3.6 DIRECTIONAL DERIVATIVES AND GRADIENTS

3.6.1 Directional Derivatives

Definition If $f(x, y)$ is a function of x and y, and if $\mathbf{u} = u_1\,\mathbf{i} + u_2\,\mathbf{j}$ is a unit vector, then the ***directional derivative of f in the direction of* u** at (x_0, y_0) is denoted by $D_{\mathbf{u}}f(x_0, y_0)$ and is defined by

$$D_{\mathbf{u}}\, f(x_0, y_0) = \frac{d}{ds}\left[f(x_0 + su_1, y_0 + su_2)\right]_{s=0}$$

provided this derivative exists.

Geometrically, $D_{\mathbf{u}}f(x_0, y_0)$ can be interpreted as the ***slope of the surface z = f (x, y) in the direction of* u** at the point $(x_0, y_0, f(x_0, y_0))$ (Figure a). Usually the value of $D_{\mathbf{u}}f(x_0, y_0)$ will depend on both the point (x_0, y_0) and the direction **u**. Thus, at a fixed point the slope of the surface may vary with the direction (Figure b). Analytically, the directional derivative represents the ***instantaneous rate of change of f (x, y) with respect to distance in the direction of* u** at the point (x_0, y_0).

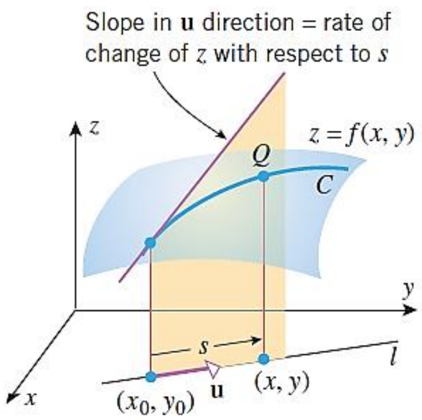


Figure a

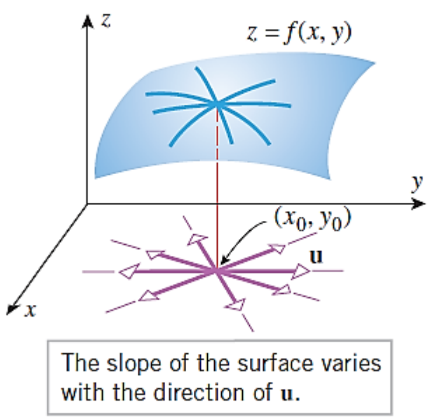


Figure b

Example 3.28 Let $f(x, y) = xy$. Find and interpret $D_{\mathbf{u}}f(1, 2)$ for the unit vector

$$\mathbf{u} = \frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}$$

Solution:

$$D_{\mathbf{u}} f(1,2) = \frac{d}{ds}\left[f\left(1+\frac{\sqrt{3}s}{2}, 2+\frac{s}{2}\right)\right]_{s=0}$$

Since

$$f\left(1+\frac{\sqrt{3}s}{2}, 2+\frac{s}{2}\right) = \left(1+\frac{\sqrt{3}s}{2}\right)\left(2+\frac{s}{2}\right) = \frac{\sqrt{3}}{4}s^2 + \left(\frac{1}{2}+\sqrt{3}\right)s + 2$$

we have

$$D_{\mathbf{u}} f(1,2) = \frac{d}{ds}\left[\frac{\sqrt{3}}{4}s^2 + \left(\frac{1}{2}+\sqrt{3}\right)s + 2\right]_{s=0}$$

$$= \left[\frac{\sqrt{3}}{2}s + \frac{1}{2} + \sqrt{3}\right]_{s=0} = \frac{1}{2}+\sqrt{3}$$

Since $1/2+\sqrt{3} \approx 2.23$, we conclude that if we move a small distance from the point $(1, 2)$ in the direction of **u**, the function $f(x, y) = xy$ will increase by about 2.23 times the distance moved.

Definition

If $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ is a unit vector, and if $f(x, y, z)$ is a function of x, y, and z, then the ***directional derivative of f in the direction of* u** at (x_0, y_0, z_0) is denoted by $D_{\mathbf{u}}f(x_0, y_0, z_0)$ and is defined by

$$D_{\mathbf{u}} f(x_0, y_0, z_0) = \frac{d}{ds}\left[f(x_0+su_1, y_0+su_2, z_0+su_3)\right]_{s=0}$$

provided this derivative exists.

