CHAPTER THREE

PARTIAL DERIVATIVES

3.1 FUNCTIONS OF TWO OR MORE VARIABLES

3.1.1 Notation and Terminology

There are many familiar formulas in which a given variable depends on two or more other variables. For example, the area A of a triangle depends on the base length b and height h by the formula $A = \frac{1}{2}bh$; the volume V of a rectangular box depends on the length l, the width w, and the height h by the formula $V = lwh$; and the arithmetic average $\bar{x}$ of n real numbers, x_1, $x_2, \dots, x_n$, depends on those numbers by the formula

$$\bar{x} = \frac{1}{n}(x_1 + x_2, + \cdots + x_n)$$

Thus, we say that

A is a function of the two variables b and h;

V is a function of the three variables l, w, and h;

$\bar{x}$ is a function of the n variables $x_1, x_2, \dots, x_n$.

The terminology and notation for functions of two or more variables is similar to that for functions of one variable. For example, the expression

$$z = f(x, y)$$

means that z is a function of x and y in the sense that a unique value of the dependent variable z is determined by specifying values for the independent variables x and y. Similarly,

$$w = f(x, y, z)$$

expresses w as a function of x, y, and z, and

$$u = f(x_1, x_2, \dots, x_n)$$

expresses u as a function of $x_1, x_2, \dots, x_n$.

As with functions of one variable, the independent variables of a function of two or more variables may be restricted to lie in some set D, which we call the ***domain*** of f.

The domain consists of all points for which the formula yields a real value for the dependent variable. We call this the ***natural domain*** of the function.

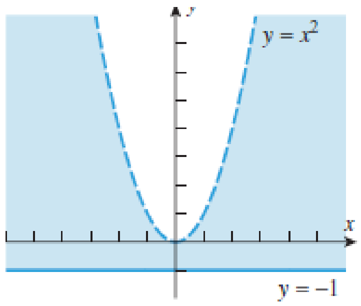
Definition 3.1 A ***function f of two variables***, x and y, is a rule that assigns a unique real number $f(x, y)$ to each point (x, y) in some set D in the xy-plane.

Definition 3.2 A ***function f of three variables***, x, y, and z, is a rule that assigns a unique real number $f(x, y, z)$ to each point (x, y, z) in some set D in three dimensional space.

Example 3.1 Let $f(x, y) = \sqrt{y + 1} + \ln(x^2 - y)$. Find $f(e, 0)$ and sketch the natural

Solution: By substitution,

$$f(e, 0) = \sqrt{0+1} + \ln(e^2 - 0) = \sqrt{1} + \ln(e^2) = 1 + 2 = 3$$



To find the natural domain of f , we note that $\sqrt{y} + 1$ is defined only when $y \geq -1$, while $\ln(x^2 - y)$ is defined only when $0 < x^2 - y$ or $y < x^2$. Thus, the natural domain of f consists of all points in the xy-plane for which $-1 \leq y < x^2$. To sketch the natural domain, we first sketch the parabola $y = x^2$ as a "dashed" curve and the line $y = -1$ as a solid curve. The natural domain of f is then the region lying above or on the line $y = -1$ and below the parabola $y = x^2$.

Example 3.2 Let $f(x, y, z) = \sqrt{1 - x^2 - y^2 - z^2}$ Find $f(0,1/2 ,-1/2)$ and the natural domain of f.

Solution: By substitution,

$$f\left(0, \tfrac{1}{2}, -\tfrac{1}{2}\right) = \sqrt{1 - (0)^2 - \left(\tfrac{1}{2}\right)^2 - \left(-\tfrac{1}{2}\right)^2} = \sqrt{\tfrac{1}{2}}$$

Because of the square root sign, we must have $0 \leq 1 - x^2 - y^2 - z^2$ in order to have a real value for f(x, y, z). Rewriting this inequality in the form

$$x^2 + y^2 + z^2 \leq 1$$

We see that the natural domain of f consists of all points on or within the sphere

$$x^2 + y^2 + z^2 = 1$$

3.1.2 Graphs of Functions of Two Variables

Example 3.3 In each part, describe the graph of the function in an *xyz*-coordinate system.

