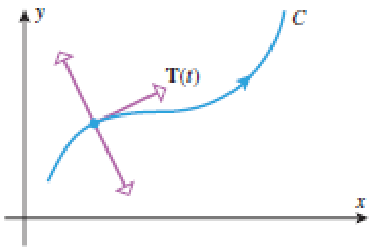
2.4.2 Unit Normal Vectors

If a vector-valued function **r**(*t*) has constant norm, then **r**(*t*) and $\mathbf{\acute{r}}(t)$are orthogonal vectors. In particular, **T**(*t*) has constant norm 1, so **T**(*t*) and $\mathbf{\acute{T}}(t)$ are orthogonal vectors. This implies that $\mathbf{\acute{T}}(t)$ is perpendicular to the tangent line to *C* at *t*, so we say that $\mathbf{\acute{T}}(t)$ is ***normal*** to *C* at *t*. It follows that if $\mathbf{\acute{T}}(t) \neq 0$, and if we normalize $\mathbf{\acute{T}}(t)$, then we obtain a unit vector

$$\mathbf{N}(t) = \frac{\mathbf{\acute{T}}(t)}{\|\mathbf{\acute{T}}(t)\|} \qquad (2)$$



That is normal to *C* and points in the same direction as $\mathbf{\acute{T}}(t)$. We call **N**(*t*) the ***principal unit normal vector*** to *C* at *t* , or more simply, the ***unit normal vector***. Observe that the unit normal vector is defined only at points where $\mathbf{\acute{T}}(t) \neq \mathbf{0}$. Unless stated otherwise, we will assume that this condition is satisfied. In particular, this *excludes* straight lines.

Example: Find **T**(*t*) and **N**(*t*) for the circular helix

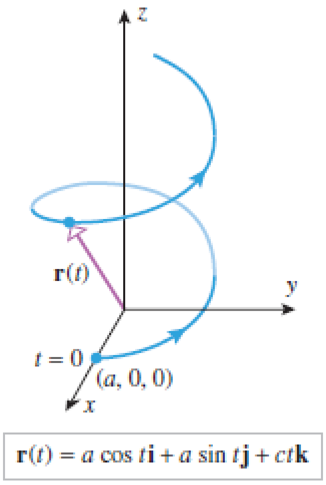
$$x = a\cos t,\ y = a\sin t,\ z = ct$$

where $a > 0$.

Solution: The radius vector for the helix is

$$\mathbf{r}(t) = a\cos t\,\mathbf{i} + a\sin t\,\mathbf{j} + ct\mathbf{k}$$

(Figure). Thus,



$$\mathbf{r}'(t) = (-a\sin t)\mathbf{i} + a\cos t\,\mathbf{j} + c\mathbf{k}$$

$$\|\mathbf{r}'(t)\| = \sqrt{(-a\sin t)^2 + (a\cos t)^2 + c^2} = \sqrt{a^2+c^2}$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = -\frac{a\sin t}{\sqrt{a^2+c^2}}\mathbf{i} + \frac{a\cos t}{\sqrt{a^2+c^2}}\mathbf{j} + \frac{c}{\sqrt{a^2+c^2}}\mathbf{k}$$

$$\mathbf{T}'(t) = -\frac{a\cos t}{\sqrt{a^2+c^2}}\mathbf{i} - \frac{a\sin t}{\sqrt{a^2+c^2}}\mathbf{j}$$

$$\|\mathbf{T}'(t)\| = \sqrt{\left(-\frac{a\cos t}{\sqrt{a^2+c^2}}\right)^2 + \left(-\frac{a\sin t}{\sqrt{a^2+c^2}}\right)^2} = \sqrt{\frac{a^2}{a^2+c^2}} = \frac{a}{\sqrt{a^2+c^2}}$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} = (-\cos t)\mathbf{i} - (\sin t)\mathbf{j} = -(\cos t\,\mathbf{i} + \sin t\,\mathbf{j})$$

2.4.3 Inward Unit Normal Vectors in 2-Space

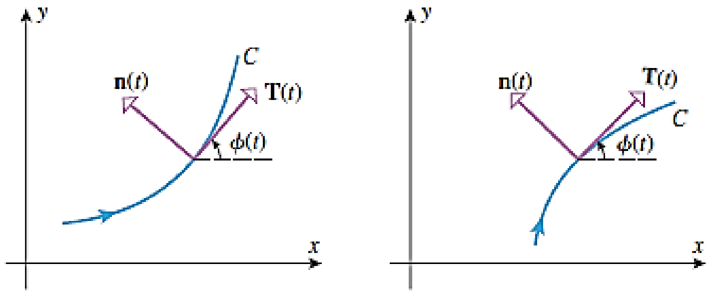
Our next objective is to show that for a nonlinear parametric curve C in 2-space the unit normal vector always points toward the concave side of C.