Theorem

(a) *If $f(x, y)$ is differentiable at (x_0, y_0), and if $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ is a unit vector, then the directional derivative $D_{\mathbf{u}}f(x_0, y_0)$ exists and is given by*

$$D_{\mathbf{u}} f(x_0, y_0) = f_x(x_0, y_0)u_1 + f_y(x_0, y_0)u_2$$

(b) *If $f(x, y, z)$ is differentiable at (x_0, y_0, z_0), and if $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ is a unit vector, then the directional derivative $D_{\mathbf{u}}f(x_0, y_0, z_0)$ exists and is given by*

$$D_{\mathbf{u}} f(x_0, y_0, z_0) = f_x(x_0, y_0, z_0)u_1 + f_y(x_0, y_0, z_0)u_2 + f_z(x_0, y_0, z_0)u_3$$

Example 3.29 Find the directional derivative of $f(x, y) = e^{xy}$ at $(-2, 0)$ in the direction of the unit vector that makes an angle of $\pi/3$ with the positive x-axis.

Solution: The partial derivatives of f are

$$f_x(x,y) = ye^{xy}, \quad f_y(x,y) = xe^{xy}$$

$$f_x(-2,0) = 0, \quad f_y(-2,0) = -2$$

The unit vector **u** that makes an angle of $\pi/3$ with the positive x-axis is

$$\mathbf{u} = \cos(\pi/3)\mathbf{i} + \sin(\pi/3)\mathbf{j} = \frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}$$

$$\begin{aligned} D_{\mathbf{u}} f(-2, 0) &= f_x(-2, 0)\cos(\pi/3) + f_y(-2, 0)\sin(\pi/3) \\ &= 0(1/2) + (-2)(\sqrt{3}/2) = -\sqrt{3} \end{aligned}$$

Example 3.30 Find the directional derivative of $f(x, y, z) = x^2y - yz^3 + z$ at the point $(1, -2, 0)$ in the direction of the vector $\mathbf{a} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$.

Solution:

$$f_x(x, y, z) = 2xy, \quad f_y(x, y, z) = x^2 - z^3, \quad f_z(x, y, z) = -3yz^2 + 1$$
$$f_x(1, -2, 0) = -4, \quad f_y(1, -2, 0) = 1, \quad f_z(1, -2, 0) = 1$$

Since **a** is not a unit vector, we normalize it, getting

$$\mathbf{u} = \frac{\mathbf{a}}{\|\mathbf{a}\|} = \frac{1}{\sqrt{9}}(2\mathbf{i} + \mathbf{j} - 2\mathbf{k}) = \frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}$$

Formula then yields

$$D_{\mathbf{u}} f(1, -2, 0) = (-4)\left(\frac{2}{3}\right) + \frac{1}{3} - \frac{2}{3} = -3$$

3.6.2 The Gradient

Definition

(a) If f is a function of x and y, then the ***gradient of f*** is defined by

$$\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j}$$

(c) If f is a function of x, y, and z, then the ***gradient of f*** is defined by

$$\nabla f(x, y, z) = f_x(x, y, z)\mathbf{i} + f_y(x, y, z)\mathbf{j} + f_z(x, y, z)\mathbf{k}$$

The symbol ∇ (read "del") is a "nabla"

Formulas can now be written as

$D_{\mathbf{u}}f(x_0, y_0) = \nabla f(x_0, y_0) \,.\, \mathbf{u}$

$D_{\mathbf{u}}f(x_0, y_0, z_0) = \nabla f(x_0, y_0, z_0) \,.\, \mathbf{u}$

For example, using above formula our solution to **Example 3.30** would take the form

$$D_{\mathbf{u}}f(1, -2, 0) = \nabla f(1, -2, 0) \,.\, \mathbf{u} = (-4\mathbf{i} + \mathbf{j} + \mathbf{k}) = 2/3\,\mathbf{i} + 1/3\mathbf{j} - 2/3\mathbf{k}$$
$$= (-4)(2/3) + 1/3 - 2/3 = -3$$

3.6.3 Properties of the Gradient

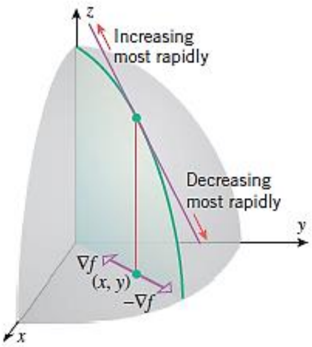
Theorem

Let f be a function of either two variables or three variables, and let P denote the point $P(x_0, y_0)$ or $P(x_0, y_0, z_0)$, respectively. Assume that f is differentiable at P.

(a) If $\nabla f = \mathbf{0}$ at P, then all directional derivatives of f at P are zero.

(b) If $\nabla f \neq \mathbf{0}$ at P, then among all possible directional derivatives of f at P, the derivative in the direction of ∇f at P has the largest value. The value of this largest directional derivative is $\|\nabla f\|$ at P.

(c) If $\nabla f \neq \mathbf{0}$ at P, then among all possible directional derivatives of f at P, the derivative in the direction opposite to that of ∇f at P has the smallest value. The value of this smallest directional derivative is $-\|\nabla f\|$ at P.



Example 3.31 Let $f(x, y) = x^2e^y$. Find the maximum value of a directional derivative at $(-2, 0)$, and find the unit vector in the direction in which the maximum value occurs.

