Chapter 1

(Measurements)

To describe natural phenomena, we must make measurements of various aspects of nature. Each measurement is associated with a physical quantity, such as the length of an object. The laws of physics are expressed as mathematical relationships among physical quantities.

In 1960, an international committee established a set of standards for the **fundamental quantities** of science. It is called the **SI** (System International), and its fundamental units of length, mass, and time are the *meter*, *kilogram*, and *second*, respectively. Other standards for SI fundamental units established by the committee are those for temperature (*kelvin*), electric current (*ampere*), luminous intensity (*candela*), and the amount of substance (*mole*).

In mechanics, **the fundamental quantities are length, mass, and time.** All other quantities in mechanics can be expressed in terms of these three.

Most other variables are **derived quantities**, those that can be expressed as a mathematical combination of fundamental quantities. Common examples are **area** (a product of two lengths) and **speed** (a ratio of a length to a time interval). Another example of a derived quantity is **density.**

The density ρ (Greek letter rho) of any substance is defined as its *mass per unit volume:*

$$\boldsymbol{\rho = \frac{m}{V}}$$

Chapter 2

(Motion in One Dimension)

2.1 Position, Velocity, and Speed

- A particle's **position** (x) is (The location of the particle with respect to a chosen reference point that we can consider to be the origin of a coordinate system).
- The **displacement** (Δx) of a particle is defined as (its change in position in some time interval). As the particle moves from an initial position (x_i) to a final position (x_f), its displacement is given by:

  $$\Delta x = x_f - x_i \qquad \text{Displacement} \qquad (2.1)$$

  We use the capital Greek letter delta (Δ) to denote the *change* in a quantity.
- From this definition, we see that (Δx) is positive if (x_f) is greater than (x_i) and negative if (x_f)is less than (x_i).

It is very important to recognize the difference between displacement and distance traveled.

- **Distance** (is the length of a path followed by a particle).
- Distance is always represented as a positive number, whereas displacement can be either positive or negative.
- Displacement is an example of a vector quantity. Many other physical quantities, including position, velocity, and acceleration, also are vectors.
- In general, a **vector quantity** requires the specification of both direction and magnitude. **Scalar quantity** has a numerical value and no direction.

- The **average velocity** ($v_{x,avg}$) of a particle is defined as (the particle's displacement (Δx) divided by the time interval (Δt) during which that displacement occurs):

$$v_{x,avg} = \frac{\Delta x}{\Delta t} \qquad (2.2)$$

Where, the subscript (x) indicates motion along the (x-axis).

From this definition we see that average velocity has dimensions of length divided by time, or meters per second (m/s) in SI units.

- The average velocity of a particle moving in one dimension can be positive or negative, depending on the sign of the displacement.
- The time interval (Δt) is always positive.

If the velocity of a particle is constant, its instantaneous velocity at any instant during a time interval is the same as the average velocity over the interval. That is, $v_x = v_{x,avg}$.

$$v_x = \frac{\Delta x}{\Delta t}$$

Remembering that $\Delta x = x_f - x_i$, we see that $v_x = (x_f - x_i)/\Delta t$, or

$$x_f = x_i + v_x \Delta t$$

In practice, we usually choose the time at the beginning of the interval to be $t_i = 0$ and the time at the end of the interval to be $t_f = t$, so our equation becomes: $\boxed{x_f = x_i + v_x t}$ (for constant v_x) (2.3)

- The **average speed** (v_{avg}) of a particle, a scalar quantity, is defined as (the total distance (d) traveled divided by the total time interval required to travel that distance):

$$v_{avg} = \frac{d}{\Delta t} \qquad \text{(Average speed)} \qquad (2.4)$$

The SI unit of average speed is the same as the unit of average velocity: (meters per second)(m/s).

- Average speed has no direction and is always expressed as a positive number.

2.2 Instantaneous Velocity and Speed

- The **instantaneous velocity** (v_x) equals the limiting value of the ratio ($\Delta x/\Delta t$) as (Δt) approaches zero:

$$\boldsymbol{v_x = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}} \qquad (2.5)$$

In calculus notation, this limit is called the *derivative* of (x) with respect to (t), written (dx/dt):

$$\boldsymbol{v_x = \frac{dx}{dt}} \qquad \text{(instantaneous velocity)} \qquad (2.6)$$

The instantaneous velocity can be positive, negative, or zero.

- The **instantaneous speed** of a particle is defined as (the magnitude of its instantaneous velocity). As with average speed, instantaneous speed has no direction associated with it.

Example (2.1):

A particle moves along the (x – axis). Its position varies with time according to the expression: ($x = - 4t + 2t^2$), where (x) is in meters and (t) in seconds.

(A) Determine the displacement of the particle in the time intervals (t = 0) to (t =1 s) and (t = 1 s) to (t =3 s).

(B) Calculate the average velocity during these two time intervals.

(C) Find the instantaneous velocity of the particle at (t = 2.5 s).

<u>Solution:</u>

(A): In the first time interval, (t = 0) to (t =1 s):

$$\Delta x = x_f - x_i$$

$$\Delta x = [- 4 (1) + 2 (1)^2] – [- 4 (0) + 2 (0)^2] = - 2 \text{ m.}$$

For the second time interval (t = 1 s) to (t =3 s):

$$\Delta x = [- 4 (3) + 2 (3)^2] – [- 4 (1) + 2 (1)^2] = + 8 \text{ m.}$$

(B): In the first time interval, use equation (2.2) with $\Delta t = t_f - t_i = 1$ s:

$$v_{x,avg} = \frac{\Delta x}{\Delta t} = \frac{-2m}{1\ s} = -2\ m/s$$

In the second time interval, $\Delta t = t_f - t_i = 2$ s:

$$v_{x,avg} = \frac{\Delta x}{\Delta t} = \frac{8m}{2s} = +4\ m/s$$

(C): Instantaneous velocity $v_x = \frac{dx}{dt}$, $x = -4t + 2t^2$

$$v_x = \frac{dx}{dt} = -4 + 4t\text{, at } t = 2.5\text{ s :}$$

$$v_x = -4 + 4\,(2.5) = +6\ m/s$$

2.3 Acceleration

- When the velocity of a particle changes with time, the particle is said to be *accelerating*.
- The **average acceleration** ($a_{x,avg}$) of the particle is defined as (The *change* in velocity (Δv_x) divided by the time interval (Δt) during which that change occurs):

$$a_{x,avg} = \frac{\Delta v_x}{\Delta t} = \frac{v_{xf} - v_{xi}}{t_f - t_i} \qquad \text{Average acceleration} \qquad (2.7)$$

The unit of acceleration is meters per second squared (m/s^2).

- The instantaneous acceleration equals the derivative of the velocity with respect to time:

$$a_x = \frac{dv_x}{dt} \qquad \text{Instantaneous acceleration} \qquad (2.8)$$

- For the case of motion in a straight line, the direction of the velocity of an object and the direction of its acceleration are related as follows: When the object's velocity and acceleration are in the same direction, the object is **speeding up**. On the other hand, when the object's velocity and acceleration are in opposite directions, the object is **slowing down**.
- Because $v_x = \frac{dx}{dt}$, the acceleration can also be written as:

$$a_x = \frac{dv_x}{dt} = = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2} \qquad (2.9)$$

That is, in one-dimensional motion, the acceleration equals the *second derivative* of (x) with respect to time.

Example (2.2):

The velocity of a particle moving along the (x- axis) varies according to the expression ($v_x = 40 - 5t^2$), where v_x is in meters per second and (t) in seconds.

(A) Find the average acceleration in the time interval ($t = 0$ to $t = 2$ s).

(B) Determine the acceleration at $t = 2$ s.

Solution:

(A) $\boldsymbol{v_{x1} = 40 - 5t^2 = 40 - 5\,(0)^2 = 40}$ **m/s**

$\boldsymbol{v_{x2} = 40 - 5t^2 = 40 - 5\,(2)^2 = 20}$ **m/s**

$$a_{x,\text{avg}} = \frac{\Delta v_x}{\Delta t} = \frac{20-40}{2-0} = -10 \text{ m/s}^2$$

The negative sign indicates that the particle is slowing down.

(B) $$a_x = \frac{dv_x}{dt},\; v_x = 40 - 5t^2$$

$$a_x = -10\,t = -10\,(2) = -20 \text{ m/s}^2$$

2.4 Particles under Constant Acceleration

If the acceleration of a particle varies in time, its motion can be complex and difficult to analyze. A very common and simple type of one-dimensional motion, however, is that in which the acceleration is constant. In such case, the average acceleration ($a_{x,\text{avg}}$) over any time interval is numerically equal to the instantaneous acceleration (a_x) at any instant within the interval, and the velocity changes at the same rate throughout the motion. This situation is considered to be the **particle under constant acceleration**.

If we replace ($a_{x,\text{avg}}$) by (a_x) in equation (2.7) and take $t_i = 0$ and (t_f) to be any later time (t), we find that:

$$a_x = \frac{v_{xf} - v_{xi}}{t - 0}$$

$$\boxed{v_{xf} = v_{xi} + a_x t} \quad \text{(for constant } a_x\text{)} \qquad (2.10)$$

We can express the average velocity in any time interval:

$$\boxed{v_{x,ave} = \frac{v_{xi} + v_{xf}}{2}} \quad \text{(for constant } a_x\text{)} \qquad (2.11)$$

Notice that this expression for average velocity applies *only* in situations in which the acceleration is constant.

We can now use equations 2.1, 2.2, and 2.11 to obtain the position of an object as a function of time. Recalling that Δx in equation (2.2) represents $(x_f - x_i)$ and recognizing that $\Delta t = t_f - t_i = t - 0 = t$, we find that:

$$x_f - x_i = v_{x,ave}\, t = \frac{1}{2}(v_{xi} + v_{xf})\, t$$

$$\boxed{x_f = x_i + \frac{1}{2}(v_{xi} + v_{xf})\, t} \quad \text{(for constant } a_x\text{)} \qquad (2.12)$$

This equation provides the final position of the particle at time (t) in terms of the initial and final velocities.

We can obtain another useful expression for the position of a particle under constant acceleration by substituting equation (2.10) into equation (2.12):

$$x_f = x_i + \frac{1}{2}[v_{xi} + (v_{xi} + a_x t)]\, t$$

$$\boxed{x_f = x_i + v_{xi} t + \frac{1}{2} a_x t^2} \quad \text{(for constant } a_x\text{)} \qquad (2.13)$$

This equation provides the final position of the particle at time (t) in terms of the initial position, the initial velocity, and the constant acceleration. Finally, we can obtain an expression for the final velocity that does not contain time as a variable by substituting the value of (t) from equation (2.10) into equation (2.12):

$$x_f = x_i + \frac{1}{2}(v_{xi} + v_{xf})\left(\frac{v_{xf} - v_{xi}}{ax}\right) = x_i + \left(\frac{v_{xf}^2 - v_{xi}^2}{2ax}\right)$$

$$\boxed{v_{xf}^2 = v_{xi}^2 + 2a_x (x_f - x_i)} \quad \text{(for constant } a_x\text{)} \qquad (2.14)$$

This equation provides the final velocity in terms of the initial velocity, the constant acceleration, and the position of the particle.

When the acceleration of a particle is zero, its velocity is constant and its position changes linearly with time.

