



By this procedure, a unique point P on the curve is determined when a value for s is given. For example, $s = 2$ determines the point that is 2 units along the curve in the positive direction from the reference point, and $s = -3/2$ determines the point that is 3/2 units along the curve in the negative direction from the reference point.

Let us now treat s as a variable. As the value of s changes, the corresponding point P moves along C and the coordinates of P become functions of s. Thus, in 2-space the coordinates of P are $(x(s), y(s))$, and in 3-space they are $(x(s), y(s), z(s))$. Therefore, in 2-space or 3-space the curve C is given by the parametric equations

$$x = x(s),\ y = y(s) \quad \text{or} \quad x = x(s),\ y = y(s),\ z = z(s)$$

A parametric representation of a curve with arc length as the parameter is called an ***arc length parametrization*** of the curve. Note that a given curve will generally have infinitely many different arc length parametrizations, since the reference point and orientation can be chosen arbitrarily.

Example: Find the arc length parametrization of the circle $x^2 + y^2 = a^2$ with counter-clockwise orientation and $(a, 0)$ as the reference point.

Solution: The circle with counter-clockwise orientation can be represented by the parametric equations

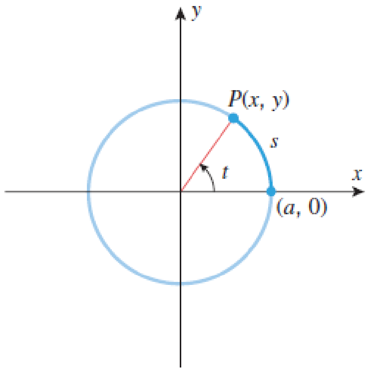
$$x = a\cos t, \quad y = a\sin t \quad (0 \le t \le 2\pi)$$

in which t can be interpreted as the angle in radian measure from the positive x-axis to the radius from the origin to the point $P(x, y)$ (see Figure). If we take the positive direction for measuring the arc length to be counter-clockwise, and we take $(a, 0)$ to be the reference point, then s and t are related by

$$s = at \text{ or } t = s/a$$

Making this change of variable and noting that s increases from 0 to $2\pi a$ as t increases from 0 to 2π yields the following arc length parametrization of the circle:

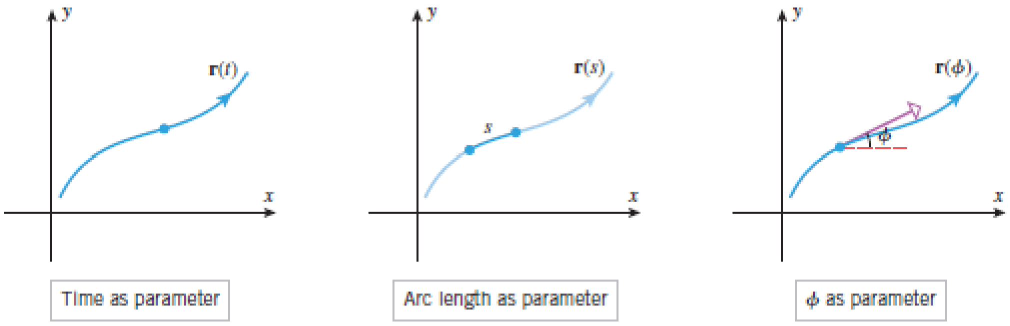
$$x = a\cos(s/a),\ y = a\sin(s/a)\ (0 \le s \le 2\pi a)$$



2.3.1 Change of Parameter

In many situations the solution of a problem can be simplified by choosing the parameter in a vector-valued function or a parametric curve in the right way. The two most common parameters for curves in 2-space or 3-space are time and arc length.

For example, in analyzing the motion of a particle in 2-space, it is often desirable to parametrize its trajectory in terms of the angle φ between the tangent vector and the positive x-axis (see below figures). Thus, our next objective is to develop methods for changing the parameter in a vector-valued function or parametric curve. This will allow us to move freely between different possible parametrizations.



