3.9 LAGRANGE MULTIPLIERS

3.9.1 Extremum Problems with Constraints

Three-Variable Extremum Problem with One Constraint

Maximize or minimize the function $f(x, y, z)$ subject to the constraint $g(x, y, z) = 0$.

Two-Variable Extremum Problem with One Constraint

Maximize or minimize the function $f(x, y)$ subject to the constraint $g(x, y) = 0$.

Theorem (***Constrained-Extremum Principle for Two Variables and One Constraint***) *Let f and g be functions of two variables with continuous first partial derivatives on some open set containing the constraint curve $g(x, y) = 0$, and assume that $\nabla g \neq \mathbf{0}$ at any point on this curve. If f has a constrained relative extremum, then this extremum occurs at a point (x_0, y_0) on the constraint curve at which the gradient vectors $\nabla f(x_0, y_0)$ and $\nabla g(x_0, y_0)$ are parallel; that is, there is some number λ such that*

$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$$

Example 3.40 At what point or points on the circle $x^2 + y^2 = 1$ does $f(x, y) = xy$ have an absolute maximum, and what is that maximum?

Solution: The circle $x^2 + y^2 = 1$ is a closed and bounded set and $f(x, y) = xy$ is a continuous function, so it follows from the Extreme-Value Theorem that f has an absolute maximum and an absolute minimum on the circle. To find these extrema, we will use Lagrange multipliers to find the constrained relative extrema, and then we will evaluate f at those relative extrema to find the absolute extrema.

We want to maximize $f(x, y) = xy$ subject to the constraint

$$g(x, y) = x^2 + y^2 - 1 = 0 \qquad (1)$$

First we will look for constrained *relative* extrema. For this purpose we will need the gradients

$$\nabla f = y\,\mathbf{i} + x\,\mathbf{j} \quad \text{and} \quad \nabla g = 2x\,\mathbf{i} + 2y\,\mathbf{j}$$

From the formula for ∇g we see that $\nabla g = \mathbf{0}$ if and only if $x = 0$ and $y = 0$, so $\nabla g \neq \mathbf{0}$ at any point on the circle $x^2 + y^2 = 1$. Thus, at a constrained relative extremum we must have

$$\nabla f = \lambda \nabla g \quad \text{or} \quad y\,\mathbf{i} + x\,\mathbf{j} = \lambda(2x\,\mathbf{i} + 2y\,\mathbf{j})$$

which is equivalent to the pair of equations

$$y = 2x\lambda \text{ and } x = 2y\lambda$$

It follows from these equations that if $x = 0$, then $y = 0$, and if $y = 0$, then $x = 0$. In either case we have $x^2 + y^2 = 0$, so the constraint equation $x^2 + y^2 = 1$ is not satisfied. Thus, we can assume that x and y are nonzero, and we can rewrite the equations as

$$\lambda = y/2x \qquad \text{and} \qquad \lambda = x/2y$$

from which we obtain

$$y/2x = x/2y$$

or

$$y^2 = x^2 \qquad (2)$$

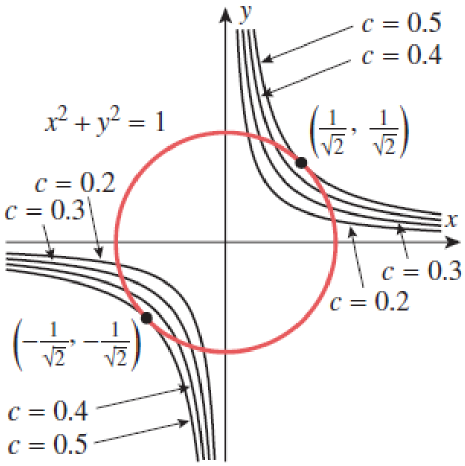
Substituting this in (1) yields

$$2x^2 - 1 = 0$$

from which we obtain $x = \pm 1/\sqrt{2}$. Each of these values, when substituted in Equation (2), produces y-values of $y = \pm 1/\sqrt{2}$. Thus, constrained relative extrema occur at the points $(1/\sqrt{2}, 1/\sqrt{2})$, $(1/\sqrt{2}, -1/\sqrt{2})$, $(-1/\sqrt{2}, 1/\sqrt{2})$, and $(-1/\sqrt{2}, -1/\sqrt{2})$. The values of xy at these points are as follows:

(x, y)	$(1/\sqrt{2}, 1/\sqrt{2})$	$(1/\sqrt{2}, -1/\sqrt{2})$	$(-1/\sqrt{2}, 1/\sqrt{2})$	$(-1/\sqrt{2}, -1/\sqrt{2})$
xy	1/2	−1/2	−1/2	1/2

Thus, the function $f(x, y) = xy$ has an absolute maximum of 1/2 occurring at the two points $(1/\sqrt{2}, 1/\sqrt{2})$ and $(-1/\sqrt{2}, -1/\sqrt{2})$. Although it was not asked for, we can also see that f has an absolute minimum of −1/2 occurring at the points $(1/\sqrt{2}, -1/\sqrt{2})$ and $(-1/\sqrt{2}, 1/\sqrt{2})$. The below figure shows some level curves $xy = c$ and the constraint curve



Example 3.41 Use the method of Lagrange multipliers to find the dimensions of a rectangle with perimeter p and maximum area.