(a) $f(x, y) = 1 - x - \frac{1}{2}y$ (b) $f(x, y) = \sqrt{1 - x^2 - y^2}$

(c) $f(x, y) = -\sqrt{x^2 + y^2}$

Solution (a): By definition, the graph of the given function is the graph of the equation

$$z = 1 - x - \frac{1}{2}y \qquad (1)$$

which is a plane. A triangular portion of the plane can be sketched by plotting the intersections with the coordinate axes and joining them with line segments (Figure *a*).

Solution (b): By definition, the graph of the given function is the graph of the equation

$$z = \sqrt{1 - x^2 - y^2} \qquad (2)$$

After squaring both sides, this can be rewritten as

$$x^2 + y^2 + z^2 = 1$$

which represents a sphere of radius 1, centered at the origin. Since (2) imposes the added condition that $z \geq 0$, the graph is just the upper hemisphere (Figure *b*).

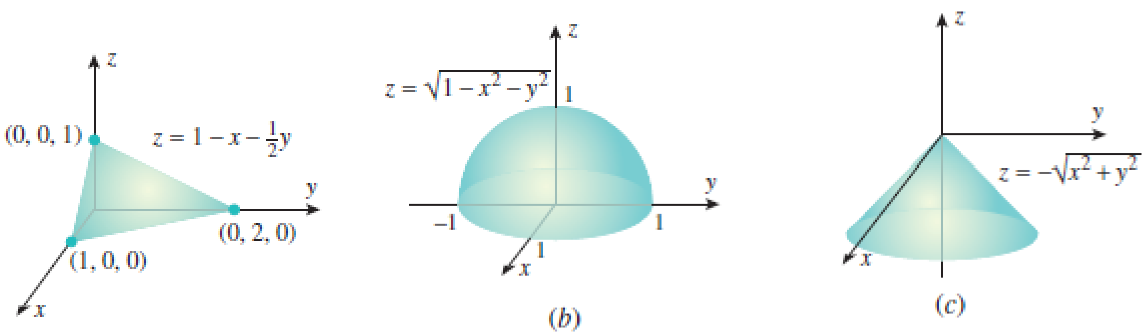
Solution (c): The graph of the given function is the graph of the equation

$$z = -\sqrt{x^2 + y^2} \qquad (3)$$

After squaring, we obtain

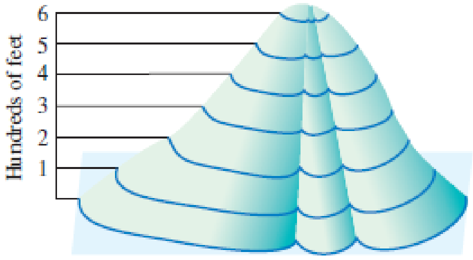
$$z^2 = x^2 + y^2$$

which is the equation of a circular cone. Since (3) imposes the condition that $z \leq 0$, the graph is just the lower nappe of the cone (Figure *c*).

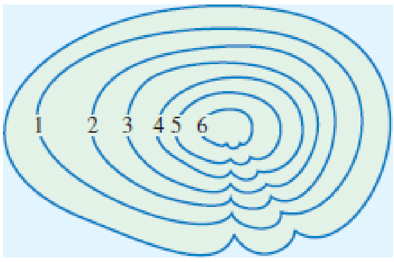


(a) (b) (c)