A ***change of parameter*** in a vector-valued function $\mathbf{r}(t)$ is a substitution $t = g(\tau)$ that produces a new vector-valued function $\mathbf{r}(g(\tau))$ having the same graph as $\mathbf{r}(t)$, but possibly traced differently as the parameter τ increases.

Example: Find a change of parameter $t = g(\tau)$ for the circle

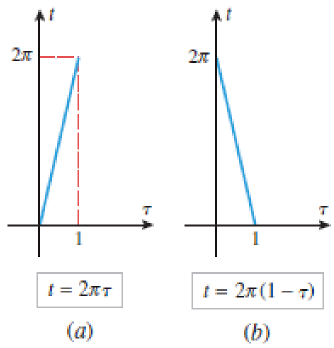
$$\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} \qquad (0 \le t \le 2\pi)$$

such that

(a) The circle is traced counter-clockwise as τ increases over the interval [0, 1];

(b) The circle is traced clockwise as τ increases over the interval [0, 1].

Solution (a): The given circle is traced counter-clockwise as t increases. Thus, if we choose g to be an increasing function, then it will follow from the relationship $t = g(\tau)$ that t increases when τ increases, thereby ensuring that the circle will be traced counter-clockwise as τ increases. We also want to choose g so that t increases from 0 to 2π as τ increases from 0 to 1. A simple choice of g that satisfies all of the required criteria is the linear function graphed in Figure a. The equation of this line is



$$t = g(\tau) = 2\pi\tau$$

which is the desired change of parameter. The resulting representation of the circle in terms of the parameter τ is

$$\mathbf{r}(g(\tau)) = \cos 2\pi\tau\ \mathbf{i} + \sin 2\pi\tau\ \mathbf{j} \qquad (0 \le \tau \le 1)$$

Solution (b): To ensure that the circle is traced clockwise, we will choose g to be a decreasing function such that t decreases from 2π to 0 as τ increases from 0 to 1. A simple choice of g that achieves this is the linear function

$$t = g(\tau) = 2\pi(1 - \tau)$$

graphed in Figure *b*. The resulting representation of the circle in terms of the parameter τ is

$$\mathbf{r}(g(\tau)) = \cos(2\pi(1 - \tau))\mathbf{i} + \sin(2\pi(1 - \tau))\mathbf{j}\ (0 \le \tau \le 1)$$

which simplifies to (verify)

$$\mathbf{r}(g(\tau)) = \cos 2\pi\tau\mathbf{i} - \sin 2\pi\tau\ \mathbf{j}\ (0 \le \tau \le 1)$$

Theorem (***Chain Rule***) *Let* $\mathbf{r}(t)$ *be a vector-valued function in 2-space or 3- space that is differentiable with respect to t. If* $t = g(\tau)$ *is a change of parameter in which g is differentiable with respect to* τ*, then* $\mathbf{r}(g(\tau))$ *is differentiable with respect to* τ *and*

$$\frac{dr}{d\tau} = \frac{dr}{dt}\frac{dt}{d\tau}$$

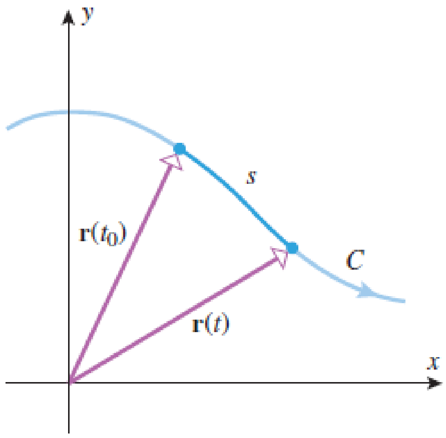
-A change of parameter $t = g(\tau)$ in which $\mathbf{r}(g(\tau))$ is smooth if $\mathbf{r}(t)$ is smooth is called a ***smooth change of parameter***.