For this purpose, let $\varphi(t)$ be the angle from the positive x-axis to $\mathbf{T}(t)$, and let $\mathbf{n}(t)$ be the unit vector that results when $\mathbf{T}(t)$ is rotated counter-clockwise through an angle of $\pi/2$ (see below figure). Since $\mathbf{T}(t)$ and $\mathbf{n}(t)$ are unit vectors, that these vectors can be expressed as

$$\mathbf{T}(t) = \cos\varphi(t)\mathbf{i} + \sin\varphi(t)\mathbf{j}$$

$$\mathbf{n}(t) = \cos[\varphi(t) + \pi/2]\mathbf{i} + \sin[\varphi(t) + \pi/2]\,\mathbf{j} = -\sin\varphi(t)\mathbf{i} + \cos\varphi(t)\mathbf{j}$$

Observe that on intervals where $\varphi(t)$ is increasing the vector $\mathbf{n}(t)$ points *toward* the concave side of C, and on intervals where $\varphi(t)$ is decreasing it points *away* from the concave side (see below figure).



$\phi(t)$ increases as t increases. $\phi(t)$ decreases as t increases.

Now let us differentiate $\mathbf{T}(t)$ by using previous formula and applying the chain rule. This Yields

$$\frac{d\mathbf{T}}{dt} = \frac{d\mathbf{T}}{d\phi}\frac{d\phi}{dt} = [(-\sin\phi)\mathbf{i} + (\cos\phi)\mathbf{j}]\frac{d\phi}{dt}$$

and thus

$$\frac{d\mathbf{T}}{dt} = \mathbf{n}(t)\frac{d\phi}{dt}$$

But $d\varphi/dt > 0$ on intervals where $\varphi(t)$ is increasing and $d\varphi/dt < 0$ on intervals where $\varphi(t)$ is decreasing. Thus, $d\mathbf{T}/dt$ has the same direction as $\mathbf{n}(t)$ on intervals where $\varphi(t)$ is increasing and the opposite direction on intervals where $\varphi(t)$ is decreasing. Therefore, $\mathbf{T}'(t) = d\mathbf{T}/dt$ points "inward" toward the concave side of the curve in all cases, and hence so does $\mathbf{N}(t)$. For this reason, $\mathbf{N}(t)$ is also called the ***inward unit normal*** when applied to curves in 2-space.

2.4.4 Computing T and N for Curves Parametrized by Arc Length

In the case where **r***(s)* is parametrized by arc length, the procedures for computing the unit tangent vector **T***(s)* and the unit normal vector **N***(s)* are simpler than in the general case. For example, we showed in Theorem that if *s* is an arc length parameter, then $\|\acute{\mathbf{r}}\,(s)\| = 1$. Thus, Formula (1) for the unit tangent vector simplifies to

$$\mathbf{T}(s) = \acute{\mathbf{r}}\,(s)$$

and consequently Formula (2) for the unit normal vector simplifies to

$$\mathbf{N}(s) = \frac{\bar{\bar{\mathbf{r}}}(s)}{\|\bar{\bar{\mathbf{r}}}(s)\|}$$

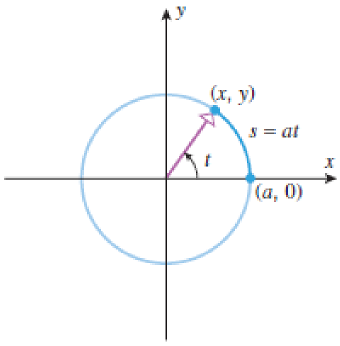
Example: The circle of radius *a* with counter-clockwise orientation and centered at the origin can be represented by the vector-valued function

$$\mathbf{r} = a \cos t\, \mathbf{i} + a \sin t\, \mathbf{j}\ (0 \leq t \leq 2\pi)$$

Parametrize this circle by arc length and find **T***(s)* and **N***(s)*.

Solution: In (8) we can interpret *t* as the angle in radian measure from the positive *x*-axis to the radius vector (below figure). This angle subtends an arc of length $s = at$ on the circle, so we can reparametrize the circle in terms of *s* by substituting *s/a* for *t*. This yields

$$\mathbf{r}(s) = a \cos(s/a)\mathbf{i} + a \sin(s/a)\mathbf{j} \qquad (0 \leq s \leq 2\pi a)$$



To find **T***(s)* and **N***(s)* from Formulas (6) and (7), we must compute $\acute{\mathbf{r}}\,(s)$, $\mathbf{r}''(s)$, and $\|\mathbf{r}''(s)\|$. Doing so, we obtain