Solution: Let

x = length of the rectangle, y = width of the rectangle, A = area of the rectangle

We want to maximize $A = xy$ on the line segment

$$2x + 2y = p,\ 0 \le x, y \qquad (1)$$

that corresponds to the perimeter constraint. This segment is a closed and bounded set, and since $f(x, y) = xy$ is a continuous function, it follows from the Extreme-Value Theorem that f has an absolute maximum on this segment. This absolute maximum must also be a constrained relative maximum since f is 0 at the endpoints of the segment and positive elsewhere on the segment. If $g(x, y) = 2x + 2y$, then we have

$$\nabla f = y\mathbf{i} + x\,\mathbf{j} \text{ and } \nabla g = 2\mathbf{i} + 2\mathbf{j}$$

Noting that $\nabla g \neq \mathbf{0}$, it follows from (4) that

$$y\,\mathbf{i} + x\,\mathbf{j} = \lambda(2\,\mathbf{i} + 2\,\mathbf{j})$$

at a constrained relative maximum. This is equivalent to the two equations

$$y = 2\lambda \quad \text{and} \quad x = 2\lambda$$

Eliminating λ from these equations we obtain $x = y$, which shows that the rectangle is actually a square. Using this condition and constraint (1), we obtain $x = p/4$, $y = p/4$.

3.9.2 Three Variables and One Constraint

Theorem (***Constrained-Extremum Principle for Three Variables and One Constraint***) *Let f and g be functions of three variables with continuous first partial derivatives on some open set containing the constraint surface $g(x, y, z) = 0$, and assume that $\nabla g \neq \mathbf{0}$ at any point on this surface. If f has a constrained relative extremum, then this extremum occurs at a point (x_0, y_0, z_0) on the constraint surface at which the gradient vectors $\nabla f(x_0, y_0, z_0)$ and $\nabla g(x_0, y_0, z_0)$ are parallel; that is, there is some number λ such that*

$$\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0)$$

Example 3.41 Find the points on the sphere $x^2 + y^2 + z^2 = 36$ that are closest to and farthest from the point (1, 2, 2).

Solution: To avoid radicals, we will find points on the sphere that minimize and maximize the *square* of the distance to (1, 2, 2). Thus, we want to find the relative extrema of

$$f(x, y, z) = (x - 1)^2 + (y - 2)^2 + (z - 2)^2$$

subject to the constraint

$$x^2 + y^2 + z^2 = 36 \qquad (1)$$

If we let $g(x, y, z) = x^2 + y^2 + z^2$, then $\nabla g = 2x\mathbf{i} + 2y\,\mathbf{j} + 2z\mathbf{k}$. Thus, $\nabla g = \mathbf{0}$ if and only if $x = y = z = 0$. It follows that $\nabla g \neq \mathbf{0}$ at any point of the sphere (1), and hence the constrained relative extrema must occur at points where

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$$

That is,

$$2(x-1)\mathbf{i} + 2(y-2)\,\mathbf{j} + 2(z-2)\mathbf{k} = \lambda(2x\mathbf{i} + 2y\,\mathbf{j} + 2z\mathbf{k})$$

which leads to the equations

$$2(x-1) = 2x\lambda,\ 2(y-2) = 2y\lambda,\ 2(z-2) = 2z\lambda \qquad (2)$$

We may assume that x, y, and z are nonzero since $x = 0$ does not satisfy the first equation, $y = 0$ does not satisfy the second, and $z = 0$ does not satisfy the third. Thus, we can rewrite (2) as

$$\frac{x-1}{x} = \lambda, \quad \frac{y-2}{y} = \lambda, \quad \frac{z-2}{z} = \lambda$$

