4.2 DOUBLE INTEGRALS OVER NONRECTANGULAR REGIONS

4.2.1 Iterated Integrals with Non-constant Limits of Integration

$$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} f(x, y)\, dy \right] dx$$

$$\int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy = \int_c^d \left[\int_{h_1(y)}^{h_2(y)} f(x, y)\, dx \right] dy$$

Example 4.4 Evaluate

$$\text{(a)} \int_0^1 \int_{-x}^{x^2} y^2 x\, dy\, dx \qquad \text{(b)} \int_0^{\pi/3} \int_0^{\cos y} x \sin y\, dx\, dy$$

Solution (a):

$$\int_{-x}^{x^2} y^2 x\, dy\, dx = \int_0^1 \left[\int_{-x}^{x^2} y^2 x\, dy \right] dx = \int_0^1 \frac{y^3 x}{3} \Bigg]_{y=-x}^{x^2} dx$$

$$= \int_0^1 \left[\frac{x^7}{3} + \frac{x^4}{3} \right] dx = \left(\frac{x^8}{24} + \frac{x^5}{15} \right) \Bigg]_0^1 = \frac{13}{120}$$

Solution (b):

$$\int_0^{\pi/3} \int_0^{\cos y} x \sin y\, dx\, dy = \int_0^{\pi/3} \left[\int_0^{\cos y} x \sin y\, dx \right] dy = \int_0^{\pi/3} \frac{x^2}{2} \sin y \Bigg]_{x=0}^{\cos y} dy$$

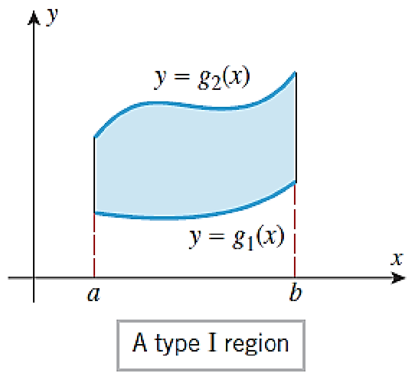
$$= \int_0^{\pi/3} \left[\frac{1}{2} \cos^2 y \sin y \right] dy = -\frac{1}{6} \cos^3 y \Bigg]_0^{\pi/3} = \frac{7}{48}$$

4.2.2 Double Integrals over Nonrectangular Regions

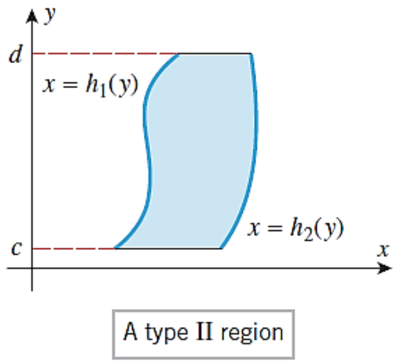
Definition

(a) A ***type I region*** is bounded on the left and right by vertical lines $x = a$ and $x = b$ and is bounded below and above by continuous curves $y = g_1(x)$ and $y = g_2(x)$, where $g_1(x) \le g_2(x)$ for $a \le x \le b$ (Figure *a*).

(b) A ***type II region*** is bounded below and above by horizontal lines $y = c$ and $y = d$ and is bounded on the left and right by continuous curves $x = h_1(y)$ and $x = h_2(y)$ satisfying $h_1(y) \le h_2(y)$ for $c \le y \le d$ (Figure *b*).



(a)



(b)

Theorem

(a) If R is a type I region on which f(x, y) is continuous, then

$$\iint_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx \tag{1}$$

(b) If R is a type II region on which f(x, y) is continuous, then

$$\iint_R f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy \tag{2}$$

Example 4.5 Each of the iterated integrals in Example 4.4 is equal to a double integral over a region R. Identify the region R in each case.

Solution: Using Theorem, the integral in Example 4.4(a) is the double integral of the function $f(x, y) = y^2x$ over the type I region R bounded on the left and right by the vertical lines $x = 0$ and $x = 1$ and bounded below and above by the curves $y = -x$ and $y = x^2$ (Figure a). The integral in Example 4.4(b) is the double integral of the function $f(x, y) = x \sin y$ over the type II region R bounded below and above by the horizontal lines $y = 0$ and $y = \pi/3$ and bounded on the left and right by the curves $x = 0$ and $x = \cos y$ (Figure b).