Solution:

$$\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j} = 2xe^y\mathbf{i} + x^2e^y\mathbf{j}$$

the gradient of f at $(-2, 0)$ is

$$\nabla f(-2, 0) = -4\mathbf{i} + 4\mathbf{j}$$

By Theorem [illegible], the maximum value of the directional derivative is

$$\|\nabla f(-2, 0)\| = \sqrt{(-4)^2 + 4^2} = \sqrt{32} = 4\sqrt{2}$$

This maximum occurs in the direction of $\nabla f(-2, 0)$. The unit vector in this direction is

$$\mathbf{u} = \frac{\nabla f(-2, 0)}{\|\nabla f(-2, 0)\|} = \frac{1}{4\sqrt{2}}(-4\mathbf{i} + 4\mathbf{j}) = -\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$$

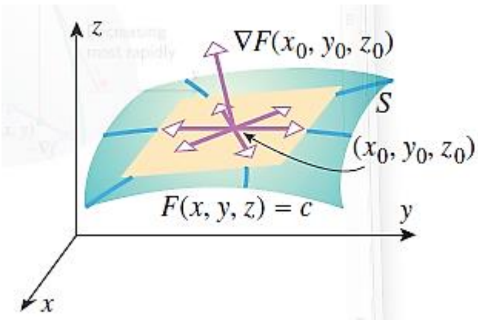
3.7 TANGENT PLANES AND NORMAL VECTORS

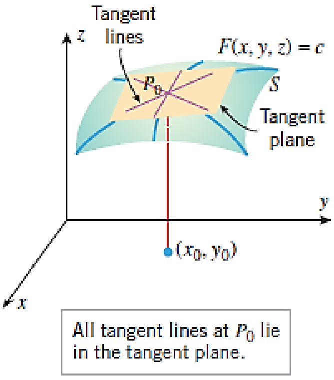
3.7.1 Tangent Planes and Normal Vectors to Level Surfaces $F(x, y, z) = c$

Definition

Assume that $F(x, y, z)$ has continuous first-order partial derivatives and that $P_0(x_0, y_0, z_0)$ is a point on the level surface S: $F(x, y, z) = c$. If $\nabla F(x_0, y_0, z_0) \neq \mathbf{0}$, then $\mathbf{n} = \nabla F(x_0, y_0, z_0)$ is a ***normal vector*** to S at P_0 and the ***tangent plane*** to S at P_0 is the plane with equation

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$$





Example 3.32 Consider the ellipsoid $x^2 + 4y^2 + z^2 = 18$.

(a) Find an equation of the tangent plane to the ellipsoid at the point (1, 2, 1).

(b) Find parametric equations of the line that is normal to the ellipsoid at the point (1, 2, 1).

(c) Find the acute angle that the tangent plane at the point (1, 2, 1) makes with the xy-plane.

Solution:

Solution (a): We apply Definition with $F(x, y, z) = x^2 + 4y^2 + z^2$ and $(x_0, y_0, z_0) = (1, 2, 1)$. Since

$$\nabla F(x, y, z) = (F_x(x, y, z), F_y(x, y, z), F_z(x, y, z)) = (2x, 8y, 2z)$$

we have

$$\mathbf{n} = \nabla F(1, 2, 1) = (2, 16, 2)$$

Hence, the equation of the tangent plane is

$$2(x - 1) + 16(y - 2) + 2(z - 1) = 0 \text{ or } x + 8y + z = 18$$

Solution (b): Since $\mathbf{n} = (2, 16, 2)$ at the point (1, 2, 1), it follows that parametric equations for the normal line to the ellipsoid at the point (1, 2, 1) are

$$x = 1 + 2t, y = 2 + 16t, z = 1 + 2t$$

Solution (c): To find the acute angle θ between the tangent plane and the xy-plane, $\mathbf{n}_1 = \mathbf{n} = (2, 16, 2)$ and $\mathbf{n}_2 = (0, 0, 1)$. This yields

$$\cos\theta = \frac{|\langle 2, 16, 2\rangle \cdot \langle 0, 0, 1\rangle|}{\|\langle 2, 16, 2\rangle\| \, \|\langle 0, 0, 1\rangle\|} = \frac{2}{(2\sqrt{66})(1)} = \frac{1}{\sqrt{66}}$$

$$\theta = \cos^{-1}\left(\frac{1}{\sqrt{66}}\right) \approx 83°$$

3.7.2 Tangent Planes to Surfaces of The Form *z* = *f* (*x*, *y*)

Example 3.33 Find an equation for the tangent plane and parametric equations for the normal line to the surface $z = x^2y$ at the point $(2, 1, 4)$.