3.1.3 Level Curves

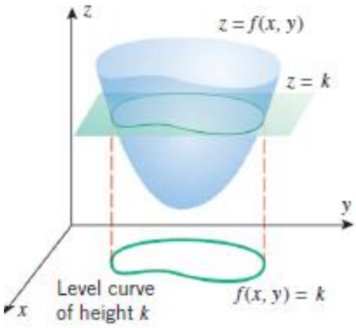


A perspective view of a model hill with two gullies



A contour map of the model hill

Contour maps are also useful for studying functions of two variables. If the surface $z = f(x, y)$ is cut by the horizontal plane $z = k$, then at all points on the intersection we have $f(x, y) = k$. The projection of this intersection onto the xy-plane is called the ***level curve of height k*** or the ***level curve with constant k*** (below figure). A set of level curves for $z = f(x, y)$ is called a ***contour plot*** or ***contour map*** of f.

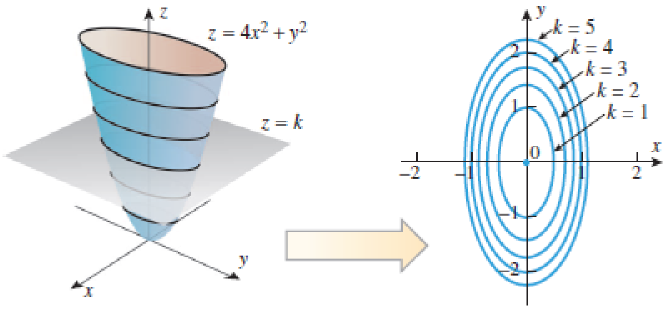


Example 3.4 Sketch the contour plot of $f(x, y) = 4x^2 + y^2$ using level curves of height $k = 0, 1, 2, 3, 4, 5$.

Solution: The graph of the surface $z = 4x^2 + y^2$ is the paraboloid shown in the left part of the below figure, so we can reasonably expect the contour plot to be a family of ellipses centered at the origin. The level curve of height k has the equation $4x^2 + y^2 = k$. If $k = 0$, then the graph is the single point $(0, 0)$. For $k > 0$ we can rewrite the equation as

$$\frac{x^2}{k/4} + \frac{y^2}{k} = 1$$

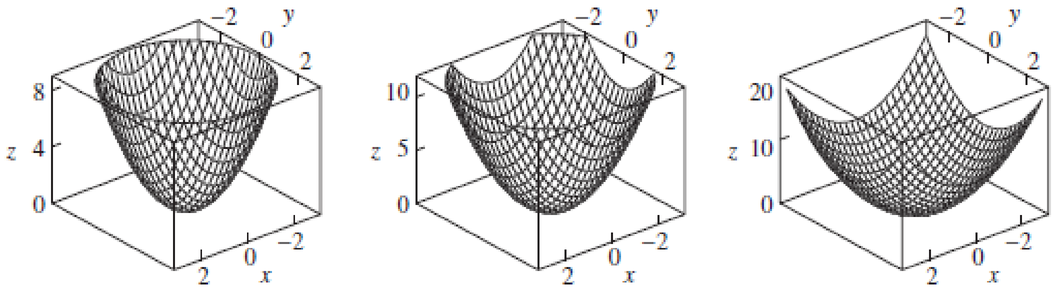
which represents a family of ellipses with x-intercepts $\pm\sqrt{k}/2$ and y-intercepts $\pm\sqrt{k}$. The contour plot for the specified values of k is shown in the right part of the following figure.



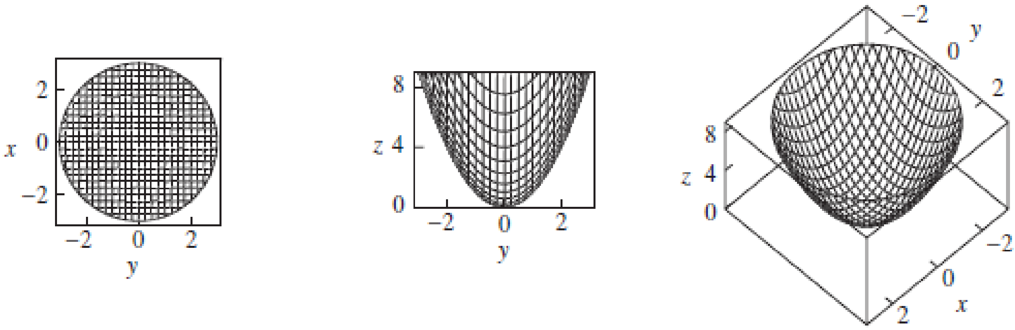
3.1.4 Graphing Functions Using Technology