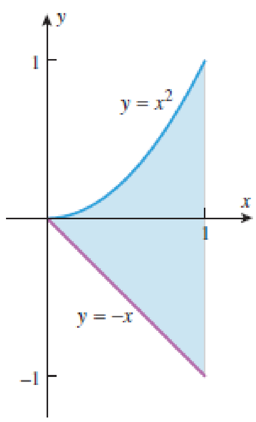


Figure a

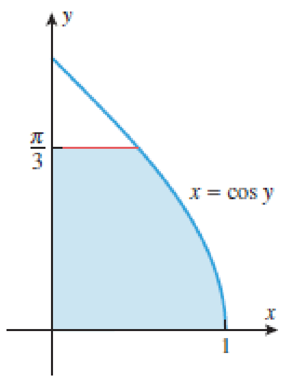
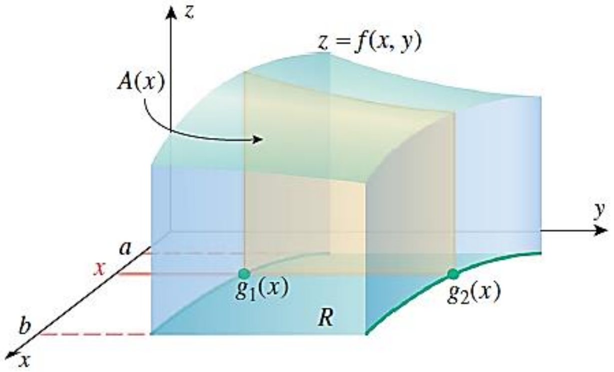


Figure b

$$A(x) = \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy$$

$$V = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx$$

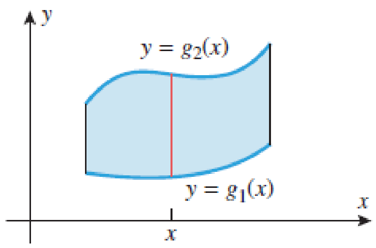


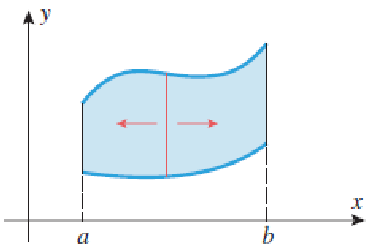
4.2.3 Setting up Limits of Integration for Evaluating Double Integrals

Determining Limits of Integration: Type I Region

Step 1. Since x is held fixed for the first integration, we draw a vertical line through the region R at an arbitrary fixed value x (below figure). This line crosses the boundary of R twice. The lower point of intersection is on the curve $y = g_1(x)$ and the higher point is on the curve $y = g_2(x)$. These two intersections determine the lower and upper y-limits of integration in Formula (1).

Step 2. Imagine moving the line drawn in Step 1 first to the left and then to the right (below figure). The leftmost position where the line intersects the region R is $x = a$, and the rightmost position where the line intersects the region R is $x = b$. This yields the limits for the x-integration in Formula (1).





Example 4.6 Evaluate

$$\iint_R xy\,dA$$

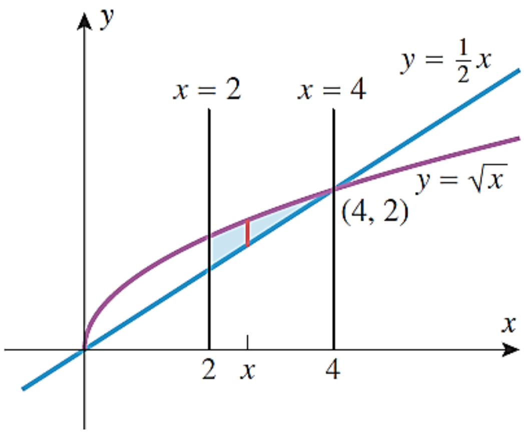
over the region R enclosed between $y = 1/2\ x$, $y = \sqrt{x}$, $x = 2$, and $x = 4$.