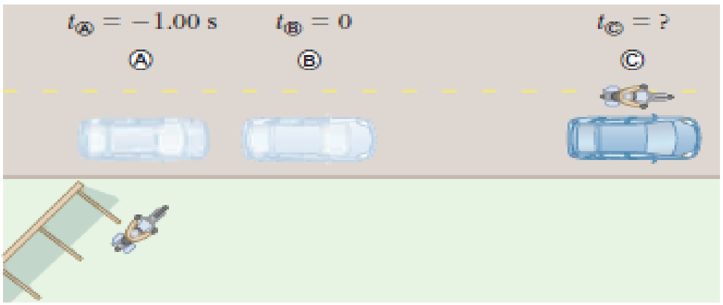
Equations (2.10), (2.12), (2.13), and (2.14) are called **Kinematic Equations** for motion of a particle under constant acceleration. These equations are listed in the table below:

Equation	Information Given by Equation
$v_{xf} = v_{xi} + a_x t$	Velocity as a function of time
$x_f = x_i + \frac{1}{2} v_{xi} + v_{xf}) t$	Position as a function of velocity and time
$x_f = x_i + v_{xi} t + \frac{1}{2} a_x t^2$	Position as a function of time
$v_{xf}^2 = v_{xi}^2 + 2a_x(x_f - x_i)$	Velocity as a function of position

Example (2.3):

A car traveling at a constant speed of (45 m/s) passes a trooper on a motorcycle hidden behind a billboard. One second after the speeding car passes the billboard; the trooper sets out from the billboard to catch the car, accelerating at a constant rate of (3 m/s^2). How long does it take him to overtake the car?

<u>Solution:</u>



First, we write expressions for the position of each vehicle as a function of time. It is convenient to choose the position of the billboard as the origin and to set $t_B = 0$ as the time the trooper begins moving. At that instant, the car has already traveled a distance of (45 m) from the billboard because it has traveled at a constant speed of $v_x = 45$ m/s for 1 s. Therefore, the initial position of the speeding car is $x_B = 45$ m.

Apply equation (2.3) to give the car's position at any time (t):

$$x_{car} = x_B + v_{x\,car}\, t$$

At ($t = 0$), this expression gives the car's correct initial position when the trooper begins to move: $x_{car} = x_B = 45\text{ m}$

The trooper starts from rest at $t_B = 0$ and accelerates at $a_x = 3\text{ m/s}^2$ away from the origin. Use equation (2.13) to give his position at any time (t):

$$x_f = x_i + v_{xi}t + \frac{1}{2}a_x t^2$$

$$x_{trooper} = 0 + 0(t) + \frac{1}{2}a_x t^2 = \frac{1}{2}a_x t^2$$

Set the positions of the car and trooper equal to represent the trooper overtaking the car at position (C): $x_{trooper} = x_{car}$

$$\frac{1}{2}a_x t^2 = x_B + v_{x\,car}\, t$$

Rearrange: $$\frac{1}{2}a_x t^2 - v_{x\,car}\, t - x_B = 0$$

$$1.5\, t^2 - 45t - 45 = 0$$

$$t = 31\text{ s.}$$

2.5 Freely Falling Objects

- A freely falling object is (any object moving freely under the influence of gravity alone, regardless of its initial motion).
- We denote the magnitude of the *free-fall acceleration* by the symbol (g). The value of (g) decreases with increasing altitude

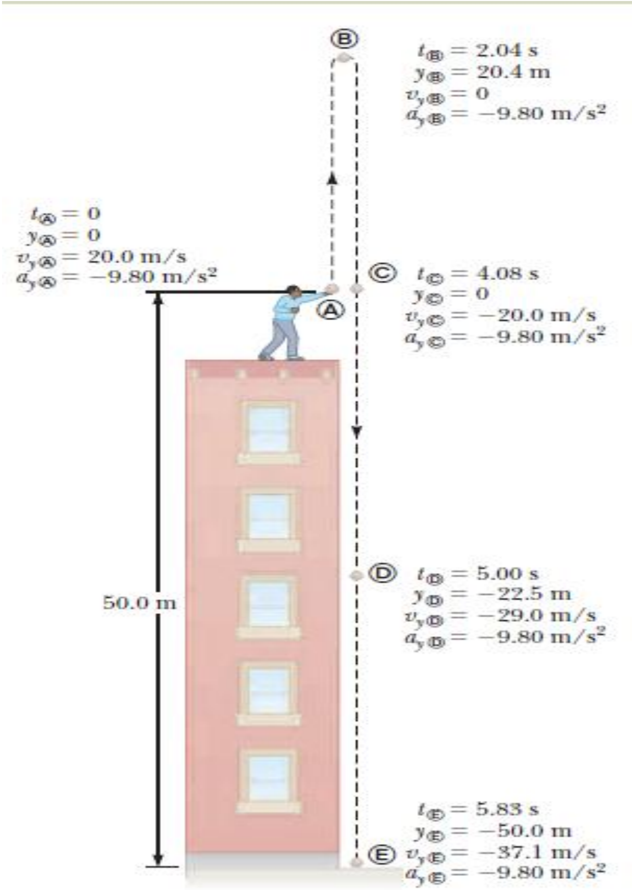
above the Earth's surface. Furthermore, slight variations in (g) occur with changes in latitude. At the Earth's surface, the value of (g) is approximately equal 9.80 m/s^2.

- Note that the motion is in the vertical direction (the *y* direction) rather than in the horizontal direction (*x*)
- We choose (g= - 9.80 m/s^2), where the negative sign means that the acceleration of a freely falling object is downward.

Example (2.4):

A stone thrown from the top of a building is given an initial velocity of 20 m/s straight upward. The stone is launched 50 m above the ground, and the stone just misses the edge of the roof on its way down as shown in the figure below.

(A) Using $t_A = 0$ as the time the stone leaves the thrower's hand at position A. Determine the time at which the stone reaches its maximum height?



<u>Solution:</u>

Use equation (2.10) to calculate the time at which the stone reaches its maximum height:

$$v_{yf} = v_{yi} + a_y t,$$

$$t = \frac{v_{yf} - v_{yi}}{a_y} = t_B$$

$$t = \frac{0-20}{-9.8} = 2.04 \text{ s}$$

(B) Find the maximum height of the stone?

As in part (A), choose the initial and final points at the beginning and the end of the upward flight. Set ($y_A = 0$) and substitute the time from part (A) into equation (2.13) to find the maximum height:

$$y_{max} = y_B = y_A + v_{yA}t + \frac{1}{2} a_y t^2$$

$$y_B = 0 + (20)(2.04) + \frac{1}{2}(-9.8)(2.04)^2 = 20.4 \text{ m}$$

(C) Determine the velocity of the stone when it returns to the height from which it was thrown?

Choose the initial point where the stone is launched and the final point when it passes this position coming down.

Substitute known values into equation (2.14):

$$v_{yC}^2 = v_{yA}^2 + 2a_y (y_C - y_A)$$

$$v_{yC}^2 = (20)^2 + 2(-9.8)(0 - 0) = 400 \text{ m}^2/\text{s}^2$$

$$v_{yC} = -20 \text{ m/s.}$$

When taking the square root, we could choose either a positive or a negative root. We choose the negative root because we know that the stone is moving downward at point C. The velocity of the stone when it arrives back at its original height is equal in magnitude to its initial velocity but is opposite in direction.

(D) Find the velocity and position of the stone at $t = 5$ s?

Choose the initial point just after the throw and the final point (5 s.) later.

Calculate the velocity at (D) from equation (2.10):

$$v_{yD} = v_{yA} + a_y t = 20 + (-9.8)(5) = -29 \text{ m/s}$$

Use equation (2.13) to find the position of the stone at $t_D = 5$ s:

$$y_D = y_A + v_{yA}t + \frac{1}{2} a_y t^2$$

$$= 0 + (20)(5) + \frac{1}{2}(-9.8)(5)^2$$

$$= -22.5 \text{ m}$$

Chapter 3

Vectors

3.1 Vector and Scalar Quantities

- A **scalar quantity** is completely specified by a single value with an appropriate unit and has no direction.

Examples of scalar quantities are temperature, volume, mass, speed, and time intervals.

- A **vector quantity** is completely specified by a number with an appropriate unit plus a direction.

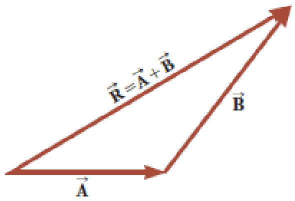
Examples of vector quantity are displacement and velocity.

3.2 Some Properties of Vectors

Adding Vectors

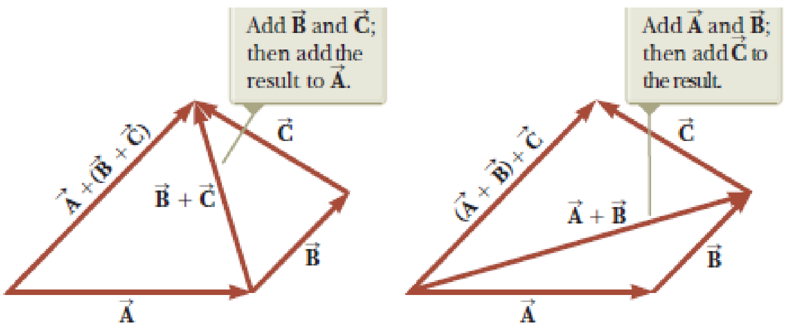
When two vectors (vector $\vec{A}$ and vector $\vec{B}$) are added, the sum is independent of the order of the addition. This property is known as the **(commutative law of addition):**

$$\vec{A} + \vec{B} = \vec{B} + \vec{A}$$



Another property is called the **associative law of addition:**

$$\vec{A} + (\vec{B} + \vec{C}) = (\vec{A} + \vec{B}) + \vec{C}$$



- A vector quantity has both magnitude and direction and also obeys the laws of vector addition.

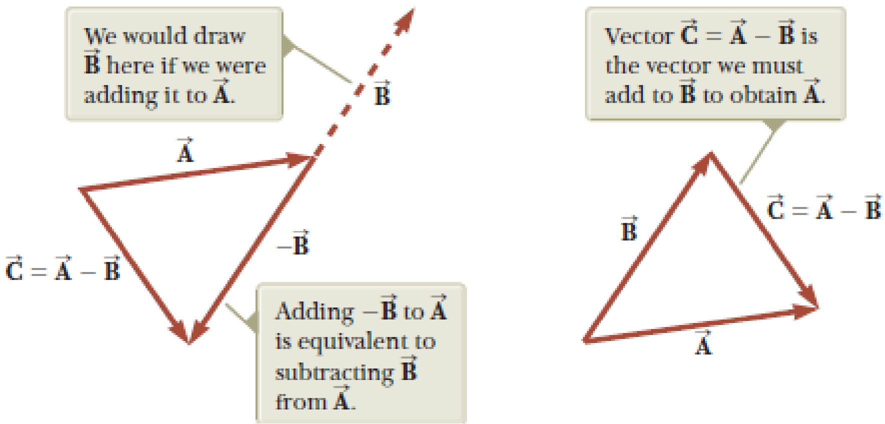
Negative of a Vector

The negative of the vector $\vec{A}$ is defined as the vector that when added to $\vec{A}$ gives zero for the vector sum. That is, $\vec{A} + (-\vec{A}) = 0$. The vectors $\vec{A}$ and (-$\vec{A}$) have the same magnitude but point in opposite directions.