Graphing utilities can only show a portion of xyz-space in a viewing screen, so the first step in graphing a surface is to determine which portion of xyz-space you want to display. This region is called the ***viewing box*** or ***viewing window***.

For example, the following figure shows the graph of the paraboloid $z = x^2 + y^2$ from three different viewpoints using the first viewing box.



Varying the viewing box.



Varying the viewpoint.

3.2 LIMITS AND CONTINUITY

3.2.1 Limits along Curves

For a function of one variable there are two one-sided limits at a point x_0, namely,

$$\lim_{x \to x_0^+} f(x) \quad \text{and} \quad \lim_{x \to x_0^-} f(x)$$

reflecting the fact that there are only two directions from which x can approach x_0, the right or the left.

For functions of two or three variables the situation is more complicated because there are infinitely many different curves along which one point can approach another.

The limit of f(x, y) as (x, y) approaches a point (x_0, y_0) along a curve C (and similarly for functions of three variables).

If C is a smooth parametric curve in 2-space or 3-space that is represented by the equations

$$x = x(t),\ y = y(t) \text{ or } x = x(t),\ y = y(t),\ z = z(t)$$

and if $x_0 = x(t_0)$, $y_0 = y(t_0)$, and $z_0 = z(t_0)$, then the limits

$$\lim_{\substack{(x,y)\to(x_0,y_0)\\ (\text{along } C)}} f(x,y) \quad \text{and} \quad \lim_{\substack{(x,y,z)\to(x_0,y_0,z_0)\\ (\text{along } C)}} f(x,y,z)$$

are defined by

$$\lim_{\substack{(x,y)\to(x_0,y_0)\\ (\text{along } C)}} f(x,y) = \lim_{t\to t_0} f(x(t), y(t)) \qquad (1)$$

$$\lim_{\substack{(x,y,z)\to(x_0,y_0,z_0)\\ (\text{along } C)}} f(x,y,z) = \lim_{t\to t_0} f(x(t), y(t), z(t)) \qquad (2)$$

In these formulas the limit of the function of t must be treated as a one-sided limit if (x_0, y_0) or (x_0, y_0, z_0) is an endpoint of C.

Example 3.5 below figure shows a computer-generated graph of the function

$$f(x,y) = -\frac{xy}{x^2+y^2}$$

The graph reveals that the surface has a ridge above the line $y = -x$, which is to be expected since $f(x, y)$ has a constant value of 1/2 for $y = -x$, except at $(0, 0)$ where f is undefined (verify). Moreover, the graph suggests that the limit of $f(x, y)$ as $(x, y) \to (0, 0)$ along a line through the origin varies with the direction of the line. Find this limit along

(a) the x-axis (b) the y-axis (c) the line $y = x$ (d) the line $y = -x$ (e) the parabola $y = x^2$

Solution (a): The x-axis has parametric equations $x = t$, $y = 0$, with (0, 0) corresponding to t = 0, so

$$\lim_{\substack{(x,y)\to(0,0)\\(\text{along } y=0)}} f(x,y) = \lim_{t\to 0} f(t,0) = \lim_{t\to 0}\left(-\frac{0}{t^2}\right) = \lim_{t\to 0} 0 = 0$$