Solution:

We view R as a type I region. The region R and a vertical line corresponding to a fixed x are shown in Figure a. This line meets the region R at the lower boundary $y = (½)x$ and the upper boundary $y = \sqrt{x}$. These are the y-limits of integration. Moving this line first left and then right yields the x-limits of integration, $x = 2$ and $x = 4$. Thus,

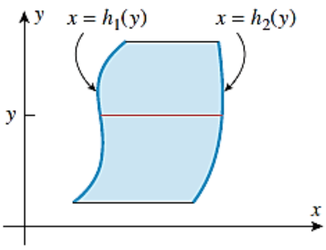
$$\iint_R xy\,dA = \int_2^4 \int_{x/2}^{\sqrt{x}} xy\,dy\,dx = \int_2^4 \left[\frac{xy^2}{2}\right]_{y=x/2}^{\sqrt{x}} dx = \int_2^4 \left(\frac{x^2}{2} - \frac{x^3}{8}\right) dx$$

$$= \left[\frac{x^3}{6} - \frac{x^4}{32}\right]_2^4 = \left(\frac{64}{6} - \frac{256}{32}\right) - \left(\frac{8}{6} - \frac{16}{32}\right) = \frac{11}{6}$$

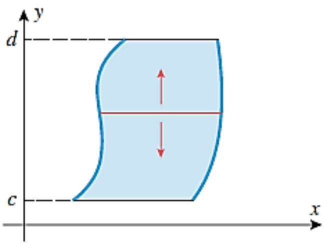


Determining Limits of Integration: Type II Region

Step 1. Since y is held fixed for the first integration, we draw a horizontal line through the region R at a fixed value y (following figure). This line crosses the boundary of R twice. The leftmost point of intersection is on the curve $x = h_1(y)$ and the rightmost point is on the curve $x = h_2(y)$. These intersections determine the x-limits of integration in (2).



Step 2. Imagine moving the line drawn in Step 1 first down and then up (following figure). The lowest position where the line intersects the region R is $y = c$, and the highest position where the line intersects the region R is $y = d$. This yields the y-limits of integration in (2).



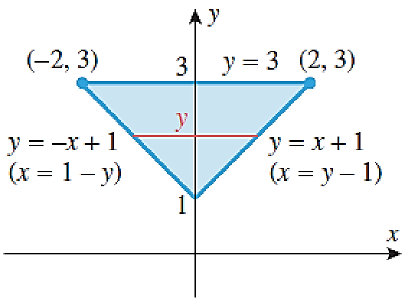
Example 4.7 Evaluate

$$\iint_R (2x - y^2)\, dA$$

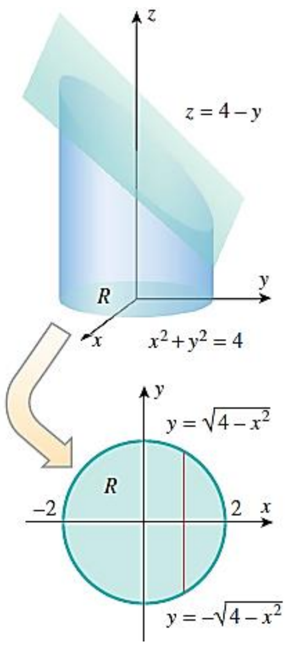
over the triangular region R enclosed between the lines $y = -x + 1$, $y = x + 1$, and $y = 3$.

Solution: We view R as a type II region. The region R and a horizontal line corresponding to a fixed y are shown in below figure. This line meets the region R at its left-hand boundary $x = 1 - y$ and its right-hand boundary $x = y - 1$. These are the x-limits of integration. Moving this line first down and then up yields the y-limits, $y = 1$ and $y = 3$. Thus,

$$\begin{aligned}\iint_R (2x - y^2)\, dA &= \int_1^3 \int_{1-y}^{y-1} (2x - y^2)\, dx\, dy = \int_1^3 \left[x^2 - y^2 x\right]_{x=1-y}^{y-1} dy \\ &= \int_1^3 [(1 - 2y + 2y^2 - y^3) - (1 - 2y + y^3)]\, dy \\ &= \int_1^3 (2y^2 - 2y^3)\, dy = \left[\frac{2y^3}{3} - \frac{y^4}{2}\right]_1^3 = -\frac{68}{3}\end{aligned}$$



Example 4.8 Find the volume of the solid bounded by the cylinder $x^2 + y^2 = 4$ and the planes $y + z = 4$ and $z = 0$.