Solution: Let $F(x, y, z) = z - x^2y$. Then $F(x, y, z) = 0$ on the surface, so we can find the find the gradient of F at the point $(2, 1, 4)$:

$$\nabla F(x, y, z) = -2xy\mathbf{i} - x^2\mathbf{j} + \mathbf{k}$$

$$\nabla F(2, 1, 4) = -4\mathbf{i} - 4\mathbf{j} + \mathbf{k}$$

the tangent plane has equation

$$-4(x - 2) - 4(y - 1) + 1(z - 4) = 0 \text{ or } -4x - 4y + z = -8$$

and the normal line has equations

$$x = 2 - 4t,\ y = 1 - 4t,\ z = 4 + t$$

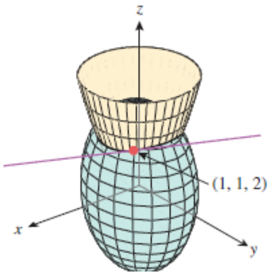
Theorem *If $f(x, y)$ is differentiable at the point (x_0, y_0), then the tangent plane to the surface $z = f(x, y)$ at the point $P_0(x_0, y_0, f(x_0, y_0))$ [or (x_0, y_0)] is the plane*

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

3.7.3 Using Gradients to Find Tangent Lines to Intersections of Surfaces

Example 3.34 Find parametric equations of the tangent line to the curve of intersection of the paraboloid $z = x^2 + y^2$ and the ellipsoid $3x^2 + 2y^2 + z^2 = 9$ at the point $(1, 1, 2)$

Solution: We begin by rewriting the equations of the surfaces as



$$x^2 + y^2 - z = 0 \text{ and } 3x^2 + 2y^2 + z^2 - 9 = 0$$

and we take

$$F(x, y, z) = x^2 + y^2 - z \text{ and } G(x, y, z) = 3x^2 + 2y^2 + z^2 - 9$$

We will need the gradients of these functions at the point $(1, 1, 2)$. The computations are

$$\nabla F(x, y, z) = 2x\mathbf{i} + 2y\,\mathbf{j} - \mathbf{k},\ \nabla G(x, y, z) = 6x\mathbf{i} + 4y\,\mathbf{j} + 2z\mathbf{k}$$

$$\nabla F(1, 1, 2) = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k},\ \nabla G(1, 1, 2) = 6\mathbf{i} + 4\mathbf{j} + 4\mathbf{k}$$

Thus, a tangent vector at $(1, 1, 2)$ to the curve of intersection is

$$\nabla F(1,1,2) \times \nabla G(1,1,2) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & -1 \\ 6 & 4 & 4 \end{vmatrix} = 12\mathbf{i} - 14\mathbf{j} - 4\mathbf{k}$$

Since any scalar multiple of this vector will do just as well, we can multiply by 1/2 to reduce the size of the coefficients and use the vector of $6\mathbf{i} - 7\mathbf{j} - 2\mathbf{k}$ to determine the direction of the tangent line. This vector and the point $(1, 1, 2)$ yield the parametric equations

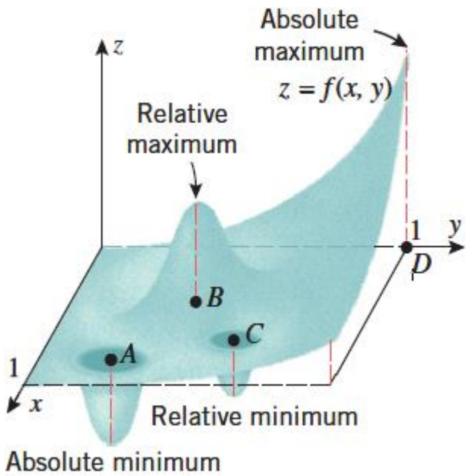
$$x = 1 + 6t,\ y = 1 - 7t,\ z = 2 - 2t$$

3.8 MAXIMA AND MINIMA OF FUNCTIONS OF TWO VARIABLES

3.8.1 Extrema

Definition A function f of two variables is said to have a ***relative maximum*** at a point (x_0, y_0) if there is a disk centered at (x_0, y_0) such that $f(x_0, y_0) \geq f(x, y)$ for all points (x, y) that lie inside the disk, and f is said to have an ***absolute maximum*** at (x_0, y_0) if $f(x_0, y_0) \geq f(x, y)$ for all points (x, y) in the domain of f.