Subtracting Vectors

The operation of vector subtraction makes use of the definition of the negative of a vector. We define the operation ($\vec{A} - \vec{B}$) as vector $(-\vec{B})$ added to vector ($\vec{A}$): $\vec{A} - \vec{B} = \vec{A} + (-\vec{B})$

The geometric construction for subtracting two vectors in this way is illustrated in the figure below:



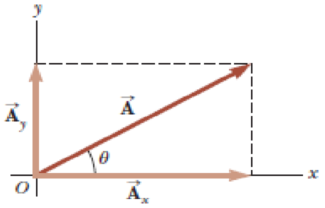
3.3 Components of a Vector and Unit Vectors

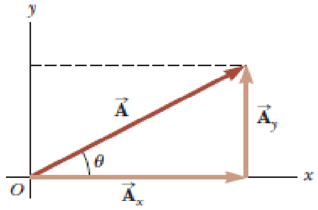
Components of a Vector

Any vector can be completely described by its components.

Consider a vector $(\vec{A})$ lying in the (*xy* plane) and making an angle (θ) with the positive (*x*-axis) as shown in the figure below. This vector can be expressed as the sum of two other *component* vectors ($\vec{A}_x$), which is parallel to the (*x*-axis), and ($\vec{A}_y$), which is parallel to the (*y*-axis).

$$\vec{A} = \vec{A}_x + \vec{A}_y$$





From the figure and the definition of sine and cosine, we see that

($\cos\theta = A_x / A$) and that ($\sin\theta = A_y / A$). Hence, the components of $\vec{A}$

$$A_x = A\cos\theta \text{ and } A_y = A\sin\theta$$

The magnitude and direction of ($\vec{A}$) are related to its components through the expressions:

$$A = \sqrt{A_x^2 + A_y^2} \qquad \text{(magnitude of } \vec{A}\text{)}$$

$$\theta = \tan^{-1}\left(\frac{Ay}{Ax}\right) \qquad \text{(direction of } \vec{A}\text{)}$$

Unit Vectors

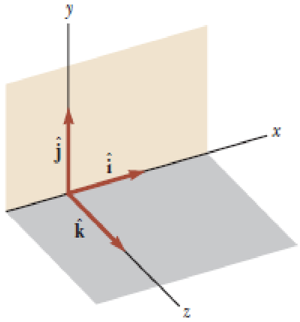
A **unit vector** is (a dimensionless vector having a magnitude of exactly one, and are used to specify a given direction).

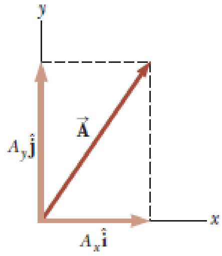
We shall use the symbols ($\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$) to represent unit vectors pointing in the positive (*x*, *y*, and *z*) directions, respectively.

The magnitude of each unit vector equals 1; that is, $|\hat{\imath}| = |\hat{\jmath}| = |\hat{k}| = 1$

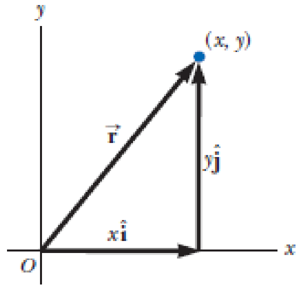
$\vec{A}_x = \hat{\imath}\, A_x$, $\vec{A}_y = \hat{\jmath}\, A_y$. Therefore, the unit-vector notation for the vector $\vec{A}$ is:

$$\vec{A} = \hat{\imath}\, A_x + \hat{\jmath}\, A_y$$





- Consider a point lying in the *xy* plane and having Cartesian coordinates (*x*, *y*) as in the figure below. The point can be specified by the **position vector** ($\vec{r}$) which in unit-vector form is given by:



$$\vec{r} = \hat{\imath}\, x + \hat{\jmath}\, y$$

- The resultant vector ($\vec{R} = \vec{A} + \vec{B}$) is:

$$\vec{R} = (\hat{\imath}\, A_x + \hat{\jmath}\, A_y) + (\hat{\imath}\, B_x + \hat{\jmath}\, B_y) \quad \text{or}$$

$$\vec{R} = \hat{\imath}\,(A_x + B_x) + \hat{\jmath}\,(A_y + B_y)$$

Because $\vec{R} = \hat{\imath}\ R_x + \hat{\jmath}\ R_y$, we see that the components of the resultant vector are: $R_x = (A_x + B_x)$ and $R_y = (A_y + B_y)$.

The magnitude of $\vec{R}$ and the angle it makes with the (*x*- axis) are obtained from its components using the relationships:

$$R = \sqrt{R_x^2 + R_y^2} = \sqrt{(A_x + B_x)^2 + (A_y + B_y)^2} \qquad \text{Magnitude of } \vec{R}$$

$$\tan\theta = \frac{R_y}{R_x} = \frac{A_y + B_y}{A_x + B_x} \qquad \text{Direction of } \vec{R}$$

- If $\vec{A}$ *and* $\vec{B}$ both have three components (*x*,*y*,*z*), they can be expressed in the form:

$$\vec{A} = \hat{\imath}\, A_x + \hat{\jmath}\, A_y + \hat{k}\, A_z$$

$$\vec{B} = \hat{\imath}\, B_x + \hat{\jmath}\, B_y + \hat{k}\, B_z$$

The sum of $\vec{A}$ *and* $\vec{B}$ is: $\vec{R} = \vec{A} + \vec{B}$ or

$$\vec{R} = \hat{\imath}\,(A_x + B_x) + \hat{\jmath}\,(A_y + B_y) + \hat{k}\,(A_z + B_z)$$

Example (3.1):

Find the sum of two displacement vectors $\vec{A}$ *and* $\vec{B}$ lying in the *xy* plane and given by: $\vec{A} = (2\hat{\imath} + 2\hat{\jmath})$ m and $\vec{B} = (2\hat{\imath} - 4\hat{\jmath})$ m.

Solution:

The resultant vector $\vec{R}$: $\vec{R} = \vec{A} + \vec{B} = \hat{\imath}\,(A_x + B_x) + \hat{\jmath}\,(A_y + B_y)$

$= \hat{\imath}\,(2 + 2) + \hat{\jmath}\,(2 - 4\,)$

The components of $\vec{R}$: $R_x = 4$ m and $R_y =$ -2 m

The magnitude of $\vec{R}$: $R = \sqrt{R_x^2 + R_y^2} = \sqrt{(4)^2 + (-2)^2} = \sqrt{20} = 4.5$ m

The direction of $\vec{R}$: $\tan\theta = \frac{R_y}{R_x} = \frac{-2}{4} = -\,0.5$

$\theta = -\,27°$

This answer is correct if we interpret it to mean 27° clockwise from the (*x* – axis).

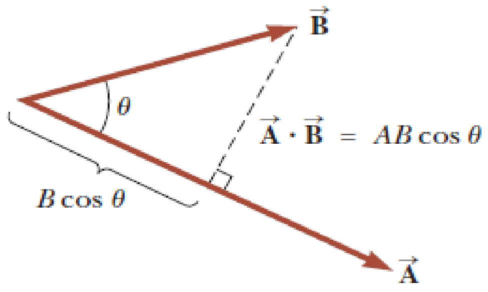
3.4 Scalar Product

The scalar product of any two vectors $\vec{A}$ *and* $\vec{B}$ is defined as (a scalar quantity equal to the product of the magnitudes of the two vectors and the cosine of the angle θ between them):

$$\vec{A} \bullet \vec{B} = AB\cos\theta$$

We write scalar product of vectors $\vec{A}$ *and* $\vec{B}$ as $\vec{A} \bullet \vec{B}$ (Because of the dot symbol, the scalar product is often called the **dot product**).

- The scalar product $(\vec{A} \bullet \vec{B})$ equals the magnitude of $\vec{A}$ multiplied by the projection of $\vec{B}$ onto $\vec{A}$:$(B\cos\theta)$ as shown in the figure below.



Properties of the scalar product:

1. Scalar product is **commutative**:

$$\vec{A} \bullet \vec{B} = \vec{B} \bullet \vec{A}$$

2. Scalar product obeys the **distributive law of multiplication**:

$$\vec{A} \bullet (\vec{B} + \vec{C}) = \vec{A} \bullet \vec{B} + \vec{A} \bullet \vec{C}$$

3. If $\vec{A}$ is perpendicular to $\vec{B}$ ($\theta = 90°$), then $\vec{A} \bullet \vec{B} = 0$.
4. If $\vec{A}$ is parallel to $\vec{B}$ ($\theta = 0°$), then $\vec{A} \bullet \vec{B} = AB$.
5. If $\theta = 180°$, then $\vec{A} \bullet \vec{B} = -AB$.
6. The scalar product is negative when ($90° < \theta \leq 180°$).

$\hat{i} \bullet \hat{i} = \hat{j} \bullet \hat{j} = \hat{k} \bullet \hat{k} = 1$

$\hat{i} \bullet \hat{j} = \hat{i} \bullet \hat{k} = \hat{j} \bullet \hat{k} = 0$

Two vectors $\vec{A}$ and $\vec{B}$ can be expressed in unit vector form as:

$$\vec{A} = \hat{i}A_x + \hat{j}A_y + \hat{k}A_z$$

$$\vec{B} = \hat{i}B_x + \hat{j}B_y + \hat{k}B_z$$

so $\vec{A} \bullet \vec{B} = A_xB_x + A_yB_y + A_zB_z$ $\quad and\ \vec{A} \bullet \vec{A} = A^2$.

Example (3.2):

The vectors $\vec{A}$ and $\vec{B}$ are given by: $\vec{A} = 2\hat{i} + 3\hat{j}$ and $\vec{B} = -\hat{i} + 2\hat{j}$

(A) Determine the scalar product $\vec{A} \bullet \vec{B}$

(B) Find the angle (θ) between $\vec{A}$ and $\vec{B}$

Solution:

(A) $\vec{A} \bullet \vec{B} = (2\hat{i} + 3\hat{j}) \bullet (-\hat{i} + 2\hat{j})$

$= -2\hat{i} \bullet \hat{i} + 2\,\hat{i} \bullet 2\hat{j} - 3\hat{j} \bullet \hat{i} + 3\hat{j} \bullet 2\hat{j}$

$= -2 + 0 - 0 + 6 = 4$

(B) The magnitude of $\vec{A}$: $A = \sqrt{A_x^2 + A_y^2} = \sqrt{(2)^2 + (3)^2} = \sqrt{13}$

The magnitude of $\vec{B}$: $B = \sqrt{B_x^2 + B_y^2} = \sqrt{(-1)^2 + (2)^2} = \sqrt{5}$

$$\cos\theta = \frac{\vec{A} \bullet \vec{B}}{AB} = \frac{4}{\sqrt{13}\sqrt{5}} = \frac{4}{\sqrt{65}}$$

$$\theta = \cos^{-1}\frac{4}{\sqrt{65}} = 60.3°$$

3.5 Vector Product

(Given any two vectors $\vec{A}$ *and* $\vec{B}$, the vector product $(\vec{A} \times \overrightarrow{B})$ is defined as a third vector $\overrightarrow{C}$, which has a magnitude of $(ABsin\theta)$).

$\vec{C} = \vec{A} \times \vec{B}$ Vector product

$C = ABsin\theta$ magnitude of vector product

- The vector product$(\vec{A} \times \vec{B})$ is also called (**cross product**).