-The $t = g(\tau)$ will be a smooth change of parameter if $dt/d\tau$ is continuous and $dt/d\tau \neq 0$ for all values of τ, since these conditions imply that $d\mathbf{r}/d\tau$ is continuous and nonzero if $d\mathbf{r}/dt$ is continuous and nonzero.

-Smooth changes of parameter fall into two categories—those for which $dt/d\tau > 0$ for all τ (called ***positive changes of parameter***) and those for which $dt/d\tau < 0$ for all τ (called ***negative changes of parameter***). A positive change of parameter preserves the orientation of a parametric curve, and a negative change of parameter reverses it.

2.3.2 Finding Arc Length Parametrizations

Theorem *Let C be the graph of a smooth vector-valued function* $\mathbf{r}(t)$ *in 2-space or 3-space, and let* $\mathbf{r}(t_0)$ *be any point on C. Then the following formula defines a positive change*



of parameter from t to s, where s is an arc length parameter having **r**(t_0) *as its reference point:*

$$s = \int_{t_0}^{t} \left\| \frac{dr}{du} \right\| du$$

Example: Find the arc length parametrization of the circular helix

$$\mathbf{r} = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\mathbf{k}$$

that has reference point **r**(0) = <1, 0, 0> and the same orientation as the given helix.

Solution: Replacing t by u in **r** for integration purposes and taking $t_0 = 0$, we obtain

$$\mathbf{r} = \cos u\mathbf{i} + \sin u\mathbf{j} + u\mathbf{k}$$

$$\frac{d\mathbf{r}}{du} = (-\sin u)\mathbf{i} + \cos u\,\mathbf{j} + \mathbf{k}$$

$$\left\| \frac{d\mathbf{r}}{du} \right\| = \sqrt{(-\sin u)^2 + \cos^2 u + 1} = \sqrt{2}$$

$$s = \int_0^t \left\| \frac{d\mathbf{r}}{du} \right\| du = \int_0^t \sqrt{2}\,du = \sqrt{2}u \Big]_0^t = \sqrt{2}t$$

Thus, $t = s/\sqrt{2}$, so (13) can be reparametrized in terms of s as

$$\mathbf{r} = \cos\left(\frac{s}{\sqrt{2}}\right)\mathbf{i} + \sin\left(\frac{s}{\sqrt{2}}\right)\mathbf{j} + \frac{s}{\sqrt{2}}\mathbf{k}$$

Example: A bug walks along the trunk of a tree following a path modeled by the circular helix in previous example. The bug starts at the reference point (1, 0, 0) and walks up the helix for a distance of 10 units. What are the bug's final coordinates?

Solution: the arc length parametrization of the helix relative to the reference point (1, 0, 0) is

$$\mathbf{r} = \cos\left(\frac{s}{\sqrt{2}}\right)\mathbf{i} + \sin\left(\frac{s}{\sqrt{2}}\right)\mathbf{j} + \frac{s}{\sqrt{2}}\mathbf{k}$$

or, expressed parametrically,

$$x = \cos\left(\frac{s}{\sqrt{2}}\right), \quad y = \sin\left(\frac{s}{\sqrt{2}}\right), \quad z = \frac{s}{\sqrt{2}}$$

Thus, at $s = 10$ the coordinates are

$$\left(\cos\left(\frac{10}{\sqrt{2}}\right), \sin\left(\frac{10}{\sqrt{2}}\right), \frac{10}{\sqrt{2}}\right) \approx (0.705, 0.709, 7.07)$$

Example: Find the arc length parametrization of the line

$$x = 2t + 1, \qquad y = 3t - 2$$

that has the same orientation as the given line and uses (1,−2) as the reference point.