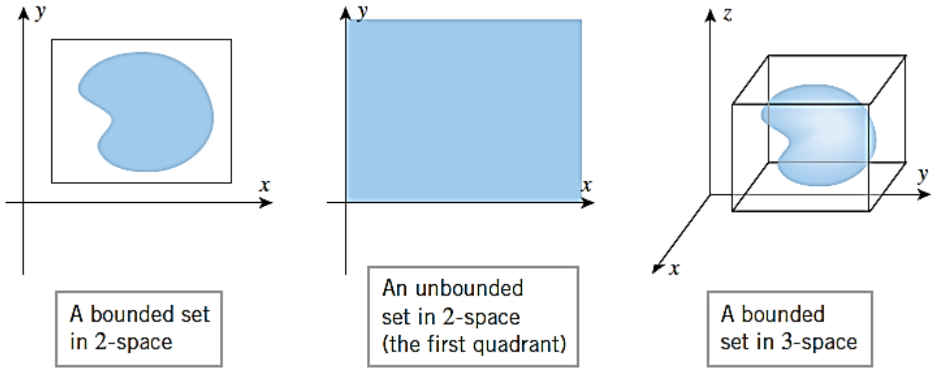
Definition A function f of two variables is said to have a ***relative minimum*** at a point (x_0, y_0) if there is a disk centered at (x_0, y_0) such that $f(x_0, y_0) \leq f(x, y)$ for all points (x, y) that lie inside the disk, and f is said to have an ***absolute minimum*** at (x_0, y_0) if $f(x_0, y_0) \leq f(x, y)$ for all points (x, y) in the domain of f.



If f has a relative maximum or a relative minimum at (x_0, y_0), then we say that f has a ***relative extremum*** at (x_0, y_0), and if f has an absolute maximum or absolute minimum at (x_0, y_0), then we say that f has an ***absolute extremum*** at (x_0, y_0).

3.8.2 Bounded Sets

- (finite intervals and infinite intervals on the real line),
- Distinguish between regions of "finite extent" and regions of "infinite extent" in 2-space and 3-space.
- A set of points in 2-space is called ***bounded*** if the entire set can be contained within some rectangle,
- called ***unbounded*** if there is no rectangle that contains all the points of the set.
- Similarly, a set of points in 3-space is ***bounded*** if the entire set can be contained within some box, and is **unbounded** otherwise (see below Figure).



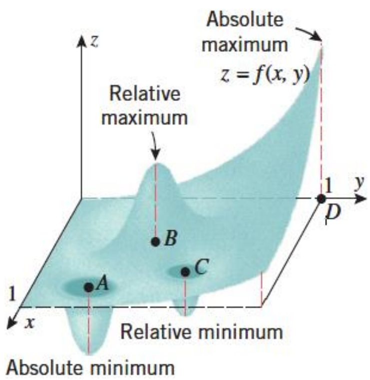
3.8.3 The Extreme-Value Theorem

Theorem (***Extreme-Value Theorem***) *If f(x, y) is continuous on a closed and bounded set R, then f has both an absolute maximum and an absolute minimum on R.*

Example 3.35 The square region R whose points satisfy the inequalities

$$0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1$$

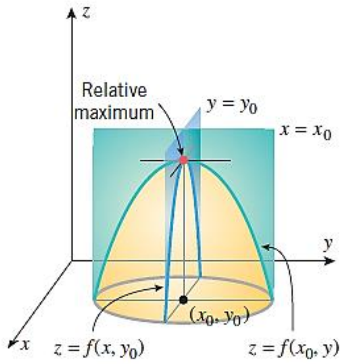
is a closed and bounded set in the xy-plane. The function f whose graph is shown in Figure is continuous on R; thus, it is guaranteed to have an absolute maximum and minimum on R by the last theorem. These occur at points D and A that are shown in the figure.



3.8.4 Finding Relative Extrema

Theorem *If f has a relative extremum at a point (x_0, y_0), and if the first-order partial derivatives of f exist at this point, then*

$$f_x(x_0, y_0) = 0 \text{ and } f_y(x_0, y_0) = 0$$



Definition A point (x_0, y_0) in the domain of a function $f(x, y)$ is called a ***critical point*** of the function if $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$ or if one or both partial derivatives do not exist at (x_0, y_0).

Example: consider the function

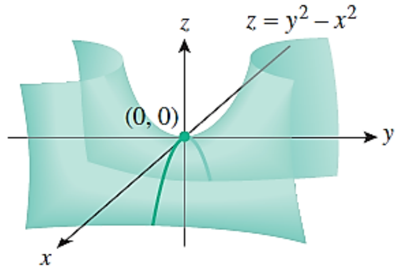
$$f(x, y) = y^2 - x^2$$

This function, whose graph is the hyperbolic paraboloid shown in the figure, has a critical point at (0, 0), since