Solution: The solid shown in below figure is bounded above by the plane $z = 4 - y$ and below by the region R within the circle $x^2 + y^2 = 4$. The volume is given by

$$V = \iint_R (4 - y)\, dA$$

Treating R as a type I region we obtain

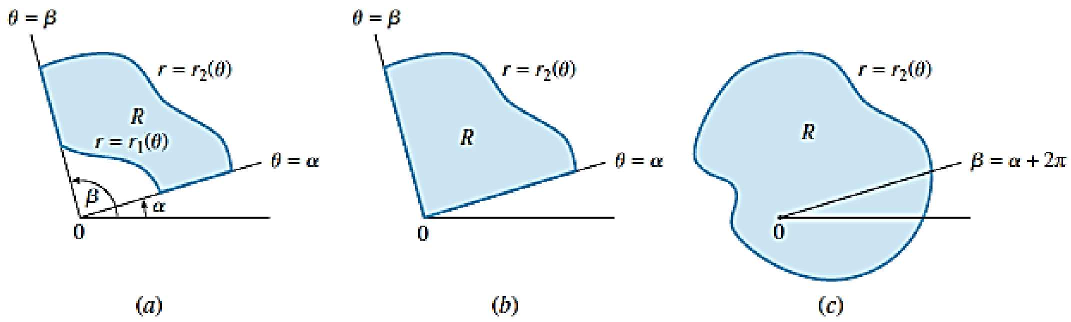
$$\begin{aligned}V &= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4 - y)\, dy\, dx = \int_{-2}^{2} \left[4y - \frac{1}{2}y^2\right]_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx \\ &= \int_{-2}^{2} 8\sqrt{4 - x^2}\, dx = 8(2\pi) = 16\pi\end{aligned}$$

4.3 DOUBLE INTEGRALS IN POLAR COORDINATES

4.3.1 Simple Polar Regions

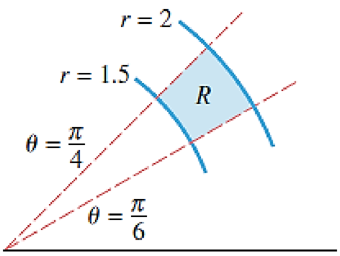
Definition A ***simple polar region*** in a polar coordinate system is a region that is enclosed between two rays, $\theta = \alpha$ and $\theta = \beta$, and two continuous polar curves, $r = r_1(\theta)$ and $r = r_2(\theta)$, where the equations of the rays and the polar curves satisfy the following conditions:

(i) $\alpha \le \beta$ (ii) $\beta - \alpha \le 2\pi$ (iii) $0 \le r_1(\theta) \le r_2(\theta)$



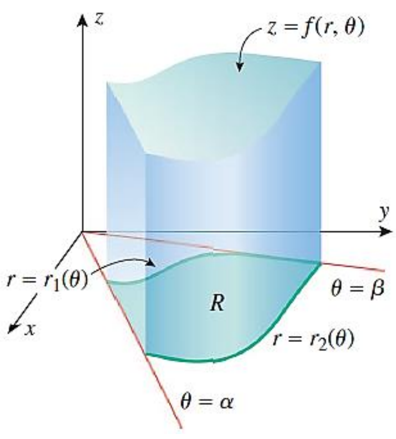
Simple polar regions

A ***polar rectangle*** is a simple polar region for which the bounding polar curves are circular arcs. For example, the following Figure shows the polar rectangle R given by $1.5 \le r \le 2$, $\pi/6 \le \theta \le \pi/4$



4.3.2 Double Integrals in Polar Coordinates

The volume problem in polar coordinates Given a function $f(r, \theta)$ that is continuous and non-negative on a simple polar region R, find the volume of the solid that is enclosed between the region R and the surface whose equation in cylindrical coordinates is $z = f(r, \theta)$ (see the figure).



If $f(r, \theta)$ is continuous on R and has both positive and negative values, then the limit

$$\lim_{n \to +\infty} \sum_{k=1}^{n} f(r_k^*, \theta_k^*) \Delta A_k$$

represents the net signed volume between the region R and the surface $z = f(r, \theta)$ (as with double integrals in rectangular coordinates). The sums are called ***polar Riemann sums***, and the limit of the polar Riemann sums is denoted by

$$\iint_R f(r, \theta)\, dA = \lim_{n \to +\infty} \sum_{k=1}^{n} f(r_k^*, \theta_k^*) \Delta A_k$$

which is called the ***polar double integral*** of $f(r, \theta)$ over R. If $f(r, \theta)$ is continuous and nonnegative on R, then the volume can be expressed as

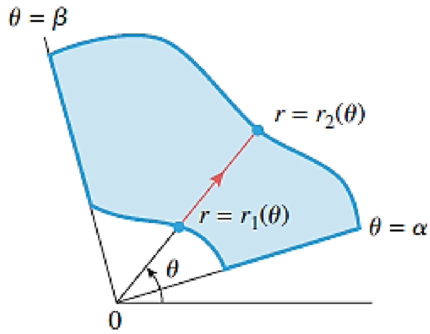
$$V = \iint_R f(r, \theta)\, dA$$

4.3.3 Evaluating Polar Double Integrals

Theorem

If R is a simple polar region whose boundaries are the rays $\theta = \alpha$ and $\theta = \beta$ and the curves $r = r_1(\theta)$ and $r = r_2(\theta)$ shown in the below figure, and if $f(r, \theta)$ is continuous on R, then