Solution (b): The y-axis has parametric equations $x = 0$, $y = t$, with (0, 0) corresponding to t = 0, so

$$\lim_{\substack{(x,y)\to(0,0)\\(\text{along } x=0)}} f(x,y) = \lim_{t\to 0} f(0,t) = \lim_{t\to 0}\left(-\frac{0}{t^2}\right) = \lim_{t\to 0} 0 = 0$$

Solution (c): The line $y = x$ has parametric equations $x = t$, $y = t$, with (0, 0) corresponding to $t = 0$, so

$$\lim_{\substack{(x,y)\to(0,0)\\(\text{along } y=x)}} f(x,y) = \lim_{t\to 0} f(t,t) = \lim_{t\to 0}\left(-\frac{t^2}{2t^2}\right) = \lim_{t\to 0}\left(-\frac{1}{2}\right) = -\frac{1}{2}$$

Solution (d): The line $y = -x$ has parametric equations $x = t$, $y = -t$, with (0, 0) corresponding to $t = 0$, so

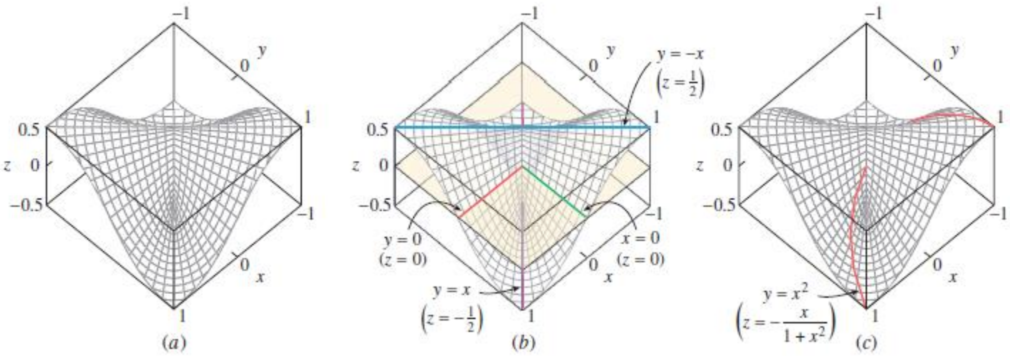
$$\lim_{\substack{(x,y)\to(0,0)\\(\text{along } y=-x)}} f(x,y) = \lim_{t\to 0} f(t,-t) = \lim_{t\to 0}\frac{t^2}{2t^2} = \lim_{t\to 0}\frac{1}{2} = \frac{1}{2}$$

Solution (e): The parabola $y = x^2$ has parametric equations $x = t$, $y = t^2$, with (0, 0) corresponding to t = 0, so

$$\lim_{\substack{(x,y)\to(0,0)\\(\text{along } y=x^2)}} f(x,y) = \lim_{t\to 0} f(t,t^2) = \lim_{t\to 0}\left(-\frac{t^3}{t^2+t^4}\right) = \lim_{t\to 0}\left(-\frac{t}{1+t^2}\right) = 0$$

This is consistent with Figure c, which shows the parametric curve

$$x = t, \quad y = t^2, \quad z = -\frac{t}{1+t^2}$$



3.2.2 Open and Closed Sets

Let C be a circle in 2-space that is centered at (x_0, y_0) and has positive radius δ.

-The set of points that are enclosed by the circle, but do not lie on the circle, is called the ***open disk.***

- The set of points that lie on the circle together with those enclosed by the circle is called the ***closed disk.***

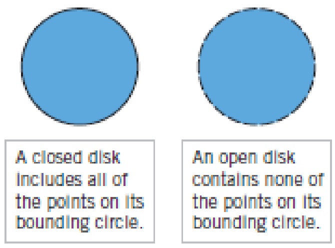
- If S is a sphere in 3-space that is centered at (x_0, y_0, z_0) and has positive radius δ:

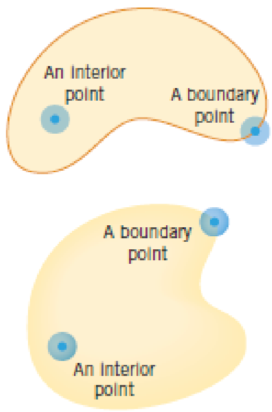
-The set of points that are enclosed by the sphere, but do not lie on the sphere, is called the ***open ball***

-The set of points that lie on the sphere together with those enclosed by the sphere is called the ***closed ball***.