$$f_x(x, y) = -2x \text{ and } f_y(x, y) = 2y$$

from which it follows that

$$f_x(0, 0) = 0 \text{ and } f_y(0, 0) = 0$$

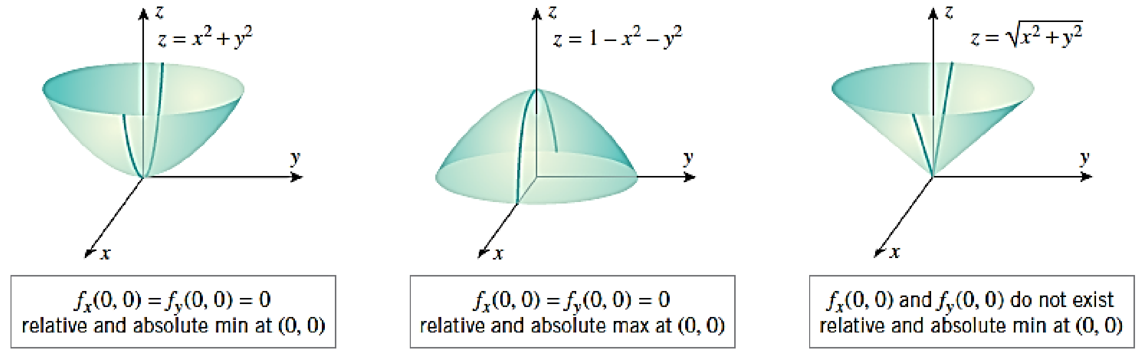


The function $f(x, y) = y^2 - x^2$ has neither a relative maximum nor a relative minimum at the critical point (0, 0).

The function f has neither a relative maximum nor a relative minimum at $(0, 0)$. For obvious reasons, the point $(0, 0)$ is called a ***saddle point*** of f.

In general, we will say that a surface $z = f(x, y)$ has a ***saddle point*** at (x_0, y_0) if there are two distinct vertical planes through this point such that the trace of the surface in one of the planes has a relative maximum at (x_0, y_0) and the trace in the other has a relative minimum at (x_0, y_0).

Example The three functions graphed in the following figure all have critical points at (0, 0). For the paraboloids, the partial derivatives at the origin are zero. You can check this algebraically by evaluating the partial derivatives at (0, 0), but you can see it geometrically by observing that the traces in the xz-plane and yz-plane have horizontal tangent lines at (0, 0).



(*a*) (*b*) (*c*)

3.8.5 The Second Partials Test

Theorem (***The Second Partials Test***) *Let f be a function of two variables with continuous second-order partial derivatives in some disk centered at a critical point* (x_0, y_0)*, and let*

$$D = f_{xx}(x_0, y_0)\, f_{yy}(x_0, y_0) - f^2_{xy}(x_0, y_0)$$

(*a*) *If* $D > 0$ *and* $f_{xx}(x_0, y_0) > 0$*, then f has a relative minimum at* (x_0, y_0).

(*b*) *If* $D > 0$ *and* $f_{xx}(x_0, y_0) < 0$*, then f has a relative maximum at* (x_0, y_0).

(*c*) *If* $D < 0$*, then f has a saddle point at* (x_0, y_0).

(*d*) *If* $D = 0$*, then no conclusion can be drawn.*

Example 3.36 Locate all relative extrema and saddle points of

$$f(x, y) = 3x^2 - 2xy + y^2 - 8y$$

Solution: Since $fx(x, y) = 6x - 2y$ and $fy(x, y) = -2x + 2y - 8$, the critical points of f satisfy the equations

$$6x - 2y = 0$$
$$-2x + 2y - 8 = 0$$

Solving these for x and y yields $x = 2$, $y = 6$ (verify), so (2, 6) is the only critical point.

To apply Theorem, we need the second-order partial derivatives

$$f_{xx}(x, y) = 6, \quad f_{yy}(x, y) = 2, \quad f_{xy}(x, y) = -2$$

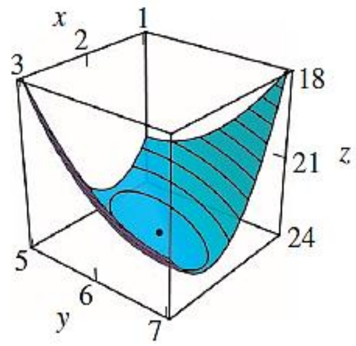
At the point (2, 6) we have

$$D = f_{xx}(2, 6)\, f_{yy}(2, 6) - f^2_{xy}(2, 6) = (6)(2) - (-2)^2 = 8 > 0$$

and

$$f_{xx}(2, 6) = 6 > 0$$

so f has a relative minimum at (2, 6) by part (*a*) of the second partials test. The below figure shows a graph of f in the vicinity of the relative minimum.



$f(x, y) = 3x^2 - 2xy + y^2 - 8y$

Example 3.37 Locate all relative extrema and saddle points of

$$f(x, y) = 4xy - x^4 - y^4$$

Solution: Since

$$f_x(x, y) = 4y - 4x^3$$
$$f_y(x, y) = 4x - 4y^3 \qquad (1)$$

the critical points of f have coordinates satisfying the equations

$$4y - 4x^3 = 0 \qquad\qquad y = x^3$$

or

$$4x - 4y^3 = 0 \qquad\qquad x = y^3 \qquad (2)$$

Substituting the top equation in the bottom yields $x = (x^3)^3$ or, equivalently, $x^9 - x = 0$ or $x(x^8 - 1) = 0$, which has solutions $x = 0$, $x = 1$, $x = -1$. Substituting these values in the top equation of (2), we obtain the corresponding y-values $y = 0$, $y = 1$, $y = -1$. Thus, the critical points of f are (0, 0), (1, 1), and (−1, −1).