The first two equations imply that

$$\frac{x-1}{x} = \frac{y-2}{y}$$

from which it follows that

$$y = 2x \qquad (3)$$

Similarly, the first and third equations imply that

$$z = 2x \qquad (4)$$

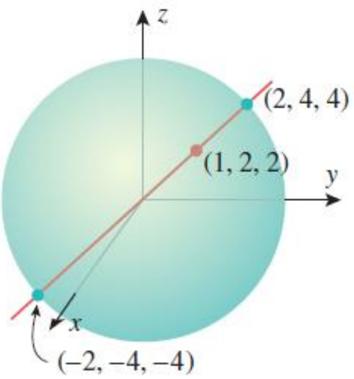
Substituting (3) and (4) in the constraint equation (1), we obtain

$$9x^2 = 36 \text{ or } x = \pm 2$$

Substituting these values in (3) and (4) yields two points:

$$(2, 4, 4) \text{ and } (-2, -4, -4)$$

Since $f(2, 4, 4) = 9$ and $f(-2,-4,-4) = 81$, it follows that $(2, 4, 4)$ is the point on the sphere closest to (1, 2, 2), and $(-2,-4,-4)$ is the point that is farthest (the following figure).



Example 3.42 Use Lagrange multipliers to determine the dimensions of a rectangular box, open at the top, having a volume of 32 ft^3, and requiring the least amount of material for its construction.

Solution: the problem is to minimize the surface area

$$S = xy + 2xz + 2yz$$

subject to the volume constraint

$$xyz = 32 \qquad (1)$$

If we let $f(x, y, z) = xy + 2xz + 2yz$ and $g(x, y, z) = xyz$, then

$$\nabla f = (y + 2z)\,\mathbf{i} + (x + 2z)\,\mathbf{j} + (2x + 2y)\,\mathbf{k} \text{ and } \nabla g = yz\,\mathbf{i} + xz\,\mathbf{j} + xy\,\mathbf{k}$$

It follows that $\nabla g \neq \mathbf{0}$ at any point on the surface $xyz = 32$, since x, y, and z are all nonzero on this surface. Thus, at a constrained relative extremum we must have $\nabla f = \lambda \nabla g$, that is,

$$(y + 2z)\mathbf{i} + (x + 2z)\mathbf{j} + (2x + 2y)\mathbf{k} = \lambda(yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k})$$

This condition yields the three equations

$$y + 2z = \lambda yz,\ x + 2z = \lambda xz,\ 2x + 2y = \lambda xy$$

Because x, y, and z are nonzero, these equations can be rewritten as

$$\frac{1}{z} + \frac{2}{y} = \lambda, \quad \frac{1}{z} + \frac{2}{x} = \lambda, \quad \frac{2}{y} + \frac{2}{x} = \lambda$$

From the first two equations,

$$y = x \qquad (2)$$

and from the first and third equations,

$$z = (½)\,x \qquad (3)$$

Substituting (2) and (3) in the volume constraint (1) yields

$$(1/2)\,x^3 = 32$$

This equation, together with (13) and (14), yields

$$x = 4,\ y = 4,\ z = 2$$

CHAPTER FOUR

DOUBLE INTEGRALS

4.1 DOUBLE INTEGRALS

4.1.1 Volume

Recall that the definite integral of a function of one variable

$$\int_a^b f(x)\,dx = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k = \lim_{n\to+\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x_k \qquad (1)$$

The volume problem Given a function *f* of two variables that is continuous and nonnegative on a region *R* in the *xy*-plane, find the volume of the solid enclosed between the surface $z = f(x, y)$ and the region *R* (Figure 1).

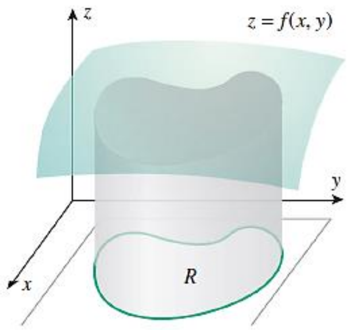


Figure 1

Definition 4.1 (***Volume Under a Surface***) If *f* is a function of two variables that is continuous and nonnegative on a region *R* in the *xy*-plane, then the volume of the solid enclosed between the surface $z = f(x, y)$ and the region *R* is defined by

$$V = \lim_{n\to+\infty} \sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta A_k \qquad (2)$$

Here, $n \to +\infty$ indicates the process of increasing the number of sub-rectangles of the rectangle enclosing *R* in such a way that both the lengths and the widths of the sub-rectangles approach zero.