Properties of the vector product:

1. It is not commutative $(\vec{A} \times \vec{B} = -\vec{B} \times \vec{A})$ Therefore, if you change the order of the vectors in a vector product, you must change the sign.
2. If $\vec{A}$ is parallel to $\vec{B}$ (θ = 0 or 180°), then

$$\vec{A} \times \vec{B} = 0 \quad \text{and} \quad \vec{A} \times \vec{A} = 0$$

3. If $\vec{A}$ is perpendicular to $\vec{B}$ (θ = 90°), then

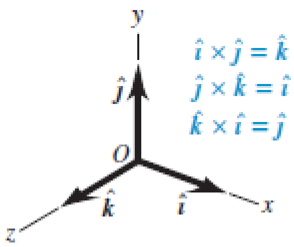
$$|\vec{A} \times \vec{B}| = AB$$

4. The vector product obeys the distributive law:

$$\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$$

5. The derivative of the vector product with respect to some variable such as (t) is: $\frac{d}{dt}(\vec{A} \times \vec{B}) = \frac{d\vec{A}}{dt} \times \vec{B} + \vec{A} \times \frac{d\vec{B}}{dt}$

- The cross products of the unit vectors ($\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$) obey the following rules: $\hat{\imath} \times \hat{\imath} = \hat{\jmath} \times \hat{\jmath} = \hat{k} \times \hat{k} = 0$
- $\hat{\imath} \times \hat{\jmath} = \hat{k}$, $\hat{\jmath} \times \hat{\imath} = -\hat{k}$
- $\hat{\jmath} \times \hat{k} = \hat{\imath}$, $\hat{k} \times \hat{\jmath} = -\hat{\imath}$
- $\hat{k} \times \hat{\imath} = \hat{\jmath}$, $\hat{\imath} \times \hat{k} = -\hat{\jmath}$



The cross product of any two vectors $\overrightarrow{A}$ *and* $\vec{B}$ can be expressed in the following determinant form:

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \hat{\imath}\begin{vmatrix} A_y & A_z \\ B_y & B_z \end{vmatrix} + \hat{\jmath}\begin{vmatrix} A_z & A_x \\ B_z & B_x \end{vmatrix} + \hat{k}\begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix}$$

$$= \hat{\imath}\,(A_yB_z - A_zB_y) + \hat{\jmath}\,(A_zB_x - A_xB_z) + \hat{k}\,(A_xB_y - A_yB_x)$$

Example (3.3):

Two vectors lying in the (*xy* plane) are given by the equations:

$$\vec{A} = 2\hat{\imath} + 3\hat{\jmath} \text{ and } \vec{B} = -\hat{\imath} + 2\hat{\jmath}$$

Find $\vec{A}\times\vec{B}$ and verify that ($\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}$).

Solution:

$\vec{A}\times\vec{B} = (2\hat{\imath} + 3\hat{\jmath}) \times (-\hat{\imath} + 2\hat{\jmath})$

$= 2\hat{\imath} \times (-\hat{\imath}) + 2\,\hat{\imath} \times 2\hat{\jmath} + 3\hat{\jmath} \times (-\hat{\imath}) + 3\hat{\jmath} \times 2\hat{\jmath}$

$= 0 + 4\,\hat{k} + 3\,\hat{k} + 0 = 7\,\hat{k}$

To verify that $\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}$:

$\vec{B}\times\vec{A} = (-\hat{\imath} + 2\hat{\jmath}) \times (2\hat{\imath} + 3\hat{\jmath})$

$= (-\hat{\imath}) \times 2\hat{\imath} + (-\hat{\imath}) \times 3\hat{\jmath} + 2\hat{\jmath} \times 2\hat{\imath} + 2\hat{\jmath} \times 3\hat{\jmath}$

$= 0 - 3\hat{k} - 4\,\hat{k} + 0 = -7\,\hat{k}$

Therefore $\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}$

Example (3.4):

Vector $\vec{A}$ has a magnitude of 6 units and it is in the direction of positive *x*- axis. Vector $\vec{B}$ has a magnitude of 4 units and lies in *xy*-plane making an angle 30° with *x*- axis. Find $\vec{A}\times\vec{B}$?

Solution:

$\vec{A} = 6\,\hat{\imath} + 0\,\hat{\jmath} + 0\,\hat{k}$

$\vec{B} = 4\,\hat{\imath}\cos 30 + 4\,\hat{\jmath}\,sin\,30 + 0\,\hat{k} = 2\sqrt{3}\,\hat{\imath} + 2\,\hat{\jmath}$

$$\vec{C} = \vec{A}\times\vec{B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 6 & 0 & 0 \\ 2\sqrt{3} & 2 & 0 \end{vmatrix} = 12\,\hat{k}$$

Chapter 4

(Motion in Two Dimensions)

4.1 The Position, Velocity, and Acceleration Vectors

We begin by describing the position of the particle by its **position vector** ($\vec{\mathbf{r}}$), drawn from the origin of some coordinate system to the location of the particle in the (xy plane) as shown in the figure.

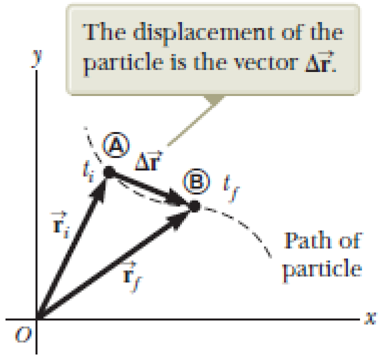
At time t_i, the particle is at point (A), described by position vector $\vec{\mathbf{r}}_i$.

At some later time t_f , it is at point (B), described by position vector $\vec{\mathbf{r}}_f$. The path from (A) to (B) is not necessarily a straight line. As the particle moves from (A) to (B) in the time interval ($\Delta t = t_f - t_i$), its position vector changes from $\vec{\mathbf{r}}_i$ to $\vec{\mathbf{r}}_f$.

We now define the **displacement vector (**$\Delta\vec{\mathbf{r}}$) for a particle as being (the difference between its final position vector and its initial position vector):

$$\Delta\vec{\mathbf{r}} = \vec{\mathbf{r}}_i - \vec{\mathbf{r}}_f \qquad \textbf{Displacement vector} \qquad (4.1)$$

As we see from the figure, the magnitude of ($\Delta\vec{\mathbf{r}}$) is *less* than the distance traveled along the curved path followed by the particle.



- The **average velocity** ($\vec{\mathbf{v}}_{ave}$) of a particle during the time interval (Δt) as the displacement of the particle divided by the time interval:

  $$\vec{\mathbf{v}}_{ave} = \frac{\Delta\vec{\mathbf{r}}}{\Delta t} \qquad \textbf{Average velocity} \qquad (4.2)$$

- Multiplying or dividing a vector quantity by a positive scalar quantity such as (Δt) changes only the magnitude of the vector, not its direction. Because displacement is a vector quantity and the time interval is a positive scalar quantity, we conclude that the average velocity is a vector quantity directed along ($\Delta\vec{\mathbf{r}}$).

- The average velocity between points is independent of the path taken.
- The **instantaneous velocity** ($\vec{\mathbf{v}}$) is defined as (the limit of the average velocity $\frac{\Delta\vec{\mathbf{r}}}{\Delta t}$ as Δt approaches zero):

$\vec{\mathbf{v}} = \lim_{\Delta t\to 0} \frac{\Delta\vec{\mathbf{r}}}{\Delta t} = \frac{d\vec{\mathbf{r}}}{dt}$ **Instantaneous velocity** (4.3)

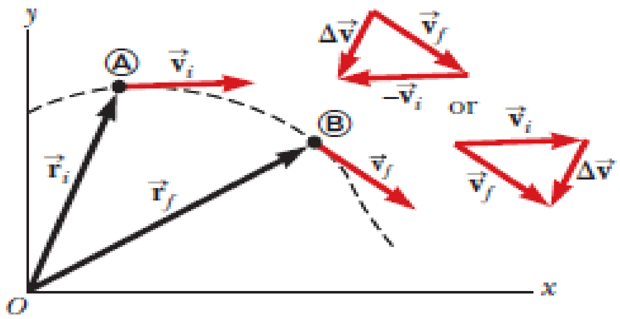
- The magnitude of the instantaneous velocity vector ($v = |\vec{\mathbf{v}}|$) of a particle is called the **speed** of the particle, which is a scalar quantity.
- The **average acceleration** ($\vec{a}_{ave}$) of a particle is defined as (the change in its instantaneous velocity vector **(**$\Delta\vec{\mathbf{v}}$) divided by the time interval Δt during which that change occurs):

$\vec{a}_{ave} = \frac{\Delta\vec{\mathbf{v}}}{\Delta t} = \frac{\vec{\mathbf{v}}_f - \vec{\mathbf{v}}_i}{t_f - t_i}$ **Average acceleration** (4.4)

Average acceleration is a vector quantity.

- The **instantaneous acceleration** ($\vec{a}$) is defined as (the limiting value of the ratio $\frac{\Delta\vec{\mathbf{v}}}{\Delta t}$ as Δt approaches zero):

$\vec{a} = \lim_{\Delta t\to 0} \frac{\Delta\vec{\mathbf{v}}}{\Delta t} = \frac{d\vec{\mathbf{v}}}{dt}$ **Instantaneous acceleration** (4.5)



4.2 Two-Dimensional Motion with Constant Acceleration

Motion in two dimensions can be modeled as two *independent* motions in each of the two perpendicular directions associated with the *x* and *y* axes. That is, any influence in the *y* direction does not affect the motion in the *x* direction and vice versa.

The position vector for a particle moving in the *xy* plane can be written:

$$\vec{\mathbf{r}} = \hat{\imath}x + \hat{\jmath}y \qquad (4.6)$$

The velocity of the particle:

$$\vec{\mathbf{v}} = \frac{d\vec{\mathbf{r}}}{dt} = \hat{\imath}\frac{dx}{dt} + \hat{\jmath}\frac{dy}{dt}$$

$$\vec{\mathbf{v}} = \hat{\imath}\, v_x + \hat{\jmath}\, v_y \qquad (4.7)$$

To determine the final velocity at any time *t*, we obtain:

$$\vec{\mathbf{v}}_f = (v_{ix} + a_x t)\,\hat{\imath} + (v_{iy} + a_y t)\,\hat{\jmath} = (v_{ix}\,\hat{\imath} + v_{iy}\,\hat{\jmath}) + (a_x\,\hat{\imath} + a_y\,\hat{\jmath})\, t$$

$$\boxed{\vec{\mathbf{v}}_f = \vec{\mathbf{v}}_i + \vec{\mathbf{a}}\, t}$$ Velocity vector as a function of time (4.8)

Similarly,

$$x_f = x_i + v_{ix}\, t + \frac{1}{2} a_x t^2 \quad \text{and} \quad y_f = y_i + v_{iy}\, t + \frac{1}{2} a_y t^2$$

Substituting these expressions into equation (4.6) (and labeling the final position vector ($\vec{\mathbf{r}}_f$) gives:

$$\vec{\mathbf{r}}_f = \left(x_i + v_{ix}\, t + \frac{1}{2} a_x t^2\right)\hat{\imath} + \left(y_i + v_{iy}\, t + \frac{1}{2} a_y t^2\right)\hat{\jmath}$$

$$= (x_i\,\hat{\imath} + y_i\,\hat{\jmath}) + (v_{ix}\,\hat{\imath} + v_{iy}\,\hat{\jmath})\, t + \frac{1}{2}(a_x\,\hat{\imath} + a_y\,\hat{\jmath})\, t^2$$

$$\boxed{\vec{\mathbf{r}}_f = \vec{\mathbf{r}}_i + \vec{\mathbf{v}}_i t + \frac{1}{2}\vec{\mathbf{a}}\, t^2}$$ Position vector as a function of time (4.9)

Example (4.1):

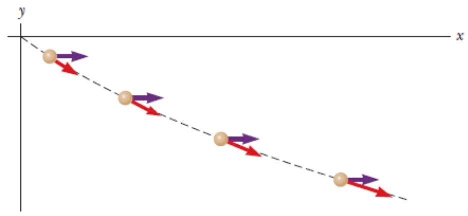
A particle moves in the *xy* plane, starting from the origin at ($t = 0$) with an initial velocity having an *x* - component of (20 m/s) and *y*- component of (-15 m/s). The particle experiences an acceleration in the *x* - direction, given by ($a_x = 4.0$ m/s^2).