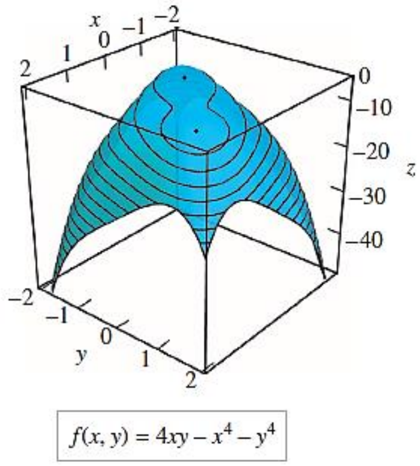
From (1),

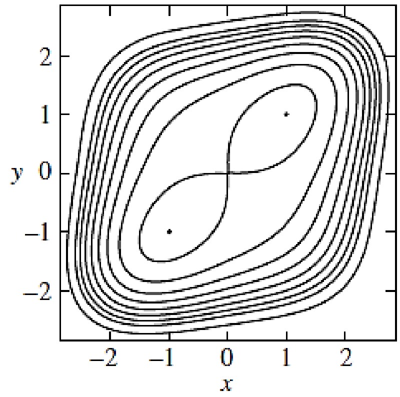
$$f_{xx}(x, y) = -12x^2, \quad f_{yy}(x, y) = -12y^2, \quad f_{xy}(x, y) = 4$$

which yields the following table:

CRITICAL POINT (x_0, y_0)	$f_{xx}(x_0, y_0)$	$f_{yy}(x_0, y_0)$	$f_{xy}(x_0, y_0)$	$D = f_{xx}f_{yy} - f_{xy}^2$
(0, 0)	0	0	4	−16
(1, 1)	−12	−12	4	128
(−1, −1)	−12	−12	4	128

At the points (1, 1) and (−1,−1), we have D > 0 and $f_{xx} < 0$, so relative maxima occur at these critical points. At (0, 0) there is a saddle point since D < 0. The surface and a contour plot are shown in the below figure.





Theorem *If a function f of two variables has an absolute extremum (either an absolute maximum or an absolute minimum) at an interior point of its domain, then this extremum occurs at a critical point.*

3.8.6 Finding Absolute Extrema on Closed and Bounded Sets

How to Find the Absolute Extrema of a Continuous Function f of Two Variables on a Closed and Bounded Set R

Step 1. Find the critical points of f that lie in the interior of R.

Step 2. Find all boundary points at which the absolute extrema can occur.

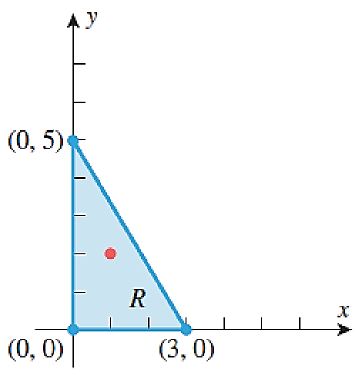
Step 3. Evaluate $f(x, y)$ at the points obtained in the preceding steps. The largest of these values is the absolute maximum and the smallest the absolute minimum.

Example 3.38 Find the absolute maximum and minimum values of

$$f(x, y) = 3xy - 6x - 3y + 7 \qquad (1)$$

on the closed triangular region R with vertices (0, 0), (3, 0), and (0, 5).

Solution: The region R is shown in Figure.



Step 1: find critical points

$$\partial f/\partial x = 3y - 6 \text{ and}$$

$$\partial f/\partial y = 3x - 3$$

so all critical points occur where

$$3y - 6 = 0 \text{ and } 3x - 3 = 0$$

Solving these equations yields $x = 1$ and $y = 2$, so (1, 2) is the only critical point. As shown in Figure, this critical point is in the interior of R.

Step 2: Determine the locations of the points on the boundary of R at which the absolute extrema might occur. The boundary of R consists of three line segments, each of which we will treat separately:

*The line segment between (*0, 0*) and (*3, 0*)*: On this line segment we have $y = 0$, so (1) simplifies to a function of the single variable x,

$$u(x) = f(x, 0) = -6x + 7, \ 0 \le x \le 3$$

This function has no critical points because $u(x) = -6$ is nonzero for all x. Thus the extreme values of $u'(x)$ occur at the endpoints $x = 0$ and $x = 3$, which correspond to the points (0, 0) and (3, 0) of R.

*The line segment between (*0, 0*) and (*0, 5*)*: On this line segment we have $x = 0$, so (1) simplifies to a function of the single variable y,

$$v(y) = f(0, y) = -3y + 7, \ 0 \le y \le 5$$

This function has no critical points because $v'(y) = -3$ is nonzero for all y. Thus, the extreme values of $v(y)$ occur at the endpoints $y = 0$ and $y = 5$, which correspond to the points $(0, 0)$ and $(0, 5)$ of R.