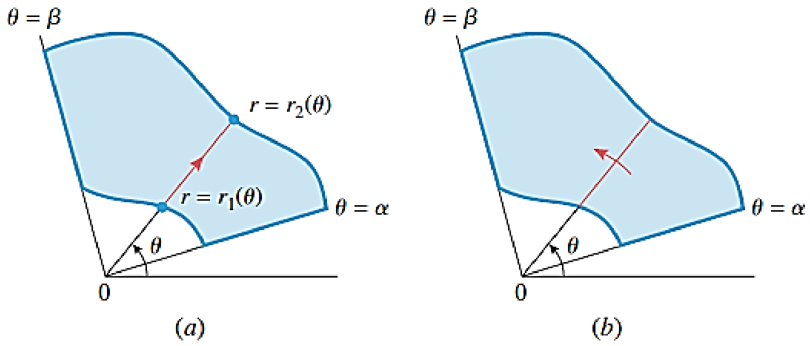
$$\iint_R f(r, \theta)\, dA = \int_{\alpha}^{\beta} \int_{r_1(\theta)}^{r_2(\theta)} f(r, \theta) r\, dr\, d\theta \qquad (1)$$



Determining Limits of Integration for a Polar Double Integral: Simple Polar Region

Step 1. Since θ is held fixed for the first integration, draw a radial line from the origin through the region R at a fixed angle θ (Figure *a*). This line crosses the boundary of R at most twice. The innermost point of intersection is on the inner boundary curve $r = r_1(\theta)$ and the outermost point is on the outer boundary curve $r = r_2(\theta)$. These intersections determine the r-limits of integration in (1).

Step 2. Imagine rotating the radial line from Step 1 about the origin, thus sweeping out the region R. The least angle at which the radial line intersects the region R is $\theta = \alpha$and the greatest angle is $\theta = \beta$ (Figure b). This determines the θ-limits of integration.

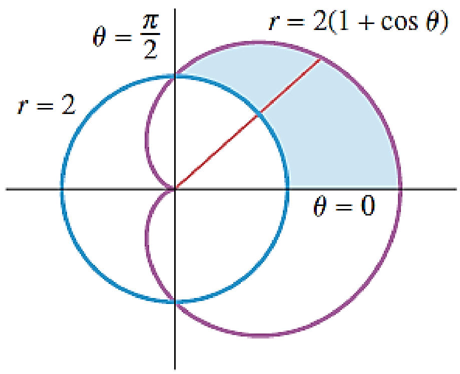


Example 4.11 Evaluate

$$\iint_R \sin\theta \, dA$$

where R is the region in the first quadrant that is outside the circle $r = 2$ and inside the cardioid $r = 2(1 + \cos\theta)$.

Solution: The region R is sketched in the following figure. Following the two steps outlined above we obtain



$$\iint_R \sin\theta \, dA = \int_0^{\pi/2}\int_2^{2(1+\cos\theta)} (\sin\theta) r \, dr \, d\theta$$

$$= \int_0^{\pi/2} \left[\frac{1}{2}r^2 \sin\theta\right]_{r=2}^{2(1+\cos\theta)} d\theta$$

$$= 2\int_0^{\pi/2} [(1+\cos\theta)^2 \sin\theta - \sin\theta] \, d\theta$$

$$= 2\left[-\frac{1}{3}(1+\cos\theta)^3 + \cos\theta\right]_0^{\pi/2}$$

$$= 2\left[-\frac{1}{3} - \left(-\frac{5}{3}\right)\right] = \frac{8}{3}$$

Example 4.12 The sphere of radius a centered at the origin is expressed in rectangular coordinates as $x^2 + y^2 + z^2 = a^2$, and hence its equation in cylindrical coordinates is $r^2 + z^2 = a^2$. Use this equation and a polar double integral to find the volume of the sphere.