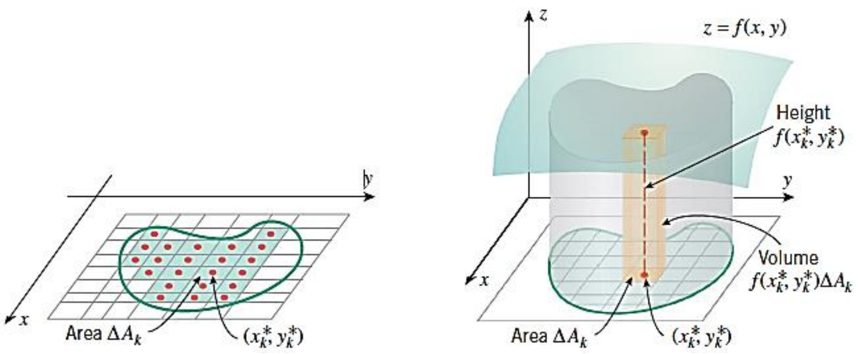


Figure 2

4.1.2 Definition of a Double Integral

As in Definition 4.1, the notation $n \to +\infty$ encapsulate a process in which the enclosing rectangle for R is repeatedly subdivided in such a way that both the lengths and the widths of the sub-rectangles approach zero.

$$\iint_R f(x, y)\, dA = \lim_{n \to +\infty} \sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta A_k$$

which is called the ***double integral*** of $f(x, y)$ over R.

If f is continuous and nonnegative on the region R, then the volume formula in (2) can be expressed as

$$V = \iint_R f(x, y)\, dA$$

4.1.3 Evaluating Double Integrals

The partial derivatives of a function $f(x, y)$ are calculated by holding one of the variables fixed and differentiating with respect to the other variable. Let us consider the reverse of this process, ***partial integration***. The symbols

$$\int_a^b f(x, y)\, dx \quad \text{and} \quad \int_c^d f(x, y)\, dy$$

denote ***partial definite integrals***; the first integral, called the ***partial definite integral with respect to x***, is evaluated by holding y fixed and integrating with respect to x, and the second integral, called the ***partial definite integral with respect to y***, is evaluated by holding x fixed and integrating with respect to y. As the following example shows, the partial definite integral with respect to x is a function of y, and the partial definite integral with respect to y is a function of x.

Example 4.1

$$\int_0^1 xy^2\,dx = y^2\int_0^1 x\,dx = \frac{y^2x^2}{2}\Big]_{x=0}^{1} = \frac{y^2}{2}$$

$$\int_0^1 xy^2\,dy = x\int_0^1 y^2\,dy = \frac{xy^3}{3}\Big]_{y=0}^{1} = \frac{x}{3}$$

A partial definite integral with respect to x is a function of y and hence can be integrated with respect to y; similarly, a partial definite integral with respect to y can be integrated with respect to x. This two-stage integration process is called ***iterated*** (or ***repeated***) ***integration***. We introduce the following notation:

$$\int_c^d\int_a^b f(x,y)\,dx\,dy = \int_c^d\left[\int_a^b f(x,y)\,dx\right]dy$$

$$\int_a^b\int_c^d f(x,y)\,dy\,dx = \int_a^b\left[\int_c^d f(x,y)\,dy\right]dx$$

These integrals are called ***iterated integrals***.

Example 4.2 Evaluate

(a) $\displaystyle\int_1^3\int_2^4 (40-2xy)\,dy\,dx$ (b) $\displaystyle\int_2^4\int_1^3 (40-2xy)\,dx\,dy$

Solution (a):

$$\begin{aligned}\int_1^3\int_2^4 (40-2xy)\,dy\,dx &= \int_1^3\left[\int_2^4 (40-2xy)\,dy\right]dx\\ &= \int_1^3 (40y-xy^2)\Big]_{y=2}^{4}\,dx\\ &= \int_1^3 [(160-16x)-(80-4x)]\,dx\\ &= \int_1^3 (80-12x)\,dx\\ &= (80x-6x^2)\Big]_1^3 = 112\end{aligned}$$

Solution (b):

$$\begin{aligned}\int_2^4\int_1^3 (40-2xy)\,dx\,dy &= \int_2^4\left[\int_1^3 (40-2xy)\,dx\right]dy\\ &= \int_2^4 (40x-x^2y)\Big]_{x=1}^{3}\,dy\\ &= \int_2^4 [(120-9y)-(40-y)]\,dy\\ &= \int_2^4 (80-8y)\,dy\\ &= (80y-4y^2)\Big]_2^4 = 112\end{aligned}$$