- If D is a set of points in 2-space, then a point (x_0, y_0) is called an ***interior point*** of D if there is *some* open disk centered at (x_0, y_0) that contains only points of D,

- (x_0, y_0) is called a ***boundary point*** of D if *every* open disk centered at (x_0, y_0) contains both points in D and points not in D.





3.2.3 General Limits of Functions of Two Variables

Definition Let f be a function of two variables, and assume that f is defined at all points of some open disk centered at (x_0, y_0), except possibly at (x_0, y_0). We will write

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L$$

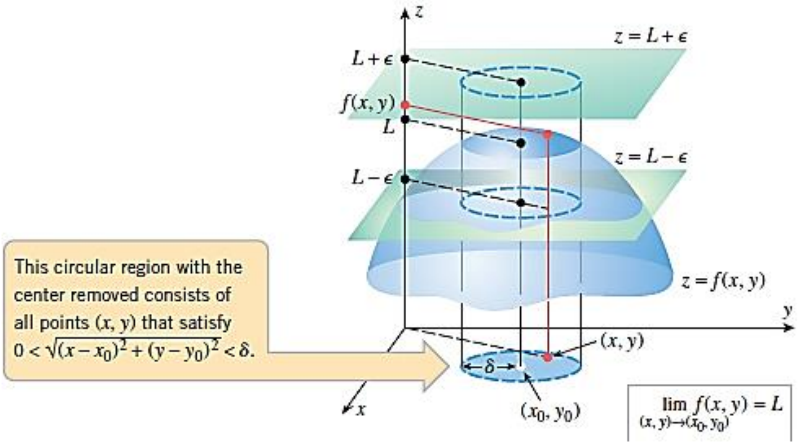
if given any number $\epsilon > 0$, we can find a number $\delta > 0$ such that $f(x, y)$ satisfies

$$|f(x, y) - L| < \epsilon$$

whenever the distance between (x, y) and (x_0, y_0) satisfies

$$0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta$$

As in below figure, this figure is intended to convey the idea that the values of $f(x, y)$ can be forced within ϵ units of L on the z-axis by restricting (x, y) to lie within δ units of (x_0, y_0) in the xy-plane. We used a white dot at (x_0, y_0) to suggest that the epsilon condition need not hold at this point.



Example 3.6

$$\begin{aligned}\lim_{(x,y)\to(1,4)} [5x^3y^2 - 9] &= \lim_{(x,y)\to(1,4)} [5x^3y^2] - \lim_{(x,y)\to(1,4)} 9 \\ &= 5\left[\lim_{(x,y)\to(1,4)} x\right]^3\left[\lim_{(x,y)\to(1,4)} y\right]^2 - 9 \\ &= 5(1)^3(4)^2 - 9 = 71\end{aligned}$$

3.2.4 Relationships between General Limits and Limits along Smooth Curves

Theorem

(a) *If $f(x, y)\to L$ as $(x, y)\to(x_0, y_0)$, then $f(x, y)\to L$ as $(x, y)\to(x_0, y_0)$ along any smooth curve.*

(b) *If the limit of $f(x, y)$ fails to exist as $(x, y)\to(x_0, y_0)$ along some smooth curve, or if $f(x, y)$ has different limits as $(x, y)\to(x_0, y_0)$ along two different smooth curves, then the limit of $f(x, y)$ does not exist as $(x, y)\to(x_0, y_0)$.*