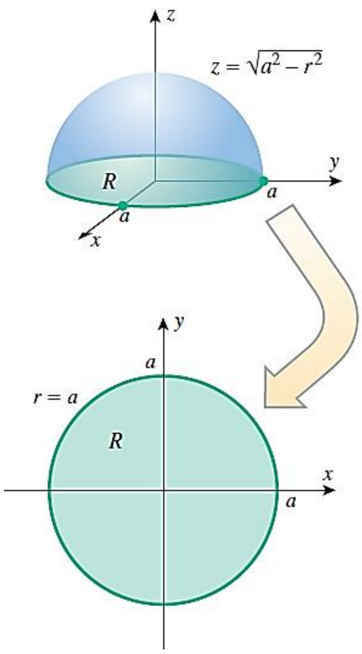
Solution: In cylindrical coordinates the upper hemisphere is given by the equation

$$z = \sqrt{a^2 - r^2}$$

so the volume enclosed by the entire sphere is

$$V = 2\iint_R \sqrt{a^2 - r^2}\, dA$$

where R is the circular region shown in following figure. Thus,



$$V = 2\iint_R \sqrt{a^2 - r^2}\, dA = \int_0^{2\pi}\int_0^a \sqrt{a^2 - r^2}(2r)\, dr\, d\theta$$

$$= \int_0^{2\pi}\left[-\frac{2}{3}(a^2 - r^2)^{3/2}\right]_{r=0}^{a} d\theta = \int_0^{2\pi} \frac{2}{3}a^3\, d\theta$$

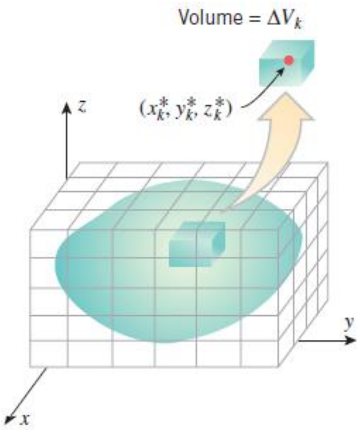
$$= \left[\frac{2}{3}a^3\theta\right]_0^{2\pi} = \frac{4}{3}\pi a^3$$

4.4 TRIPLE INTEGRALS

4.4.1 Definition of a Triple Integral

To define the triple integral of $f(x, y, z)$ over G, we first divide the box into n "sub-boxes" by planes parallel to the coordinate planes. We then discard those sub-boxes that contain any points outside of G and choose an arbitrary point in each of the remaining sub-boxes. As shown in the figure, we denote the volume of the kth remaining sub-box by ΔV_k and the point selected in the kth sub-box by (x_k^*, y_k^*, z_k^*). Next we form the product

$$f(x_k^*, y_k^*, z_k^*)\Delta V_k$$



for each sub-box, then add the products for all of the sub-boxes to obtain the ***Riemann sum***

$$\sum_{k=1}^{n} f(x_k^*, y_k^*, z_k^*)\Delta V_k$$

Finally, we repeat this process with more and more subdivisions in such a way that the length, width, and height of each sub-box approach zero, and n approaches $+\infty$. The limit

$$\iiint_G f(x, y, z)\, dV = \lim_{n\to+\infty} \sum_{k=1}^{n} f(x_k^*, y_k^*, z_k^*)\Delta V_k$$

is called the ***triple integral*** of *f(x, y, z)* over the region *G*.

4.4.2 Properties of Triple Integrals

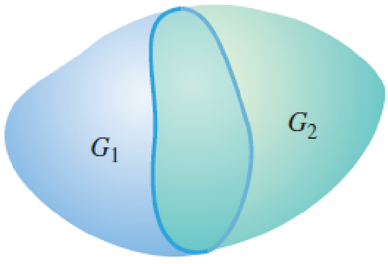
$$\iiint_G cf(x, y, z)\, dV = c\iiint_G f(x, y, z)\, dV \quad (c \text{ a constant})$$

$$\iiint_G [f(x, y, z) + g(x, y, z)]\, dV = \iiint_G f(x, y, z)\, dV + \iiint_G g(x, y, z)\, dV$$

$$\iiint_G [f(x, y, z) - g(x, y, z)]\, dV = \iiint_G f(x, y, z)\, dV - \iiint_G g(x, y, z)\, dV$$

Moreover, if the region G is subdivided into two sub-regions G_1 and G_2 (following figure), then

$$\iiint_G f(x, y, z)\, dV = \iiint_{G_1} f(x, y, z)\, dV + \iiint_{G_2} f(x, y, z)\, dV$$



4.4.3 Evaluating Triple Integrals over Rectangular Boxes

Theorem (***Fubini's Theorem****) *Let G be the rectangular box defined by the inequalities*

$$a \le x \le b,\ c \le y \le d,\ k \le z \le l$$

If f is continuous on the region G, then

$$\iiint_G f(x, y, z)\, dV = \int_a^b \int_c^d \int_k^l f(x, y, z)\, dz\, dy\, dx \qquad (1)$$

Moreover, the iterated integral on the right can be replaced with any of the five other iterated integrals that result by altering the order of integration.