Solution: The line passes through the point $(1, -2)$ and is parallel to $\mathbf{v} = 2\mathbf{i} + 3\mathbf{j}$. To find the arc length parametrization of the line, we need only rewrite the given equations using $\mathbf{v}/\|\mathbf{v}\|$ rather than $\mathbf{v}$ to determine the direction and replace t by s. Since

$$\frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{2\mathbf{i} + 3\mathbf{j}}{\sqrt{13}} = \frac{2}{\sqrt{13}}\mathbf{i} + \frac{3}{\sqrt{13}}\mathbf{j}$$

it follows that the parametric equations for the line in terms of s are

$$x = \frac{2}{\sqrt{13}}s + 1, \quad y = \frac{3}{\sqrt{13}}s - 2$$

2.3.3 Properties of Arc Length Parametrizations

Theorem

(a) If C is the graph of a smooth vector-valued function $\mathbf{r}(t)$ in 2-space or 3-space, where t is a general parameter, and if s is the arc length parameter for C defined by previous formula, then for every value of t the tangent vector has length

$$\left\|\frac{d\mathbf{r}}{dt}\right\| = \frac{ds}{dt}$$

(b) If C is the graph of a smooth vector-valued function $\mathbf{r}(s)$ in 2-space or 3-space, where s is an arc length parameter, then for every value of s the tangent vector to C has length

$$\left\|\frac{d\mathbf{r}}{ds}\right\| = 1$$

(c) If C is the graph of a smooth vector-valued function $\mathbf{r}(t)$ in 2-space or 3-space, and if $\|d\mathbf{r}/dt\| = 1$ for every value of t, then for any value of t_0 in the domain of $\mathbf{r}$, the parameter $s = t - t_0$ is an arc length parameter that has its reference point at the point on C where $t = t_0$.

$$\frac{ds}{dt} = \left\|\frac{d\mathbf{r}}{dt}\right\| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \qquad \text{2-space}$$

$$\frac{ds}{dt} = \left\|\frac{d\mathbf{r}}{dt}\right\| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \qquad \text{3-space}$$

$$\left\|\frac{d\mathbf{r}}{ds}\right\| = \sqrt{\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2} = 1 \qquad \text{2-space}$$

$$\left\|\frac{d\mathbf{r}}{ds}\right\| = \sqrt{\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2} = 1 \qquad \text{3-space}$$

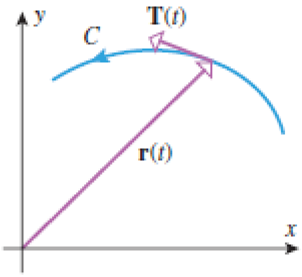
2.4 UNIT TANGENT, NORMAL, AND BINORMAL VECTORS

2.4.1 Unit Tangent Vectors

If C is the graph of a *smooth* vector-valued function $\mathbf{r}(t)$ in 2-space or 3-space, then the vector $\mathbf{r}'(t)$ is nonzero, tangent to C, and points in the direction of increasing parameter. Thus, by normalizing $\mathbf{r}'(t)$ we obtain a unit vector

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} \qquad (1)$$

that is tangent to C and points in the direction of increasing parameter. We call $\mathbf{T}(t)$ the ***unit tangent vector*** to C at t.



Example: Find the unit tangent vector to the graph of $\mathbf{r}(t) = t^2\,\mathbf{i} + t^3\,\mathbf{j}$ at the point where $t = 2$.

Solution: Since

$$\mathbf{r}'(t) = 2t\mathbf{i} + 3t^2\mathbf{j}$$

we obtain

$$\mathbf{T}(2) = \frac{\mathbf{r}'(2)}{\|\mathbf{r}'(2)\|} = \frac{4\mathbf{i} + 12\mathbf{j}}{\sqrt{160}} = \frac{4\mathbf{i} + 12\mathbf{j}}{4\sqrt{10}} = \frac{1}{\sqrt{10}}\mathbf{i} + \frac{3}{\sqrt{10}}\mathbf{j}$$