Example 3.7 The limit

$$\lim_{(x,y)\to(0,0)} -\frac{xy}{x^2 + y^2}$$

does not exist because in previous example we found two different smooth curves along which this limit had different values. Specifically,

$$\lim_{\substack{(x,y)\to(0,0)\\(\text{along } x=0)}} -\frac{xy}{x^2 + y^2} = 0 \quad \text{and} \quad \lim_{\substack{(x,y)\to(0,0)\\(\text{along } y=x)}} -\frac{xy}{x^2 + y^2} = -\frac{1}{2}$$

3.2.5 CONTINUITY

Definition

A function $f(x, y)$ is said to be ***continuous at*** (x_0, y_0) if $f(x_0, y_0)$ is defined and if

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0)$$

In addition, if f is continuous at every point in an open set D, then we say that f is ***continuous on D***, and if f is continuous at every point in the xy-plane, then we say that f is ***continuous everywhere***.

Theorem

(a) If $g(x)$ is continuous at x_0 and $h(y)$ is continuous at y_0, then $f(x, y) = g(x)h(y)$ is continuous at (x_0, y_0).

(b) If $h(x, y)$ is continuous at (x_0, y_0) and $g(u)$ is continuous at $u = h(x_0, y_0)$, then the composition $f(x, y) = g(h(x, y))$ is continuous at (x_0, y_0).

(c) If $f(x, y)$ is continuous at (x_0, y_0), and if $x(t)$ and $y(t)$ are continuous at t_0 with $x(t_0) = x_0$ and $y(t_0) = y_0$, then the composition $f(x(t), y(t))$ is continuous at t_0.

Example 3.8 Use Theorem to show that the functions $f(x, y) = 3x^2y^5$ and $f(x, y) = \sin(3x^2y^5)$ are continuous everywhere.

Solution: The polynomials $g(x) = 3x^2$ and $h(y) = y^5$ are continuous at every real number, and therefore by part (*a*) of Theorem, the function $f(x, y) = 3x^2y^5$ is continuous at every point (x, y) in the xy-plane. Since $3x^2y^5$ is continuous at every point in the xy-plane and $\sin u$ is continuous at every real number u, it follows from part (*b*) of Theorem that the composition $f(x, y) = \sin(3x^2y^5)$ is continuous everywhere.

Recognizing Continuous Functions

- A composition of continuous functions is continuous.
- A sum, difference, or product of continuous functions is continuous.
- A quotient of continuous functions is continuous, except where the denominator is zero.

Example 3.9 Evaluate

$$\lim_{(x,y)\to(-1,2)} \frac{xy}{x^2 + y^2}$$

Solution: Since $f(x, y) = xy/(x^2 + y^2)$ is continuous at $(-1, 2)$ (why?), it follows from the definition of continuity for functions of two variables that

$$\lim_{(x,y)\to(-1,2)} \frac{xy}{x^2+y^2} = \frac{(-1)(2)}{(-1)^2+(2)^2} = -\frac{2}{5}$$

Example 3.10 Since the function

$$f(x,y) = \frac{x^3y^2}{1-xy}$$

is a quotient of continuous functions, it is continuous except where $1 - xy = 0$. Thus, $f(x, y)$ is continuous everywhere except on the hyperbola $xy = 1$.

3.2.6 Limits at Discontinuities

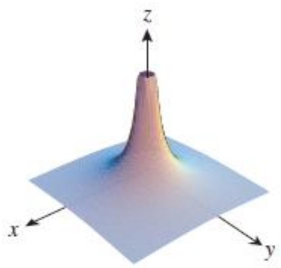
Sometimes it is easy to recognize when a limit does not exist. For example, it is evident that

$$\lim_{(x,y)\to(0,0)} \frac{1}{x^2+y^2} = +\infty$$

which implies that the values of the function approach $+\infty$ as $(x, y)\to(0, 0)$ along any smooth curve (below figure). However, it is not evident whether the limit

$$\lim_{(x,y)\to(0,0)} (x^2+y^2)\ln(x^2+y^2)$$

exists because it is an indeterminate form of type $0 \cdot \infty$. Although L'Hôpital's rule cannot be applied directly, the following example illustrates a method for finding this limit by converting to polar coordinates.