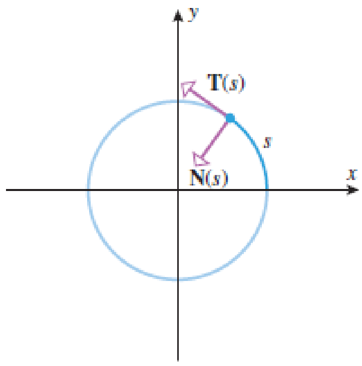
$$\mathbf{r}'(s) = -\sin(s/a)\mathbf{i} + \cos(s/a)\mathbf{j}$$

$$\mathbf{r}''(s) = -(1/a)\cos(s/a)\mathbf{i} - (1/a)\sin(s/a)\mathbf{j}$$

$$\|\mathbf{r}''(s)\| = \sqrt{(-1/a)^2\cos^2(s/a) + (-1/a)^2\sin^2(s/a)} = 1/a$$

$$\mathbf{T}(s) = \mathbf{r}'(s) = -\sin(s/a)\mathbf{i} + \cos(s/a)\mathbf{j}$$

$$\mathbf{N}(s) = \mathbf{r}''(s)/\|\mathbf{r}''(s)\| = -\cos(s/a)\mathbf{i} - \sin(s/a)\mathbf{j}$$



2.4.5 Binormal Vectors In 3-Space

If C is the graph of a vector-valued function $\mathbf{r}(t)$ in 3-space, then we define the ***binormal vector*** to C at t to be

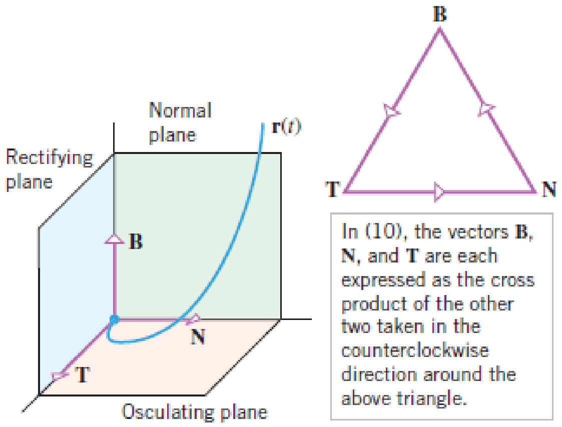
$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t) \qquad (9)$$

It follows from properties of the cross product that $\mathbf{B}(t)$ is orthogonal to both $\mathbf{T}(t)$ and $\mathbf{N}(t)$ and is oriented relative to $\mathbf{T}(t)$ and $\mathbf{N}(t)$ by the right-hand rule. Moreover, $\mathbf{T}(t) \times \mathbf{N}(t)$ is a unit vector since

$$\|\mathbf{T}(t) \times \mathbf{N}(t)\| = \|\mathbf{T}(t)\| \, \|\mathbf{N}(t)\| \sin(\pi/2) = 1$$

Thus, $\{\mathbf{T}(t), \mathbf{N}(t), \mathbf{B}(t)\}$ is a set of three mutually orthogonal unit vectors.

Just as the vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ determine a right-handed coordinate system in 3-space, so do the vectors $\mathbf{T}(t)$, $\mathbf{N}(t)$, and $\mathbf{B}(t)$. At each point on a smooth parametric curve C in 3-space, these vectors determine three mutually perpendicular planes that pass through the point— the **TB**-plane (called the ***rectifying plane***), the **TN**-plane (called the ***osculating plane***), and the **NB**-plane (called the ***normal plane***) (Figure). Moreover, one can show that a coordinate system determined by $\mathbf{T}(t)$, $\mathbf{N}(t)$, and $\mathbf{B}(t)$ is right-handed in the sense that each of these vectors is related to the other two by the right-hand rule (see figure):



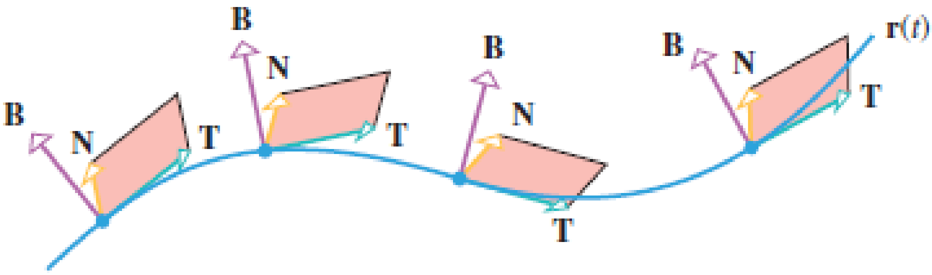
$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t), \quad \mathbf{N}(t) = \mathbf{B}(t) \times \mathbf{T}(t), \quad \mathbf{T}(t) = \mathbf{N}(t) \times \mathbf{B}(t)$$

The coordinate system determined by **T***(t)*, **N***(t)*, and **B***(t)* is called the ***TNB-frame*** or sometimes the ***Frenet frame*** in honor of the French mathematician Jean Frédéric Frenet (1816–1900) who pioneered its application to the study of space curves. Typically, the *xyz*-coordinate system determined by the unit vectors **i**, **j**, and **k** remains fixed, whereas the **TNB**-frame changes as its origin moves along the curve *C* (Figure). Formula expresses **B***(t)* in terms of **T***(t)* and **N***(t)*. Alternatively, the binormal **B***(t)* can be expressed directly in terms of **r***(t)* as

$$\mathbf{B}(t) = \frac{\mathbf{r}'(t) \times \mathbf{r}''(t)}{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}$$

and in the case where the parameter is arc length it can be expressed in terms of **r***(s)* as