(A) Determine the total velocity vector at any time.

(B) Calculate the velocity and speed of the particle at (t = 5.0 s) and the angle the velocity vector makes with the x- axis.

(C) Determine the x and y coordinates of the particle at any time t and its position vector at this time.

Solution:



(A) The components of the initial velocity tell us that the particle starts by moving toward the right and downward.

The x- component of velocity starts at 20 m/s and increases by 4.0 m/s every second. The y - component of velocity never changes from its initial value of (-15 m/s).

$\vec{v}_f = \vec{v}_i + \vec{a}\, t = (v_{ix} + a_x t)\,\hat{i} + (v_{iy} + a_y t)\,\hat{j}$

$\vec{v}_f = [20 + 4t]\,\hat{i} + [-15 + 0\,t]\,\hat{j}$

$\vec{v}_f = [(20 + 4t)\,\hat{i} - 15\,\hat{j}]$ m/s

(B) $\vec{v}_f = [(20 + 4t)\,\hat{i} - 15\,\hat{j}] = [\{20 + 4(5)\}\,\hat{i} - 15\,\hat{j}] = (40\,\hat{i} - 15\,\hat{j})$ m/s

The angle θ: $\theta = \tan^{-1}\frac{v_{yf}}{v_{xf}} = \tan^{-1}\frac{-15}{40} = -21°$

The negative sign for the angle θ indicates that the velocity vector is directed at an angle of 21° below the positive x - axis.

The speed of the particle as the magnitude of $\vec{v}_f$:

$$v_f = |\vec{v}_f| = \sqrt{v_{xf}^2 + v_{yf}^2} = \sqrt{(40)^2 + (-15)^2}\ v_f = 43 \text{ m/s}$$

(C) $x_f = v_{xi}t + \frac{1}{2}a_x t^2$

$x_f = (20\,t + 2\,t^2)$ m

$y_f = v_{yi}t = (-15\,t)$ m

The position vector of the particle at any time t :

$$\vec{\mathbf{r}}_f = (x_f\,\hat{i} + y_f\,\hat{j}) = [(20\,t + 2\,t^2)\,\hat{i} - 15\,t\,\hat{j}] \text{ m}$$

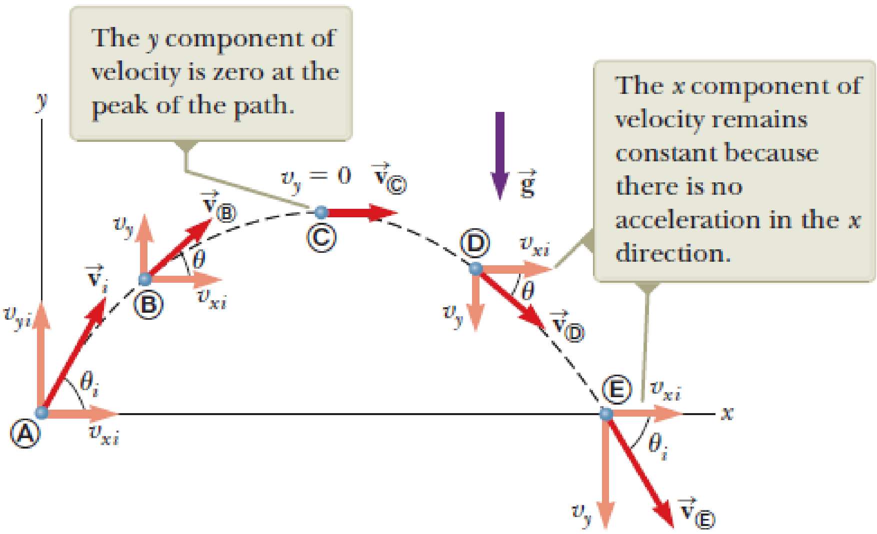
4.3 Projectile Motion

Anyone who has observed a baseball in motion has observed projectile motion. The ball moves in a curved path and returns to the ground. The path of a projectile, which we call its *trajectory*, is always a ***parabola***. The expression for the position vector of the projectile as a function of time follows directly from equation 4.9, with its acceleration being that due to gravity, $\vec{a} = \vec{g}$

$$\vec{\mathbf{r}}_f = \vec{\mathbf{r}}_i + \vec{\mathbf{v}}_i t + \frac{1}{2}\vec{\mathbf{g}}\, t^2 \qquad (4.10)$$

Where the initial *x* and *y* components of the velocity of the projectile are:

$$v_{xi} = v_i \cos\theta_i \qquad\qquad v_{yi} = v_i \sin\theta_i$$



When analyzing projectile motion, model it to be the superposition of two motions: (1) motion of a particle under constant velocity in the horizontal direction and (2) motion of a particle under constant acceleration (free fall) in the vertical direction.

Horizontal Range and Maximum Height of a Projectile

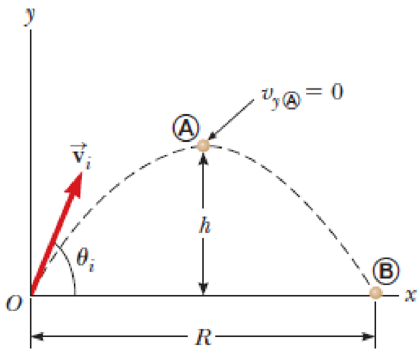
Let us assume a projectile is launched from the origin at $t_i = 0$ with a positive v_{yi} component as shown in figure above, and returns to the the same horizontal level. This situation is common in sports, where baseballs, footballs, and golf balls often land at the same level from which they were launched.

Two points in this motion are especially interesting to analyze:

- The peak point A, which has Cartesian coordinates (R/2, h), and
- The point B, which has coordinates (R, 0).
- The distance (R) is called the **horizontal range** of the projectile, and the distance (h) is its **maximum height.**

Let us find (h) and (R) mathematically in terms of v_i, θ_i, and g :

We can determine (h) by noting that at the peak v_{yA} =0. Therefore, we can use the y component of equation (4.8) to determine the time t_A at which the projectile reaches the peak:



$$\vec{v}_{yf} = \vec{v}_{yi} + a_y\, t$$

$$0 = v_i \sin\theta\, i - g\, t_A$$

$$t_A = \frac{v_i \sin \theta i}{g}$$

Substituting this expression for t_A into the y component of equation (4.9) and replacing $y = y_A$ with h, we obtain an expression for h in terms of the magnitude and direction of the initial velocity vector:

$$h = (v_i \sin\theta\, i\,)\left(\frac{v_i \sin \theta i}{g}\right) - \frac{1}{2}\, g \left(\frac{v_i^2 \sin^2 \theta i}{g^2}\right)$$

$$\boldsymbol{h} = \frac{v_i^2\, sin^2\, \theta i}{2g} \qquad \text{Maximum height for the projectile} \qquad (4.11)$$

- The range R is the horizontal position of the projectile at a time that is <u>twice</u> the time at which it reaches its peak, that is, at time ($t_B = 2t_A$).

Using the x component of equation (4.9), noting that:

$v_{xi} = v_{xB} = v_i \cos\theta\, i$, and setting $x_B = R$ at $t = 2t_A$, we find that:

$R = v_{xi}\, t_B = (v_i \cos\theta\, i)\,(2t_A)$

$R = (v_i \cos\theta\, i)\left(\frac{2v_i \sin\Theta i}{g}\right) = \frac{2{v_i}^2\ \sin\Theta i\ \cos\Theta i}{g}$

Using the identity $\sin 2\theta = 2sin\theta\ \cos\theta$, so

$$\boldsymbol{R = \frac{{v_i}^2 \sin 2\Theta i}{g}}$$ Horizontal range of the projectile (4.12)

The maximum value of R from equation (4.12) is:

$$\boldsymbol{R_{max} = \frac{{v_i}^2}{g}}$$ because the maximum value of $(\sin 2\theta\, i = 1)$, which occurs when $2\theta i = 90°$. Therefore, R is a maximum when $\theta i = 45°$.

Example (4.2):

A long jumper leaves the ground at an angle of 20° above the horizontal and at a speed of 11.0 m/s.

(A) How far does he jump in the horizontal direction?

(B) What is the maximum height reached?

Solution:

(A): Use equation (4.12) to find the range of the jumper:

$$R = \frac{{v_i}^2 \sin 2\Theta i}{g} = \frac{(11)^2\ \sin(2X20°)}{9.8} = 7.94 \text{ m}$$

(B): The maximum height reached by using equation 4.11:

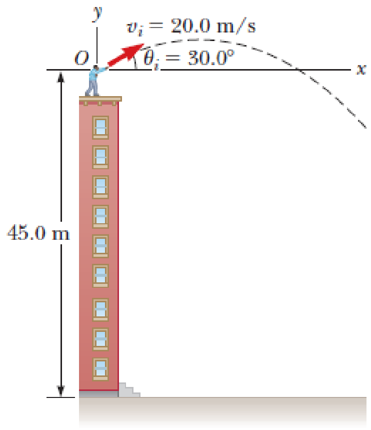
$$h = \frac{{v_i}^2\ sin^2\,\Theta i}{2g} = \frac{(11)^2\ sin^2\ 20°}{2\ (9.8)} = 0.722 \text{ m}$$

Example (4.3):

A stone is thrown from the top of a building upward at an angle of (30°) to the horizontal with an initial speed of (20 m/s) as shown in the figure. The height from which the stone is thrown is (45 m) above the ground.

(A) How long does it take the stone to reach the ground?

Solution: (A) We have the information



$x_i = y_i = 0$, y_f = - 45 m, a_y = - g , and v_i = 20 m/s

The initial *x* and *y* components of the stone's velocity:

$v_{xi} = v_i \cos \Theta i$ = 20 cos 30° = 17.3 m/s

$v_{yi} = v_i \sin \Theta i$ = 20 sin 30° = 10 m/s

The vertical position of the stone from the vertical component:

$y_f = y_i + v_{yi} t + \frac{1}{2} a_y t^2$

$-45 = 0 + 10\, t + \frac{1}{2} (-9.8)\, t^2$

t = 4.22 s

(B) What is the speed of the stone just before it strikes the ground?