Example 4.15 Evaluate the triple integral

$$\iiint_G 12xy^2z^3\, dV$$

over the rectangular box G defined by the inequalities $-1 \le x \le 2,\ 0 \le y \le 3,\ 0 \le z \le 2$.

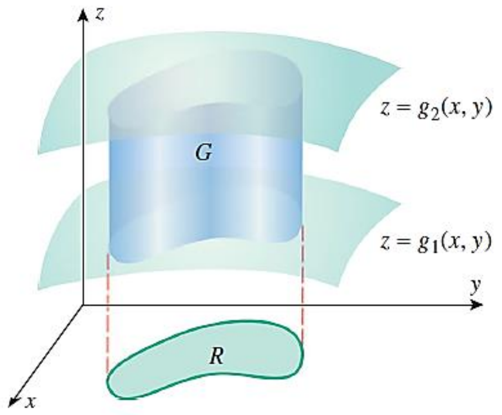
Solution: Of the six possible iterated integrals we might use, we will choose the one in (1). Thus, we will first integrate with respect to z, holding x and y fixed, then with respect to y, holding x fixed, and finally with respect to x.

$$\begin{aligned}\iiint_G 12xy^2z^3\, dV &= \int_{-1}^{2}\int_0^3\int_0^2 12xy^2z^3\, dz\, dy\, dx \\ &= \int_{-1}^{2}\int_0^3 \left[3xy^2z^4\right]_{z=0}^{2} dy\, dx = \int_{-1}^{2}\int_0^3 48xy^2\, dy\, dx \\ &= \int_{-1}^{2} \left[16xy^3\right]_{y=0}^{3} dx = \int_{-1}^{2} 432x\, dx \\ &= 216x^2\Big]_{-1}^{2} = 648\end{aligned}$$

4.4.4 Evaluating Triple Integrals over More General Regions

Theorem *Let G be a simple xy-solid with upper surface $z = g_2(x, y)$ and lower surface $z = g_1(x, y)$, and let R be the projection of G on the xy-plane. If $f(x, y, z)$ is continuous on G, then*

$$\iiint_G f(x, y, z)\, dV = \iint_R \left[\int_{g_1(x,y)}^{g_2(x,y)} f(x, y, z)\, dz\right] dA \qquad (2)$$



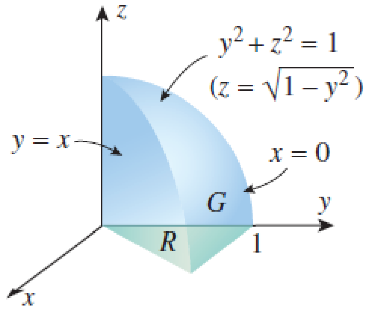
Determining Limits of Integration: Simple xy-Solid

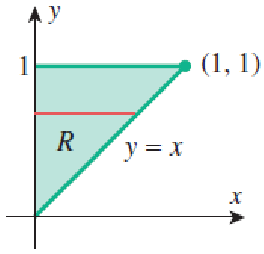
Step 1. Find an equation $z = g_2(x, y)$ for the upper surface and an equation $z = g_1(x, y)$ for the lower surface of G. The functions $g_1(x, y)$ and $g_2(x, y)$ determine the lower and upper z-limits of integration.

Step 2. Make a two-dimensional sketch of the projection R of the solid on the xy-plane. From this sketch determine the limits of integration for the double integral over R in (2).

Example 4.16 Let G be the wedge in the first octant that is cut from the cylindrical solid $y^2 + z^2 \le 1$ by the planes $y = x$ and $x = 0$. Evaluate

$$\iiint_G z\, dV$$





Solution. The solid G and its projection R on the xy-plane are shown in the figure. The upper surface of the solid is formed by the cylinder and the lower surface by the xy-plane. Since the portion of the cylinder $y^2 + z^2 = 1$ that lies above the xy-plane has the equation $z = \sqrt{1 - y^2}$, and the xy-plane has the equation $z = 0$, it follows from (2) that

$$\iiint_G z\, dV = \iint_R \left[\int_0^{\sqrt{1-y^2}} z\, dz\right] dA$$

For the double integral over R, the x- and y-integrations can be performed in either order, since R is both a type I and type II region. We will integrate with respect to x first. With this choice, yields