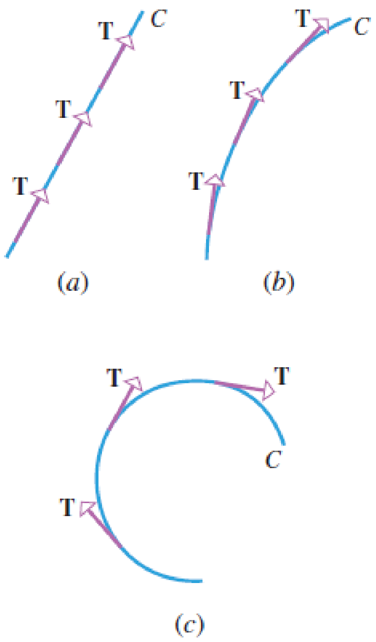
$$\mathbf{B}(s) = \frac{\mathbf{r}'(s) \times \mathbf{r}''(s)}{\|\mathbf{r}''(s)\|}$$



2.5 CURVATURE

2.5.1 Definition of Curvature

Suppose that C is the graph of a smooth vector-valued function in 2-space or 3-space that is parametrized in terms of arc length. Figure suggests that for a curve in 2-space the "sharpness" of the bend in C is closely related to $d\mathbf{T}/ds$, which is the rate of change of the unit tangent vector $\mathbf{T}$ with respect to s. (Keep in mind that $\mathbf{T}$ has constant length, so only its direction changes.) If C is a straight line (no bend), then the direction of $\mathbf{T}$ remains constant (Figure a); if C bends slightly, then $\mathbf{T}$ undergoes a gradual change of direction (Figure b); and if C bends sharply, then $\mathbf{T}$ undergoes a rapid change of direction (Figure c).



The situation in 3-space is more complicated because bends in a curve are not limited to a single plane—they can occur in all directions. To describe the bending characteristics of a curve in 3-space completely, one must take into account $d\mathbf{T}/ds$, $d\mathbf{N}/ds$, and $d\mathbf{B}/ds$. A complete study of this topic would take us too far afield, so we will limit our discussion to $d\mathbf{T}/ds$, which is the most important of these derivatives in applications.

Definition If C is a smooth curve in 2-space or 3-space that is parametrized by arc length, then the ***curvature*** of C, denoted by $\kappa = \kappa(s)$ (κ = Greek "kappa"), is defined by

$$k(s) = \left\|\frac{d\mathbf{T}}{ds}\right\| = \|\mathbf{r}''(s)\| \qquad (1)$$

Observe that $\kappa(s)$ is a real-valued function of s, since it is the *length* of $d\mathbf{T}/ds$ that measures the curvature. In general, the curvature will vary from point to point along a curve; however, the following example shows that the curvature is constant for circles in 2-space, as you might expect.

Example: the circle of radius a, centered at the origin, can be parametrized in terms of arc length as

$$\mathbf{r}(s) = a\cos(s/a)\mathbf{i} + a\sin(s/a)\mathbf{j} \qquad (0 \le s \le 2\pi a)$$

$$\mathbf{r}''(s) = -\frac{1}{a}\cos\left(\frac{s}{a}\right)\mathbf{i} - \frac{1}{a}\sin\left(\frac{s}{a}\right)\mathbf{j}$$

$$\kappa(s) = \|\mathbf{r}''(s)\| = \sqrt{\left[-\frac{1}{a}\cos\left(\frac{s}{a}\right)\right]^2 + \left[-\frac{1}{a}\sin\left(\frac{s}{a}\right)\right]^2} = \frac{1}{a}$$

so the circle has constant curvature 1/a.

2.5.2 Formulas for Curvature

Formula (1) is only applicable if the curve is parametrized in terms of arc length. The following theorem provides two formulas for curvature in terms of a general parameter *t*.