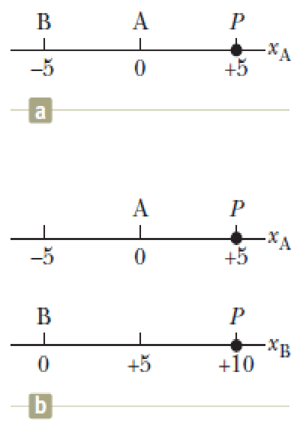
$v_{yf} = v_{yi} + a_y\, t$

= 10 + (- 9.8)(4.22) = - 31.3 m/s

4.4 Relative Velocity

We describe how observations made by different observers in different frames of reference are related to one another. A frame of reference can be described by a Cartesian coordinate system for which an observer is at rest with respect to the origin.

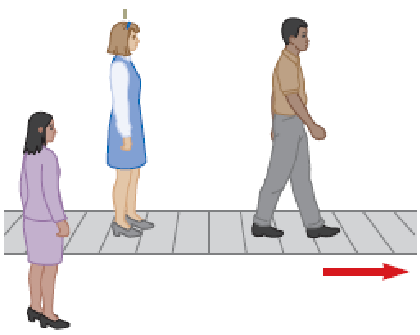
Consider the two observers A and B along the number line in figure a.



Observer A is located at the origin of a one-dimensional x_A axis, while observer B is at the position x_A = -5. We denote the position variable as x_A because observer A is at the origin of this axis. Both observers measure the position of point *P*, which is located at x_A = +5. Suppose observer B decides that he is located at the origin of an x_B axis as in Figure b. Notice that the two observers disagree on the value of the position of point *P*. Observer A claims point *P* is located at a position with a value of +5, whereas observer B claims it is located at a position with a value of +10. Both observers are correct, even though they make different measurements. Their measurements differ because they are making the measurement from different frames of reference.

Imagine now that observer B in (figure b) is moving to the right along the x_B axis. Now the two measurements are even more different. Observer A claims point *P* remains at rest at a position with a value of +5, whereas observer B claims the position of *P* continuously changes with time, even passing him and moving behind him! Again, both observers are correct, with the difference in their measurements arising from their different frames of reference.

We explore this phenomenon further by considering two observers watching a man walking on a moving beltway at an airport in figure below. The woman standing on the moving beltway sees the man moving at a normal walking speed. The woman observing from the stationary floor sees the man moving with a higher speed because the beltway speed combines with his walking speed. Both observers look at the same man and arrive at different values for his speed. Both are correct; the difference in their measurements results from the **relative velocity** of their frames of reference.



In a more general situation, consider a particle located at point *P* in the figure. Imagine that the motion of this particle is being described by two observers, observer A in a reference frame S_A fixed relative to the Earth and a second observer B in a reference frame S_B moving to the right relative to S_A (and therefore relative to the Earth) with a constant velocity $\vec{v}_{BA}$. In this discussion of relative velocity, we use a double-subscript notation; the first subscript represents what is being observed, and the second represents who is doing the observing. Therefore, the notation $\vec{v}_{BA}$ means the velocity of observer B (and the attached frame S_B) as measured by observer A. With this notation, observer B measures A to be moving to the left with velocity ($\vec{v}_{AB} = -\vec{v}_{BA}$). For purposes of this discussion, let us place each observer at his respective origin. We define the time $t = 0$ as the instant at which the origins of the two reference frames coincide in space. Therefore, at time t, the origins of the reference frames will be separated by a distance ($v_{BA}\ t$). We label the position P of the particle relative to observer A with the position vector $\vec{\mathbf{r}}_{PA}$ and that relative to observer B with the position vector $\vec{\mathbf{r}}_{PB}$, both at time t. We see that the vectors $\vec{\mathbf{r}}_{PA}$ and $\vec{\mathbf{r}}_{PB}$ are related to each other through the expression:

$$\vec{\mathbf{r}}_{PA} = \vec{\mathbf{r}}_{PB} + \vec{v}_{BA}\, t \qquad (4.13)$$

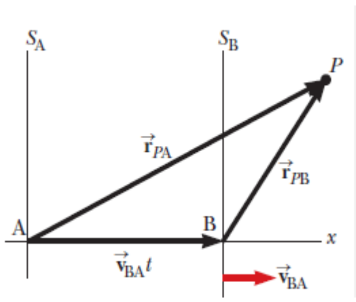
By differentiating (equation 4.13) with respect to time, noting that $\vec{v}_{BA}$ is constant, we obtain: $\frac{d\vec{\mathbf{r}}PA}{dt} = \frac{d\vec{\mathbf{r}}PB}{dt} + \vec{v}_{BA}$

$$\vec{\mathbf{u}}_{PA} = \vec{\mathbf{u}}_{PB} + \vec{v}_{BA} \qquad (4.14)$$

Where $\vec{\mathbf{u}}_{PA}$ is the velocity of the particle at P measured by observer A and $\vec{\mathbf{u}}_{PB}$ is its velocity measured by B. (We use the symbol $\vec{\mathbf{u}}$ for particle velocity rather than $(\vec{v})$, which we have already used for the relative velocity of two reference frames.) Equations 4.13 and 4.14 are known as **Galilean transformation equations.**

Although observers in two frames measure different velocities for the particle, they measure the *same acceleration* when $\vec{v}_{BA}$ is constant.

We can verify that by taking the time derivative of equation 4.14:



$$\frac{d\vec{\mathbf{u}}PA}{dt} = \frac{d\vec{\mathbf{u}}PB}{dt} + \frac{d\vec{v}BA}{dt}$$

Because $\vec{v}_{BA}$ is constant, $\frac{d\vec{v}BA}{dt} = 0$.

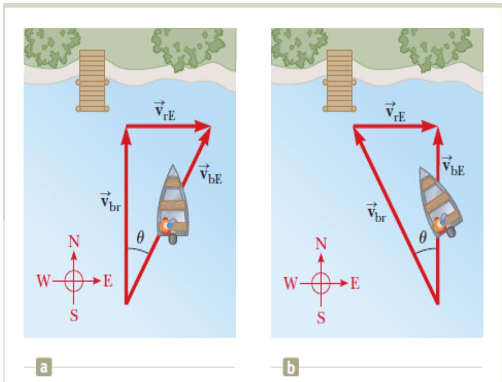
Therefore, we conclude that ($\vec{a}_{PA} = \vec{a}_{PB}$). That is, the acceleration of the particle measured by an observer in one frame of reference is the same as that measured by any other observer moving with constant velocity relative to the first frame.

Example (4.4):

A boat crossing a wide river moves with a speed of 10 km/h relative to the water. The water in the river has a uniform speed of 5km/h due east relative to the Earth.

(A) If the boat heads due north, determine the velocity of the boat relative to an observer standing on either bank.

Solution:



We know $\vec{v}_{br}$, the velocity of the boat relative to the river, and $\vec{v}_{rE}$, the velocity of the river relative to the Earth.

What we must find is $\vec{v}_{bE}$, the velocity of the boat relative to the Earth. The relationship between these three quantities is $\vec{v}_{bE} = \vec{v}_{br} + \vec{v}_{rE}$.

The quantity $\vec{v}_{br}$ is due north; $\vec{v}_{rE}$ is due east; and the vector sum of the two, $\vec{v}_{bE}$, is at an angle θ as defined in the figure a.

$$v_{bE} = \sqrt{v_{br}^2 + v_{rE}^2} = \sqrt{(10)^2 + (5)^2} = 11.2 \text{ km/h}$$

Find the direction of $\vec{v}_{bE}$: $\theta = \tan^{-1}\frac{v_{rE}}{v_{br}} = \tan^{-1}\frac{5}{10} = 26.6°$

The boat is moving at a speed of 11.2 km/h in the direction 26.6° east of north relative to the Earth.

Chapter 5

(Force and Motion)

The Laws of Motion

5.1 Newton's First Law of Motion:

Newton's First Law of Motion Sometimes called the (*law of inertia*). The term inertia is described as (the tendency of an object to resist changes in its motion). Another statement of Newton's first law is

(In the absence of external forces, an object at rest remains at rest and an object in motion continue in motion with a constant velocity in a straight line).

In other words, when no force acts on an object, the acceleration of the object is zero; the object is treated with the **particle in equilibrium** model. In this model, the net force on the object is zero:

$$\boldsymbol{\Sigma \vec{F} = 0} \qquad (5.1)$$

- **Force**: From the first law, we can define **force** as that which causes a change in motion of an object.
- **Mass:** we can define mass is that property of an object that specifies how much resistance an object exhibits to changes in its velocity. Mass is a scalar quantity. The SI unit of mass is the kilogram. Mass should not be confused with weight. Mass and weight are two different quantities. The mass of an object is the same everywhere.
- **Weight:** The weight of an object is equal to the magnitude of the gravitational force exerted on the object and varies with location.

  For example, a person weighing (84 kg) on the Earth weighs only about (14 kg) on the Moon, that means (1/6) his weighs on the Earth.

5.2 Newton's Second Law

Newton's first law explains what happens to an object when no forces act on it: it either remains at rest or moves in a straight line with constant speed. Newton's second law answers the question of what happens to an object when one or more forces act on it.

- The acceleration of an object is directly proportional to the force acting on it: $\vec{\mathbf{F}} \propto \vec{\mathbf{a}}$
- The magnitude of the acceleration of an object is inversely proportional to its mass: $|\vec{\mathbf{a}}| \propto \mathbf{1}/\boldsymbol{m}$

Newton's second law: The acceleration of an object is directly proportional to the net force acting on it and inversely proportional to its mass:

$$\vec{\mathbf{a}} \propto \frac{\Sigma \vec{\mathbf{F}}}{m}$$

If we choose a proportionality constant of 1, we can relate mass, acceleration, and force through the following mathematical statement of Newton's second law:

$$\Sigma \vec{\mathbf{F}} = m\,\vec{\mathbf{a}} \qquad \text{Newton's second law} \qquad (5.2)$$

- The net force ($\Sigma \vec{\mathrm{F}}$) on an object is the vector sum of all forces acting on the object.
- The SI unit of force is the **newton** (N).
- The definition of the newton is: A force of 1 N is the force that, when acting on an object of mass 1 kg, produces an acceleration of 1 m/s^2.

$$1\ \mathrm{N} = 1\ \mathrm{kg} \cdot \mathrm{m/s^2}$$

5.3 The Gravitational Force and Weight

All objects are attracted to the Earth. The attractive force exerted by the Earth on an object is called the **gravitational force** $\vec{F}_g$. This force is directed toward the center of the Earth, and its magnitude is called the **weight** of the object.

A freely falling object experiences an acceleration ($\vec{g}$) acting toward the center of the Earth. Applying Newton's second law $\boldsymbol{\Sigma}\vec{F} = m\,\vec{a}$ to a freely falling object of mass *m*, with $\vec{a} = \vec{g}$ and $\boldsymbol{\Sigma}\vec{F} = \vec{F}_g$, gives

$$\vec{\mathbf{F}}_g = \boldsymbol{m}\,\vec{\mathbf{g}} \qquad (5.3)$$

- The weight of an object is equal to *mg*: $F = m\,g$
- Because it depends on *g*, weight varies with geographic location. Because *g* decreases with increasing distance from the center of the Earth, objects weigh less at higher altitudes than at sea level.