*The line segment between (*3, 0*) and (*0, 5*)*: In the xy-plane, an equation for this line segment is

$$y = -\tfrac{5}{3}x + 5, \quad 0 \le x \le 3$$

so (1) simplifies to a function of the single variable x,

$$\begin{aligned} w(x) &= f\left(x, -\tfrac{5}{3}x + 5\right) = 3x\left(-\tfrac{5}{3}x + 5\right) - 6x - 3\left(-\tfrac{5}{3}x + 5\right) + 7 \\ &= -5x^2 + 14x - 8, \qquad 0 \le x \le 3 \end{aligned}$$

Since $w'(x) = -10x + 14$, the equation $w(x) = 0$ yields $x = 7/5$ as the only critical point of w. Thus, the extreme values of w occur either at the critical point $x = 7/5$ or at the endpoints $x = 0$ and $x = 3$. The endpoints correspond to the points (0, 5) and (3, 0) of R, and from (4) the critical point corresponds to (7/5, 8/3).

Final step: the below table lists the values of $f(x, y)$ at the interior critical point and at the points on the boundary where an absolute extremum can occur. From the table we conclude that the absolute maximum value of f is $f(0, 0) = 7$ and the absolute minimum value is $f(3, 0) = -11$.

(x, y)	$(0, 0)$	$(3, 0)$	$(0, 5)$	$\left(\frac{7}{5}, \frac{8}{3}\right)$	$(1, 2)$
$f(x, y)$	7	-11	-8	$\frac{9}{5}$	1

Example 3.39 Determine the dimensions of a rectangular box, open at the top, having a volume of 32 ft^3, and requiring the least amount of material for its construction.

Solution: Let

x = length of the box (in feet)

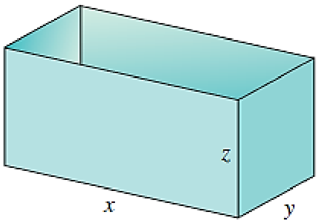
y = width of the box (in feet)

z = height of the box (in feet)

S = surface area of the box (in square feet)

We may reasonably assume that the box with least surface area requires the least amount of material, so our objective is to minimize the surface area

$$S = xy + 2xz + 2yz \qquad (1)$$



Two sides each have area xz.
Two sides each have area yz.
The base has area xy.

(Figure) subject to the volume requirement

$$xyz = 32 \qquad (2)$$

From (2) we obtain $z = 32/xy$, so (1) can be rewritten as

$$S = xy + \frac{64}{y} + \frac{64}{x} \qquad (3)$$

Differentiating (3) we obtain

$$\frac{\partial S}{\partial x} = y - \frac{64}{x^2}, \quad \frac{\partial S}{\partial y} = x - \frac{64}{y^2}$$

so the coordinates of the critical points of S satisfy

$$y - \frac{64}{x^2} = 0, \quad x - \frac{64}{y^2} = 0$$

Solving the first equation for y yields

$$y = \frac{64}{x^2}$$

and substituting this expression in the second equation yields

$$x - \frac{64}{(64/x^2)^2} = 0$$

which can be rewritten as

$$x\left(1 - \frac{x^3}{64}\right) = 0$$

The solutions of this equation are $x = 0$ and $x = 4$. Since we require $x > 0$, the only solution of significance is $x = 4$. Substituting this value into ($y=64/x^2$) yields $y = 4$. We conclude that the point $(x, y) = (4, 4)$ is the only critical point of S in the first quadrant. Since $S = 48$ if $x = y = 4$, this suggests we try to show that the minimum value of S on the open first quadrant is 48.

It immediately follows from Equation (3) that $48 < S$ at any point in the first quadrant for which at least one of the inequalities

$$xy > 48, \quad 64/y > 48, \quad 64/x > 48$$

is satisfied. Therefore, to prove that $48 \le S$, we can restrict attention to the set of points in the first quadrant that satisfy the three inequalities

$$xy \le 48, \quad 64/y \le 48, \quad 64/x \le 48$$

These inequalities can be rewritten as

$$xy \le 48, \qquad y \ge 4/3, \qquad x \ge 4/3$$

and they define a closed and bounded region R within the first quadrant (below figure). The function S is continuous on R, so Theorem guarantees that S has an absolute minimum value somewhere on R. Since the point $(4, 4)$ lies within R, and $48 < S$ on the boundary of R (why?), the minimum value of S on R must occur at an interior point. It then follows from Theorem that the mimimum value of S on R must occur at a critical point of S. Hence, the absolute minimum of S on R (and therefore on the entire open first quadrant) is $S = 48$ at the point $(4, 4)$. Substituting $x = 4$ and $y = 4$ into (6) yields $z = 2$, so the box using the least material has a height of 2 ft and a square base whose edges are 4ft long.