Theorem *If* **r**(*t*) *is a smooth vector-valued function in 2-space or 3-space, then for each value of t at which* $\mathbf{T}'(t)$ *and* $\mathbf{r}''(t)$ *exist, the curvature* κ *can be expressed*

a) $$k(t) = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} \qquad (2)$$

b) $$k(t) = \frac{\|\mathbf{r}'(t)\times\mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} \qquad (3)$$

Proof a:

$$\kappa(t) = \left\|\frac{d\mathbf{T}}{ds}\right\| = \left\|\frac{d\mathbf{T}/dt}{ds/dt}\right\| = \left\|\frac{d\mathbf{T}/dt}{\|d\mathbf{r}/dt\|}\right\| = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|}$$

Proof b:

$$\mathbf{r}'(t) = \|\mathbf{r}'(t)\|\mathbf{T}(t)$$
$$\mathbf{r}''(t) = \|\mathbf{r}'(t)\|'\mathbf{T}(t) + \|\mathbf{r}'(t)\|\mathbf{T}'(t)$$

$$\mathbf{T}'(t) = \|\mathbf{T}'(t)\|\mathbf{N}(t) \quad \text{and} \quad \|\mathbf{T}'(t)\| = \kappa(t)\|\mathbf{r}'(t)\|$$

$$\mathbf{T}'(t) = \kappa(t)\|\mathbf{r}'(t)\|\mathbf{N}(t)$$

$$\mathbf{r}''(t) = \|\mathbf{r}'(t)\|'\mathbf{T}(t) + \kappa(t)\|\mathbf{r}'(t)\|^2\mathbf{N}(t)$$

$$\mathbf{r}'(t)\times\mathbf{r}''(t) = \|\mathbf{r}'(t)\|\|\mathbf{r}'(t)\|'(\mathbf{T}(t)\times\mathbf{T}(t)) + \kappa(t)\|\mathbf{r}'(t)\|^3(\mathbf{T}(t)\times\mathbf{N}(t))$$

$$\mathbf{r}'(t)\times\mathbf{r}''(t) = \kappa(t)\|\mathbf{r}'(t)\|^3(\mathbf{T}(t)\times\mathbf{N}(t)) = \kappa(t)\|\mathbf{r}'(t)\|^3\mathbf{B}(t)$$

$$\|\mathbf{r}'(t)\times\mathbf{r}''(t)\| = \kappa(t)\|\mathbf{r}'(t)\|^3$$

Example: Find κ(t) for the circular helix

$x = a\cos t,\ y = a\sin t,\ z = ct$ where $a > 0$.

Solution: The radius vector for the helix is

$$\mathbf{r}(t) = a\cos t\,\mathbf{i} + a\sin t\,\mathbf{j} + ct\mathbf{k}$$

Thus,

$$\mathbf{r}'(t) = (-a\sin t)\mathbf{i} + a\cos t\,\mathbf{j} + c\mathbf{k}$$

$$\mathbf{r}''(t) = (-a\cos t)\mathbf{i} + (-a\sin t)\mathbf{j}$$

$$\mathbf{r}'(t)\times\mathbf{r}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a\sin t & a\cos t & c \\ -a\cos t & -a\sin t & 0 \end{vmatrix} = (ac\sin t)\mathbf{i} - (ac\cos t)\mathbf{j} + a^2\mathbf{k}$$

Therefore,

$$\|\mathbf{r}'(t)\| = \sqrt{(-a\sin t)^2 + (a\cos t)^2 + c^2} = \sqrt{a^2 + c^2}$$

and

$$\|\mathbf{r}'(t)\times\mathbf{r}''(t)\| = \sqrt{(ac\sin t)^2 + (-ac\cos t)^2 + a^4}$$
$$= \sqrt{a^2c^2 + a^4} = a\sqrt{a^2 + c^2}$$

so

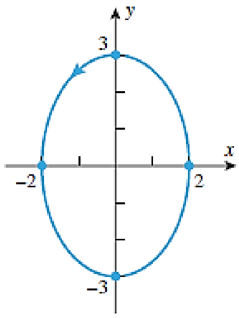
$$\kappa(t) = \frac{\|\mathbf{r}'(t)\times\mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \frac{a\sqrt{a^2+c^2}}{\left(\sqrt{a^2+c^2}\right)^3} = \frac{a}{a^2+c^2}$$

Note that κ does not depend on t, which tells us that the helix has constant curvature.

Example: The graph of the vector equation

$$\mathbf{r} = 2\cos t\,\mathbf{i} + 3\sin t\,\mathbf{j} \qquad (0 \le t \le 2\pi)$$

is the ellipse as shown in Figure. Find the curvature of the ellipse at the endpoints of the major and minor axes, and use a graphing utility to generate the graph of $\kappa(t)$.



$\mathbf{r} = 2\cos t\mathbf{i} + 3\sin t\mathbf{j}$

Solution: To apply Formula (3), we must treat the ellipse as a curve in the xy-plane of an xyz-coordinate system by adding a zero $\mathbf{k}$ component and writing its equation as

$$\mathbf{r} = 2\cos t\,\mathbf{i} + 3\sin t\,\mathbf{j} + 0\mathbf{k}$$

It is not essential to write the zero $\mathbf{k}$ component explicitly as long as you assume it to be there when you calculate a cross product. Thus,

$$\mathbf{r}'(t) = (-2\sin t)\mathbf{i} + 3\cos t\,\mathbf{j}$$

$$\mathbf{r}''(t) = (-2\cos t)\mathbf{i} + (-3\sin t)\mathbf{j}$$

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2\sin t & 3\cos t & 0 \\ -2\cos t & -3\sin t & 0 \end{vmatrix} = [(6\sin^2 t) + (6\cos^2 t)]\mathbf{k} = 6\mathbf{k}$$