5.4 Newton's Third Law

When your finger pushes on the book, the book pushes back on your finger. This important principle is known as **Newton's third law:**

(If two objects interact, the force $\vec{F}_{12}$ exerted by object 1 on object 2 is equal in magnitude and opposite in direction to the force $\vec{F}_{21}$ exerted by object 2 on object 1):

$$\vec{\mathbf{F}}_{12} = -\,\vec{\mathbf{F}}_{21} \qquad (5.4)$$

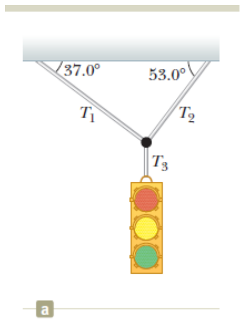
- The force that object 1 exerts on object 2 is popularly called the *action force,* and the force of object 2 on object 1 is called the *reaction force.*

- The action and reaction forces act on *different* objects and must be of the same type (gravitational, electrical, etc.).

Some Applications of Newton's laws:

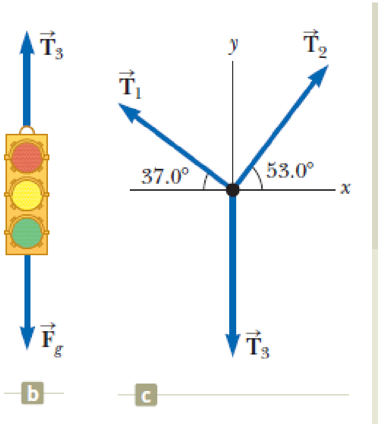
Example (5.1):

A traffic light weighing 122 N hangs from a cable tied to two other cables fastened to a support as in the figure a. The upper cables make angles of 37° and 53° with the horizontal. These upper cables are not as strong as the vertical cable and will break if the tension in them exceeds 100 N. Does the traffic light remain hanging in this situation, or will one of the cables break?



Solution:

We construct a diagram of the forces acting on the traffic light, shown in the figure b, and a free-body diagram for the knot that holds the three cables together, shown in the figure c. This knot is a convenient object to choose because all the forces of interest act along lines passing through the knot.



- Apply equation (5.1) for the traffic light in the *y* direction:

$$\sum F_y = 0 \rightarrow T_3 - F_g = 0$$
$$T_3 = F_g = 122 \text{ N}$$

Choose the coordinate axes as shown in the figure c and resolve the forces acting on the knot into their components:

Force	x Component	y Component
$\vec{T}_1$	$-T_1 \cos 37.0°$	$T_1 \sin 37.0°$
$\vec{T}_2$	$T_2 \cos 53.0°$	$T_2 \sin 53.0°$
$\vec{T}_3$	0	-122 N

Apply the particle in equilibrium model to the knot:

$$(1)\ \sum F_x = -T_1 \cos 37.0° + T_2 \cos 53.0° = 0$$

$$(2)\ \sum F_y = T_1 \sin 37.0° + T_2 \sin 53.0° + (-122\text{ N}) = 0$$

$$(3)\ T_2 = T_1\left(\frac{\cos 37.0°}{\cos 53.0°}\right) = 1.33T_1$$

Substitute this value for T_2 into equation (2):

$$T_1 \sin 37.0° + (1.33T_1)(\sin 53.0°) - 122\text{ N} = 0$$

$$T_1 = 73.4\text{ N}$$

$$T_2 = 1.33T_1 = 97.4\text{ N}$$

Both values are less than 100 N, so the cables will not break.

Example (5.2):

A car of mass *m* is on an icy driveway inclined at an angle u as in the figure a.

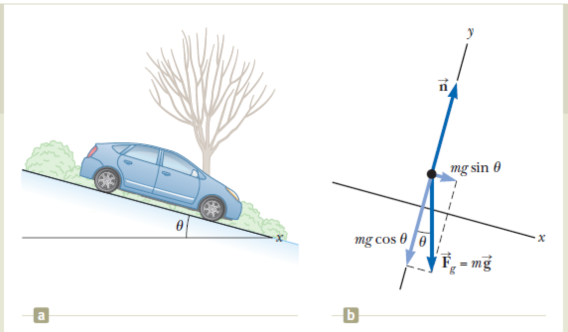
(A) Find the acceleration of the car, assuming the driveway is frictionless.

Solution:

$$(1)\ \sum F_x = mg \sin\theta = ma_x$$

$$(2)\ \sum F_y = n - mg\cos\theta = 0$$

$$(3)\ a_x = g\sin\theta$$



Note that the acceleration component a_x is independent of the mass of the car! It depends only on the angle of inclination and on g.

(B) Suppose the car is released from rest at the top of the incline and the distance from the car's front bumper to the bottom of the incline is d. How long does it take the front bumper to reach the bottom of the hill, and what is the car's speed as it arrives there?

Solution:

Apply equation: $x_f = x_i + v_{xi}t + \frac{1}{2} a_x t^2$

$x_f = d$, $x_i = 0$ and $v_{xi} = 0$ then $d = \frac{1}{2} a_x t^2$

Solve for t :

$$t = \sqrt{\frac{2d}{a_x}} = \sqrt{\frac{2d}{g \sin \theta}}$$

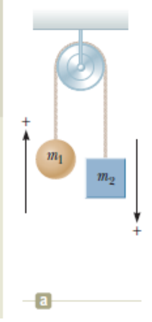
Use equation: $v_{xf}^2 = v_{xi}^2 + 2a_x (x_f - x_i)$

with $v_{xi} = 0$, to find the final velocity of the car:

$$v_{xf}^2 = 2\, a_x\, d$$

$$v_{xf} = \sqrt{2a_x d} = \sqrt{2gd \sin \theta}$$

Example (5.3):



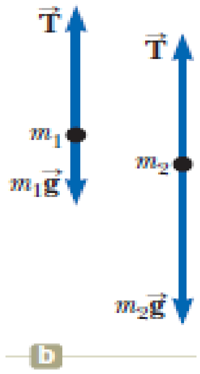
When two objects of unequal mass are hung vertically over a frictionless pulley of negligible mass as in the figure a, the arrangement is called an *Atwood machine.*

The device is sometimes used in the lab. to determine the value of g. Determine the magnitude of the acceleration of the two objects and the tension in the lightweight cord.

Solution:

The objects in the Atwood machine are subject to the gravitational force as well as to the forces exerted by the strings connected to them.

Two forces act on each object: the upward force $\vec{T}$ (tension) exerted by the string and the downward gravitational force.



Apply Newton's second law to object 1:

$$\sum F_y = T - m_1 g = m_1 a_y$$

Apply Newton's second law to object 2:

$$\sum F_y = m_2 g - T = m_2 a_y$$

$$-m_1 g + m_2 g = m_1 a_y + m_2 a_y$$

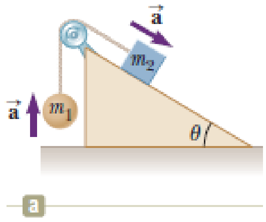
Solve for the acceleration: $a_y = \left(\frac{m_2 - m_1}{m_1 + m_2}\right) g$

To find the tension T of the string: $T = m_1(g + a_y) = \left(\frac{2 m_1 m_2}{m_1 + m_2}\right) g$

Example (5.4):

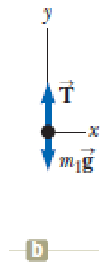
A ball of mass m_1 and a block of mass m_2 are attached by a lightweight cord that passes over a frictionless pulley of negligible mass as in the figure a. The block lies on a frictionless incline of angle θ. Find the magnitude of the acceleration of the two objects and the tension in the cord.

Solution:



If m_2 moves down the incline, then m_1 moves upward. Because the objects are connected by a cord (which we assume does not stretch), their accelerations have the same magnitude.

Apply Newton's second law in component form to the ball, choosing the upward direction as positive:

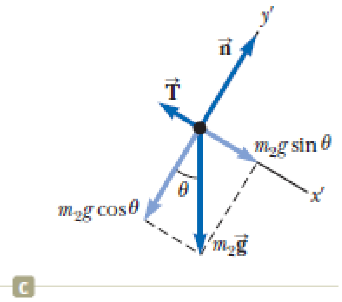


$$(1)\ \sum F_x = 0$$

$$(2)\ \sum F_y = T - m_1 g = m_1 a_y = m_1 a$$

For the ball to accelerate upward, it is necessary that $T > m_1 g$.

Apply Newton's second law in component form to the block:



$$(3)\ \sum F_{x'} = m_2 g \sin\theta - T = m_2 a_{x'} = m_2 a$$

$$(4)\ \sum F_{y'} = n - m_2 g \cos\theta = 0$$

We replaced ($a_{x'}$) with (a) because the two objects have accelerations of equal magnitude (a).

Solve equation (2) for T :

$$(5)\ T = m_1(g + a)$$

Substitute this expression for T into equation (3):

$$m_2 g \sin\theta - m_1(g + a) = m_2 a$$

Solve for (a):

$$(6)\ a = \left(\frac{m_2 \sin\theta - m_1}{m_1 + m_2}\right) g$$

Substitute this expression for (a) into equation (5) to find T :

$$(7)\ T = \left(\frac{m_1 m_2(\sin\theta + 1)}{m_1 + m_2}\right) g$$

5.5 Forces of Friction

When an object is in motion either on a surface or in a viscous medium such as air or water, there is resistance to the motion because the object interacts with its surroundings. We call such resistance a **force of friction.**

- If we apply an external horizontal force $\vec{F}$ to a block for example, acting to the right, the block can remains stationary when $\vec{F}$ is small. The force on the block that counteracts $\vec{F}$ and keeps it from moving acts toward the left and is called the **force of static friction** $\vec{f}_s$. As long as the block is not moving, $f_s = F$. Therefore, if $\vec{F}$ is increased, $\vec{f}_s$ also increases. Likewise, if $\vec{F}$ decreases, $\vec{f}_s$ also decreases.
- We call the friction force for an object in motion the **force of kinetic friction** $\vec{f}_k$.
- The magnitude of the force of static friction between any two surfaces in contact can have the values:

$$f_s \leq \mu_s n \qquad (5.5)$$

Where the dimensionless constant (μ_s) is called the **coefficient of static friction** and (n) is the magnitude of the normal force exerted by one surface on the other.

- The equality in equation (5.5) holds when the surfaces are on the verge of slipping, that is, when $f_s = f_{s,\max} = \mu_s n$. This situation is called *impending motion.*
- The inequality holds when the surfaces are not on the verge of slipping.
- The magnitude of the force of kinetic friction acting between two surfaces is:

$$f_k = \mu_k n \qquad (5.6)$$

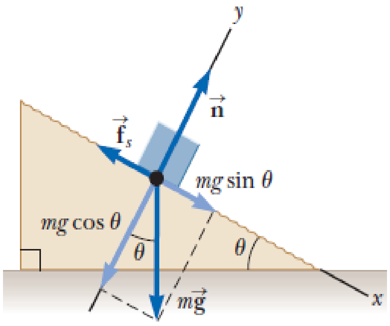
Where (μ_k) is the **coefficient of kinetic friction.**

- The values of μ_k and μ_s depend on the nature of the surfaces.
- **μ_k is generally less than μ_s.**

  Typical values range from around (0.03 to 1).
- The direction of the friction force on an object is parallel to the surface with which the object is in contact and opposite to the actual motion (kinetic friction) or the impending motion (static friction) of the object relative to the surface.

Example (5.5):

A block is placed on a rough surface inclined relative to the horizontal as shown in the figure. The incline angle is increased until the block starts to move. Show that you can obtain μ_s by measuring the critical angle θ at which this slipping just occurs?