$$\iiint_G z\,dV = \int_0^1 \int_0^y \int_0^{\sqrt{1-y^2}} z\,dz\,dx\,dy = \int_0^1 \int_0^y \frac{1}{2}z^2\Big]_{z=0}^{\sqrt{1-y^2}} dx\,dy$$

$$= \int_0^1 \int_0^y \frac{1}{2}(1-y^2)\,dx\,dy = \frac{1}{2}\int_0^1 (1-y^2)x\Big]_{x=0}^{y} dy$$

$$= \frac{1}{2}\int_0^1 (y-y^3)\,dy = \frac{1}{2}\left[\frac{1}{2}y^2 - \frac{1}{4}y^4\right]_0^1 = \frac{1}{8}$$

4.4.5 Volume Calculated As a Triple Integral

$$\text{volume of } G = \iiint_G dV$$

Example 4.17 Use a triple integral to find the volume of the solid within the cylinder $x^2 + y^2 = 9$ and between the planes $z = 1$ and $x + z = 5$.

Solution: The solid G and its projection R on the xy-plane are shown in Figure. The lower surface of the solid is the plane $z = 1$ and the upper surface is the plane $x + z = 5$ or, equivalently, $z = 5 - x$. Thus,

$$\text{volume of } G = \iiint_G dV = \iint_R \left[\int_1^{5-x} dz\right] dA$$

For the double integral over R, we will integrate with respect to y first. Thus,

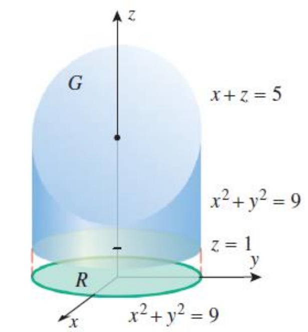
$$\text{volume of } G = \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_1^{5-x} dz\,dy\,dx = \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} z\Big]_{z=1}^{5-x} dy\,dx$$

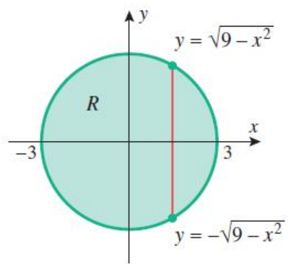
$$= \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (4-x)\,dy\,dx = \int_{-3}^{3} (8-2x)\sqrt{9-x^2}\,dx$$

$$= 8\int_{-3}^{3}\sqrt{9-x^2}\,dx - \int_{-3}^{3} 2x\sqrt{9-x^2}\,dx$$

$$= 8\left(\frac{9}{2}\pi\right) - \int_{-3}^{3} 2x\sqrt{9-x^2}\,dx$$

$$= 8\left(\frac{9}{2}\pi\right) - 0 = 36\pi \blacktriangleleft$$





Example 4.18 Find the volume of the solid enclosed between the paraboloids $z = 5x^2 + 5y^2$ and $z = 6 - 7x^2 - y^2$

Solution: The solid G and its projection R on the xy-plane are shown in Figure. The projection R is obtained by solving the given equations simultaneously to determine where the paraboloids intersect. We obtain

$$5x^2 + 5y^2 = 6 - 7x^2 - y^2$$

or

$$2x^2 + y^2 = 1$$

which tells us that the paraboloids intersect in a curve on the elliptic cylinder given by ($2x^2 + y^2 = 1$). The projection of this intersection on the xy-plane is an ellipse with this same equation. Therefore,

$$\text{volume of } G = \iiint\limits_G dV = \iint\limits_R \left[\int_{5x^2+5y^2}^{6-7x^2-y^2} dz\right] dA$$

$$= \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \int_{-\sqrt{1-2x^2}}^{\sqrt{1-2x^2}} \int_{5x^2+5y^2}^{6-7x^2-y^2} dz\, dy\, dx$$

$$= \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \int_{-\sqrt{1-2x^2}}^{\sqrt{1-2x^2}} (6 - 12x^2 - 6y^2)\, dy\, dx$$

$$= \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \Big[6(1-2x^2)y - 2y^3\Big]_{y=-\sqrt{1-2x^2}}^{\sqrt{1-2x^2}} dx$$

$$= 8 \int_{-1/\sqrt{2}}^{1/\sqrt{2}} (1-2x^2)^{3/2}\, dx = \frac{8}{\sqrt{2}} \int_{-\pi/2}^{\pi/2} \cos^4\theta\, d\theta = \frac{3\pi}{\sqrt{2}}$$