Therefore,

$$\|\mathbf{r}'(t)\| = \sqrt{(-2\sin t)^2 + (3\cos t)^2} = \sqrt{4\sin^2 t + 9\cos^2 t}$$
$$\|\mathbf{r}'(t) \times \mathbf{r}''(t)\| = 6$$

so

$$\kappa(t) = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \frac{6}{[4\sin^2 t + 9\cos^2 t]^{3/2}}$$

The endpoints of the minor axis are $(2, 0)$ and $(-2, 0)$, which correspond to $t = 0$ and $t = \pi$, respectively. Substituting these values in (7) yields the same curvature at both points, namely

$$\kappa = \kappa(0) = \kappa(\pi) = \frac{6}{9^{3/2}} = \frac{6}{27} = \frac{2}{9}$$

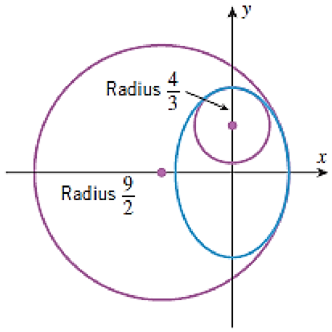
The endpoints of the major axis are $(0, 3)$ and $(0,-3)$, which correspond to $t = \pi/2$ and $t = 3\pi/2$, respectively; from (7) the curvature at these points is

$$\kappa = \kappa\left(\frac{\pi}{2}\right) = \kappa\left(\frac{3\pi}{2}\right) = \frac{6}{4^{3/2}} = \frac{3}{4}$$

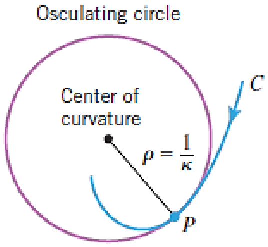
RADIUS OF CURVATURE

In the last example we found the curvature at the ends of the minor axis to be 2/9 and the curvature at the ends of the major axis to be 3/4. To obtain a better understanding of the meaning of these numbers, recall from Example 1 that a circle of radius a has a constant curvature of

$1/a$; thus, the curvature of the ellipse at the ends of the minor axis is the same as that of a circle of radius 9/2, and the curvature at the ends of the major axis is the same as that of a circle of radius 4/3 (Figure).



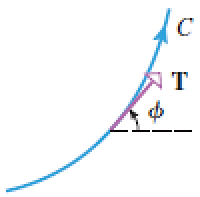
In general, if a curve C in 2-space has nonzero curvature κ at a point P, then the circle of radius $\rho = 1/\kappa$ sharing a common tangent with C at P, and centered on the concave side of the curve at P, is called the ***osculating circle*** or ***circle of curvature*** at P (Figure).



The osculating circle and the curve C not only touch at P but they have equal curvatures at that point. In this sense, the osculating circle is the circle that best approximates the curve C near P. The radius ρ of the osculating circle at P is called the ***radius of curvature*** at P, and the center of the circle is called the ***center of curvature*** at P (previous Figure).

2.5.3 An Interpretation of Curvature in 2-Space

A useful geometric interpretation of curvature in 2-space can be obtained by considering the angle φ measured counter-clockwise from the direction of the positive x-axis to the unit tangent vector $\mathbf{T}$ (see below figure). By previous formula, we can express $\mathbf{T}$ in terms of φ as



$$\mathbf{T}(\varphi) = \cos\varphi\,\mathbf{i} + \sin\varphi\,\mathbf{j}$$

Thus,

$$\frac{d\mathbf{T}}{d\phi} = (-\sin\phi)\mathbf{i} + \cos\phi\,\mathbf{j}$$

$$\frac{d\mathbf{T}}{ds} = \frac{d\mathbf{T}}{d\phi}\frac{d\phi}{ds}$$

from which we obtain

$$\kappa(s) = \left\|\frac{d\mathbf{T}}{ds}\right\| = \left|\frac{d\phi}{ds}\right|\left\|\frac{d\mathbf{T}}{d\phi}\right\| = \left|\frac{d\phi}{ds}\right|\sqrt{(-\sin\phi)^2 + \cos^2\phi} = \left|\frac{d\phi}{ds}\right|$$

In summary, we have shown that

$$\boxed{\kappa(s) = \left|\frac{d\phi}{ds}\right|}$$

which tells us that curvature in 2-space can be interpreted as the magnitude of the rate of change of φ with respect to s—the greater the curvature, the more rapidly φ changes with s (Figure a). In the case of a straight line, the angle φ is constant (Figure b) and consequently $\kappa(s) = |d\varphi/ds| = 0$, which is consistent with the fact that a straight line has zero curvature at every point.