Consider the solid S bounded above by the surface $z = 40 - 2xy$ and below by the rectangle R defined by $1 \le x \le 3$ and $2 \le y \le 4$. The volume of S is given by

$$V = \int_1^3 A(x)\,dx$$

where $A(x)$ is the area of a vertical cross section of S taken perpendicular to the x-axis (Figure 3). For a fixed value of x, $1 \le x \le 3$, $z = 40 - 2xy$ is a function of y, so the

$$A(x) = \int_2^4 (40 - 2xy)\,dy$$

represents the area under the graph of this function of y. Thus,

$$V = \int_1^3 \left[\int_2^4 (40 - 2xy)\,dy\right] dx = \int_1^3 \int_2^4 (40 - 2xy)\,dy\,dx$$

is the volume of S. Similarly, by the method of slicing with cross sections of S taken perpendicular to the y-axis, the volume of S is given by

$$V = \int_2^4 A(y)\,dy = \int_2^4 \left[\int_1^3 (40 - 2xy)\,dx\right] dy = \int_2^4 \int_1^3 (40 - 2xy)\,dx\,dy$$

(Figure 4). Thus, the iterated integrals in parts (a) and (b) of Example both measure the volume of S, which is the double integral of $z = 40 - 2xy$ over R. That is,

$$\int_1^3 \int_2^4 (40 - 2xy)\,dy\,dx = \iint_R (40 - 2xy)\,dA = \int_2^4 \int_1^3 (40 - 2xy)\,dx\,dy$$

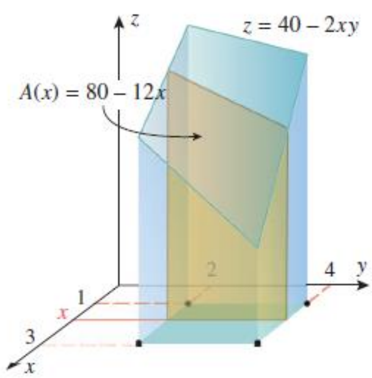


Figure 3

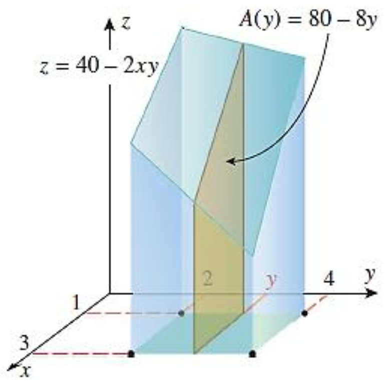


Figure 4

Theorem (***Fubini's Theorem***) *Let R be the rectangle defined by the inequalities*

$$a \le x \le b,\ c \le y \le d$$

If f(x, y) is continuous on this rectangle, then

$$\iint_R f(x, y)\,dA = \int_c^d \int_a^b f(x, y)\,dx\,dy = \int_a^b \int_c^d f(x, y)\,dy\,dx$$

Example 4.3 Use a double integral to find the volume of the solid that is bounded above by the plane $z = 4 - x - y$ and below by the rectangle $R = [0, 1] \times [0, 2]$ (Figure 5).

Solution: The volume is the double integral of $z = 4 - x - y$ over R. Using Theorem, this can be obtained from either of the iterated integrals

$$\int_0^2 \int_0^1 (4 - x - y)\, dx\, dy \quad \text{or} \quad \int_0^1 \int_0^2 (4 - x - y)\, dy\, dx$$

Using the first of these, we obtain

$$V = \iint_R (4 - x - y)\, dA = \int_0^2 \int_0^1 (4 - x - y)\, dx\, dy$$

$$= \int_0^2 \left[4x - \frac{x^2}{2} - xy\right]_{x=0}^{1} dy = \int_0^2 \left(\frac{7}{2} - y\right) dy$$

$$= \left[\frac{7}{2}y - \frac{y^2}{2}\right]_0^2 = 5$$

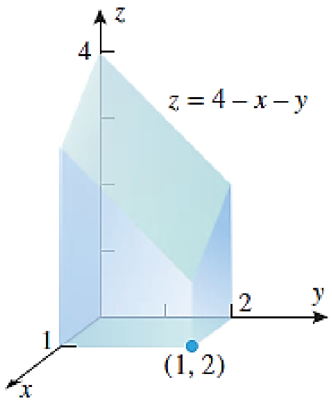


Figure 5

4.1.4 Properties of Double Integrals

$$\iint_R cf(x, y)\, dA = c \iint_R f(x, y)\, dA \quad (c \text{ a constant})$$

$$\iint_R [f(x, y) + g(x, y)]\, dA = \iint_R f(x, y)\, dA + \iint_R g(x, y)\, dA$$

$$\iint_R [f(x, y) - g(x, y)]\, dA = \iint_R f(x, y)\, dA - \iint_R g(x, y)\, dA$$

$$\iint_R f(x, y)\, dA = \iint_{R_1} f(x, y)\, dA + \iint_{R_2} f(x, y)\, dA$$