$$z = \frac{1}{x^2+y^2}$$

Example 3.11 Find

$$\lim_{(x,y)\to(0,0)} (x^2+y^2)\ln(x^2+y^2)$$

Solution: Let (r, θ) be polar coordinates of the point (x, y) with $r \geq 0$. Then we have

$$x = r\cos\theta,\ y = r\sin\theta,\ r^2 = x^2 + y^2$$

Moreover, since $r \geq 0$ we have $r = \sqrt{x^2 + y^2}$, so that $r \to 0^+$ if and only if $(x, y) \to (0, 0)$. Thus, we can rewrite the given limit as

$$\begin{aligned}
\lim_{(x,y)\to(0,0)} (x^2+y^2)\ln(x^2+y^2) &= \lim_{r\to 0^+} r^2 \ln r^2 \\
&= \lim_{r\to 0^+} \frac{2\ln r}{1/r^2} && \boxed{\text{This converts the limit to an indeterminate form of type } \infty/\infty.} \\
&= \lim_{r\to 0^+} \frac{2/r}{-2/r^3} && \boxed{\text{L'Hôpital's rule}} \\
&= \lim_{r\to 0^+} (-r^2) = 0
\end{aligned}$$

3.2.7 Continuity at Boundary Points

Recall that in our study of continuity for functions of one variable, we first defined continuity at a point, then continuity on an open interval, and then, by using one-sided limits, we extended the notion of continuity to include the boundary points of the interval. Similarly, for functions of two variables one can extend the notion of continuity of $f(x, y)$ to the boundary of its domain by modifying previous definition appropriately so that (x, y) is restricted to approach (x_0, y_0) through points lying wholly in the domain of f.

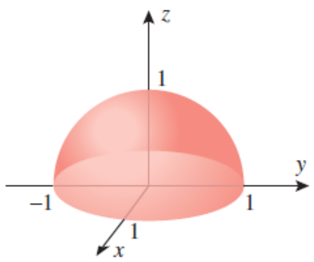
Example 3.12 The graph of the function $f(x, y) = \sqrt{1 - x^2 - y^2}$ is the upper hemisphere shown in below figure, and the natural domain of f is the closed unit disk

$$x^2 + y^2 \leq 1$$

The graph of f has no tears or holes, so it passes our "intuitive test" of continuity. In this case the continuity at a point (x_0, y_0) on the boundary reflects the fact that

$$\lim_{(x,y)\to(x_0,y_0)} \sqrt{1 - x^2 - y^2} = \sqrt{1 - x_0^2 - y_0^2} = 0$$

when (x, y) is restricted to points on the closed unit disk $x^2 + y^2 \leq 1$. It follows that f is continuous on its domain.



$z = \sqrt{1 - x^2 - y^2}$

3.2.8 Extensions to Three Variables

Definition Let f be a function of three variables, and assume that f is defined at all points within a ball centered at (x_0, y_0, z_0), except possibly at (x_0, y_0, z_0). We will write

$$\lim_{(x,y,z)\to(x_0,y_0,z_0)} f(x, y, z) = L$$

if given any number $\epsilon > 0$, we can find a number $\delta > 0$ such that $f(x, y, z)$ satisfies

$$|f(x, y, z) - L| < \epsilon$$

whenever the distance between (x, y, z) and (x_0, y_0, z_0) satisfies

$$0 < \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2} < \delta$$

As with functions of one and two variables, we define a function $f(x, y, z)$ of three variables to be continuous at a point (x_0, y_0, z_0) if the limit of the function and the value of the function are the same at this point; that is,

$$\lim_{(x,y,z)\to(x_0,y_0,z_0)} f(x, y, z) = f(x_0, y_0, z_0)$$