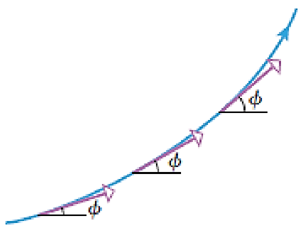


Figure a

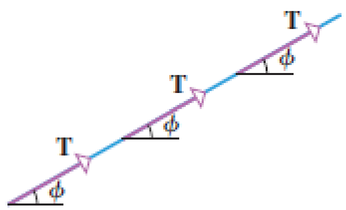


Figure b

2.6 MOTION ALONG A CURVE

2.6.1 Velocity, Acceleration, and Speed

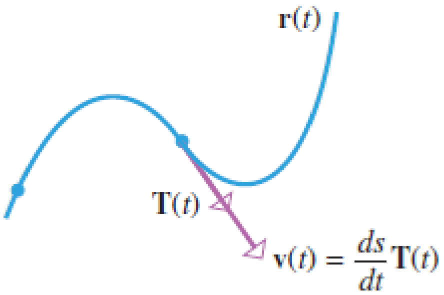
Definition

If $\mathbf{r}(t)$ is the position function of a particle moving along a curve in 2-space or 3-space, then the ***instantaneous velocity***, ***instantaneous acceleration***, and ***instantaneous speed*** of the particle at time t are defined by

$$\text{velocity} = \mathbf{v}(t) = \frac{d\mathbf{r}}{dt}$$

$$\text{acceleration} = \mathbf{a}(t) = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2}$$

$$\text{speed} = \|\mathbf{v}(t)\| = \frac{ds}{dt}$$



The length of the velocity vector is the speed of the particle, and the direction of the velocity vector is the direction of motion.

FORMULAS FOR POSITION, VELOCITY, ACCELERATION, AND SPEED

	2-SPACE	3-SPACE
POSITION	$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$	$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$
VELOCITY	$\mathbf{v}(t) = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j}$	$\mathbf{v}(t) = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k}$
ACCELERATION	$\mathbf{a}(t) = \frac{d^2x}{dt^2}\mathbf{i} + \frac{d^2y}{dt^2}\mathbf{j}$	$\mathbf{a}(t) = \frac{d^2x}{dt^2}\mathbf{i} + \frac{d^2y}{dt^2}\mathbf{j} + \frac{d^2z}{dt^2}\mathbf{k}$
SPEED	$\|\mathbf{v}(t)\| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$	$\|\mathbf{v}(t)\| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$

Example: A particle moves along a circular path in such a way that its x- and y-coordinates at time t are

$$x = 2\cos t,\ y = 2\sin t$$

(a) Find the instantaneous velocity and speed of the particle at time t .

(b) Sketch the path of the particle, and show the position and velocity vectors at time $t = \pi/4$ with the velocity vector drawn so that its initial point is at the tip of the position vector.

(c) Show that at each instant the acceleration vector is perpendicular to the velocity vector.

Solution (a). At time t , the position vector is

$$\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j}$$

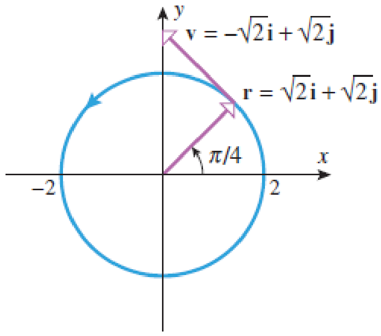
so the instantaneous velocity and speed are

$$\mathbf{v}(t) = \frac{d\mathbf{r}}{dt} = -2\sin t\mathbf{i} + 2\cos t\mathbf{j}$$
$$\|\mathbf{v}(t)\| = \sqrt{(-2\sin t)^2 + (2\cos t)^2} = 2$$

Solution (b). The graph of the parametric equations is a circle of radius 2 centered at the origin. At time $t = \pi/4$ the position and velocity vectors of the particle are

$$\mathbf{r}(\pi/4) = 2\cos(\pi/4)\mathbf{i} + 2\sin(\pi/4)\mathbf{j} = \sqrt{2}\mathbf{i} + \sqrt{2}\mathbf{j}$$
$$\mathbf{v}(\pi/4) = -2\sin(\pi/4)\mathbf{i} + 2\cos(\pi/4)\mathbf{j} = -\sqrt{2}\mathbf{i} + \sqrt{2}\mathbf{j}$$

These vectors and the circle are shown in Figure



Solution (c). At time t , the acceleration vector is

$$\mathbf{a}(t) = \frac{d\mathbf{v}}{dt} = -2\cos t\mathbf{i} - 2\sin t\mathbf{j}$$

One way of showing that $\mathbf{v}(t)$ and $\mathbf{a}(t)$ are perpendicular is to show that their dot product is zero (try it). However, it is easier to observe that $\mathbf{a}(t)$ is the negative of $\mathbf{r}(t)$, which implies that $\mathbf{v}(t)$ and $\mathbf{a}(t)$ are perpendicular, since at each point on a circle the radius and tangent line are perpendicular.

Example: A particle moves through 3-space in such a way that its velocity is

$$\mathbf{v}(t) = \mathbf{i} + t\,\mathbf{j} + t^2\,\mathbf{k}$$

Find the coordinates of the particle at time $t = 1$ given that the particle is at the point $(-1, 2, 4)$ at time $t = 0$.

Solution. Integrating the velocity function to obtain the position function yields

$$\mathbf{r}(t) = \int \mathbf{v}(t)\,dt = \int (\mathbf{i} + t\mathbf{j} + t^2\mathbf{k})\,dt = t\mathbf{i} + \frac{t^2}{2}\mathbf{j} + \frac{t^3}{3}\mathbf{k} + \mathbf{C}$$

where $\mathbf{C}$ is a vector constant of integration. Since the coordinates of the particle at time $t = 0$ are $(-1, 2, 4)$, the position vector at time $t = 0$ is

$$\mathbf{r}(0) = -\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$$

It follows on substituting $t = 0$ in (5) and equating the result with (6) that

$$\mathbf{C} = -\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$$

Substituting this value of $\mathbf{C}$ in (5) and simplifying yields

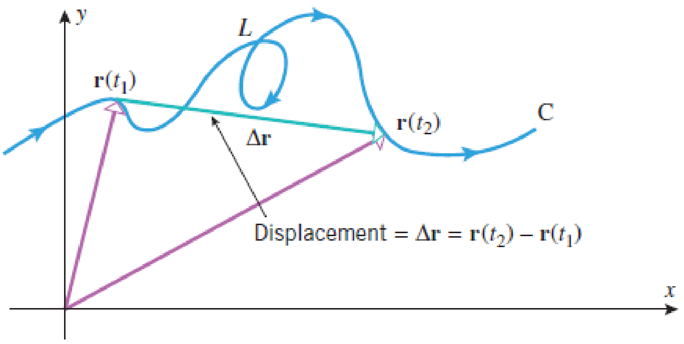
$$\mathbf{r}(t) = (t-1)\mathbf{i} + \left(\frac{t^2}{2} + 2\right)\mathbf{j} + \left(\frac{t^3}{3} + 4\right)\mathbf{k}$$

Thus, at time $t = 1$ the position vector of the particle is

$$\mathbf{r}(1) = 0\mathbf{i} + \frac{5}{2}\mathbf{j} + \frac{13}{3}\mathbf{k}$$

so its coordinates at that instant are $\left(0, \frac{5}{2}, \frac{13}{3}\right)$.

2.6.2 Displacement and Distance Traveled



$$\Delta\mathbf{r} = \int_{t_1}^{t_2} \mathbf{v}(t)\,dt = \int_{t_1}^{t_2} \frac{d\mathbf{r}}{dt}\,dt = \mathbf{r}(t)\Big]_{t_1}^{t_2} = \mathbf{r}(t_2) - \mathbf{r}(t_1)$$

Displacement

$$s = \int_{t_1}^{t_2} \left\| \frac{d\mathbf{r}}{dt} \right\| dt = \int_{t_1}^{t_2} \|\mathbf{v}(t)\| \, dt$$

Distance traveled

Example: Suppose that a particle moves along a circular helix in 3-space so that its position vector at time t is

$$\mathbf{r}(t) = (4\cos \pi t)\mathbf{i} + (4 \sin \pi t)\,\mathbf{j} + t\mathbf{k}$$

Find the distance traveled and the displacement of the particle during the time interval $1 \le t \le 5$.

Solution. We have

$$\mathbf{v}(t) = \frac{d\mathbf{r}}{dt} = (-4\pi \sin \pi t)\mathbf{i} + (4\pi \cos \pi t)\mathbf{j} + \mathbf{k}$$

$$\|\mathbf{v}(t)\| = \sqrt{(-4\pi \sin \pi t)^2 + (4\pi \cos \pi t)^2 + 1} = \sqrt{16\pi^2 + 1}$$

that the distance traveled by the particle from time $t = 1$ to $t = 5$ is

$$s = \int_1^5 \sqrt{16\pi^2 + 1}\, dt = 4\sqrt{16\pi^2 + 1}$$

that the displacement over the time interval is

$$\begin{aligned} \Delta\mathbf{r} &= \mathbf{r}(5) - \mathbf{r}(1) \\ &= (4\cos 5\pi \mathbf{i} + 4\sin 5\pi \mathbf{j} + 5\mathbf{k}) - (4\cos \pi \mathbf{i} + 4 \sin \pi \mathbf{j} + \mathbf{k}) \\ &= (-4\mathbf{i} + 5\mathbf{k}) - (-4\mathbf{i} + \mathbf{k}) = 4\mathbf{k} \end{aligned}$$

which tells us that the change in the position of the particle over the time interval was 4 units straight up.