Solution:

$$(1)\ \sum F_x = mg\sin\theta - f_s = 0$$

$$(2)\ \sum F_y = n - mg\cos\theta = 0$$

Substitute ($mg = n/\cos\theta$) from equation (2) into equation (1):

$$(3)\ f_s = mg\sin\theta = \left(\frac{n}{\cos\theta}\right)\sin\theta = n\tan\theta$$

When the incline angle is increased until the block is on the verge of slipping, the force of static friction has reached its maximum value $\mu_s n$.

$$\mu_s n = n\tan\theta$$

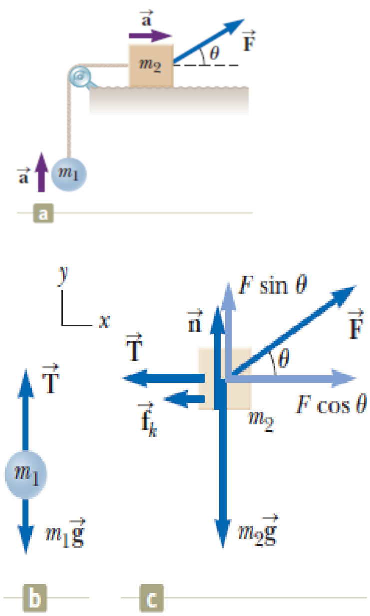
$$\mu_s = \tan\theta$$

Example (5.6):

A block of mass m_2 on a rough, horizontal surface is connected to a ball of mass m_1 by a lightweight cord over a lightweight, frictionless pulley as shown in the figure a. A force of magnitude F at an angle θ with the horizontal is applied to the block as shown, and the block slides to the right. The coefficient of kinetic friction between the block and surface is μ_k.

Determine the magnitude of the acceleration of the two objects.

Solution:



$$(1)\ \sum F_x = F\cos\theta - f_k - T = m_2 a_x = m_2 a$$

$$(2)\ \sum F_y = n + F\sin\theta - m_2 g = 0$$

$$(3)\ \sum F_y = T - m_1 g = m_1 a_y = m_1 a$$

Solve equation (2) for n:

$$n = m_2 g - F\sin\theta$$

Substitute (n) into $f_k = \mu_k n$:

$$(4)\ f_k = \mu_k(m_2 g - F\sin\theta)$$

Substitute equation (4) and the value of (T) from equation (3) into equation (1):

$$F\cos\theta - \mu_k(m_2 g - F\sin\theta) - m_1(a + g) = m_2 a$$

Solve for a:

$$(5)\ a = \frac{F(\cos\theta + \mu_k \sin\theta) - (m_1 + \mu_k m_2)g}{m_1 + m_2}$$

Chapter 6

(Uniform Circular Motion)

6.1 Particle in Uniform Circular Motion

If a car is moving on a circular path with *constant speed v*, we call it **uniform circular motion.** Even though an object moves at a constant speed in a circular path, *it still has acceleration*. To see why, consider the defining equation for acceleration,

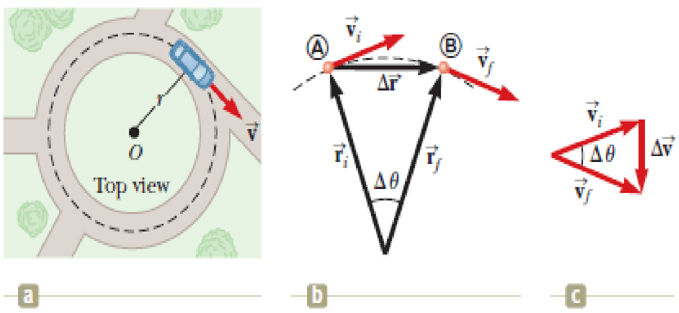
$$\vec{\mathbf{a}} = \frac{d\vec{\mathbf{v}}}{dt}$$

Notice that the acceleration depends on the change in the *velocity*. Because velocity is a vector quantity, acceleration can occur in two ways:

1- By a change in the *magnitude* of the velocity.

2- By a change in the *direction* of the velocity.

The constant-magnitude velocity vector is always tangent to the path of the object and perpendicular to the radius of the circular path. For *uniform* circular motion, the acceleration vector can only have a component perpendicular to the path, which is toward the center of the circle.



- Let us now find the magnitude of the acceleration of the particle.

The angle $\Delta\theta$ between the two position vectors in the figure (b) is the same as the angle between the velocities vectors in figure (c) because the velocity vector $\vec{v}$ is always perpendicular to the position vector $\vec{r}$. Therefore, the two triangles are *similar*; (Two triangles are similar if the angle between any two sides is the same for both triangles and if the ratio of the lengths of these sides is the same). We can now write a relationship between the lengths of the sides for the two triangles.

$$\frac{|\Delta \vec{v}|}{v} = \frac{|\Delta \vec{r}|}{r}$$

Where $v_i = v_f = v$ and $r = r_i = r_f$

The magnitude of the average acceleration over the time interval for the particle to move from A to B:

$$|\vec{a}_{avg}| = \frac{|\Delta \vec{v}|}{|\Delta t|} = \frac{v|\Delta \vec{r}|}{r \ \Delta t}$$

As A and B approach each other, Δt approaches zero, $|\Delta \vec{r}|$ approaches the distance traveled by the particle along the circular path, and the ratio $\frac{|\Delta \vec{r}|}{\Delta t}$ approaches the speed v. In addition, the average acceleration becomes the instantaneous acceleration at point A. Hence, in the limit $\Delta t \rightarrow 0$, the magnitude of the acceleration is:

$$a_c = \frac{v^2}{r}$$ **Centripetal acceleration** (6.1)

This acceleration is called a **centripetal acceleration** (*centripetal* means *center-seeking*) because $\vec{a}_c$ is directed toward the center of the circle. Furthermore, $\vec{a}_c$ is *always* perpendicular to $\vec{v}$.

In many situations, it is convenient to describe the motion of a particle moving with constant speed in a circle of radius r in terms of the **period** T, which is defined as (the time interval required for one complete revolution of the particle). In the time interval T, the particle moves a distance of $2\pi r$, which is equal to the circumference of the particle's circular path. Therefore, because its speed is equal to the circumference of the circular path divided by the period or $v = 2\pi r/T$, it follows that:

$$T = \frac{2\pi r}{v}$$ **Period of circular motion** (6.2)

Example (6.1):

What is the centripetal acceleration of the Earth as it moves in its orbit around the Sun?

Solution:

Combine equations (6.1) and (6.2):

$$a_c = \frac{v^2}{r} = \frac{\left(\frac{2\pi r}{T}\right)^2}{r} = \frac{4\pi^2 r}{T^2}$$

The period of the Earth's orbit, which we know is one year, and the radius of the Earth's orbit around the Sun, which is (1.496×10^{11} m).

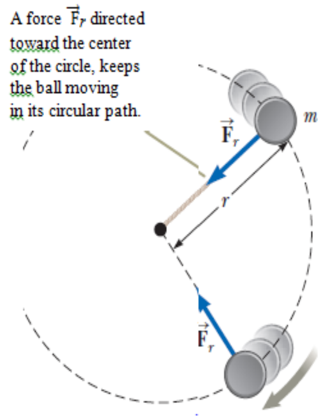
$$a_c = \frac{4\pi^2(1.496 \times 10^{11}\ \text{m})}{(1\ \text{yr})^2}\left(\frac{1\ \text{yr}}{3.156 \times 10^7\ \text{s}}\right)^2 = 5.93 \times 10^{-3}\ \text{m/s}^2$$

- Let us now consider a ball of mass *m* that is tied to a string of length *r* and moves at constant speed in a horizontal circular path:

According to Newton's first law, the ball would move in a straight line if there were no force on it; the string, however, prevents motion along a straight line by exerting on the ball a radial force $\vec{F}_r$ that makes it follow the circular path. This force is directed along the string toward the center of the circle. If Newton's second law is applied along the radial direction, the net force causing the centripetal acceleration can be related to the acceleration as follows:

$$\sum F = ma_c = m\frac{v^2}{r}$$ Centripetal force (6.3)

This force causing a centripetal acceleration acts toward the center of the circular path and causes a change in the direction of the velocity vector. If that force should vanish, the object would no longer move in its circular path; instead, it would move along a straight-line path tangent to the circle as shown in the figure.



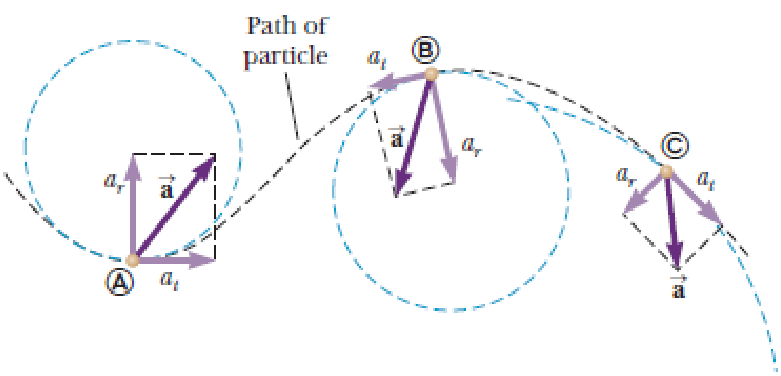
The magnitude of the centripetal force required to keep on the object in a circular path depends on the mass of the object and its acceleration.

6.2 Tangential and Radial Acceleration

Let us consider a more general motion:

A particle moves to the right along a curved path, and its velocity changes both in direction and in magnitude. In this situation, the velocity vector is always tangent to the path; the acceleration vector $\vec{a}$. The direction of the total acceleration vector $\vec{a}$ changes from point to point. At any instant, this vector can be resolved into two components as shown in the figure, based on an origin at the center of the dashed circle corresponding to that instant: a radial component a_r along the radius of the circle and a tangential component a_t perpendicular to this radius. The *total* acceleration vector $\vec{a}$ can be written as the vector sum of the component vectors:

$$\vec{a} = \vec{a}_r + \vec{a}_t \quad \text{Total acceleration} \quad (6.4)$$



Curved path

The tangential acceleration component causes a change in the speed v of the particle. This component is parallel to the instantaneous velocity, and its magnitude is given by:

$$\mathbf{a}_t = \left|\frac{dv}{dt}\right| \qquad \text{Tangential acceleration} \qquad (6.5)$$

The radial acceleration component arises from a change in direction of the velocity vector and is given by:

$$\mathbf{a}_r = -\,\mathbf{a}_c = -\frac{v^2}{r} \qquad \text{Radial acceleration} \qquad (6.6)$$

Where (r) is the radius of curvature of the path. The negative sign in equation (6.6) indicates that the direction of the centripetal acceleration is toward the center of the circle representing the radius of curvature. The direction is opposite that of the radial unit vector $\vec{r}$, which always points away from the origin at the center of the circle.

Because $\vec{a}_r$ and $\vec{a}_t$ are perpendicular component vectors of $\vec{a}$,

it follows that the magnitude of $\vec{a}$ is: $\quad a = \sqrt{a_r^2 + a_t^2}.$

At a given speed, a_r is **large** when the radius of curvature **(r)** is **small** (as at points A and B) in the figure, and **small** when **(r)** is **large** (as at point C). The direction of $\vec{a}_t$ is either in the same direction as $\vec{v}$ (if v is increasing) or opposite $\vec{v}$ (if v is decreasing, as at point B).

- In uniform circular motion, where v is constant, $a_t = 0$ and the acceleration is always **completely radial.** In other words, uniform circular motion is a special case of motion along a general curved path.

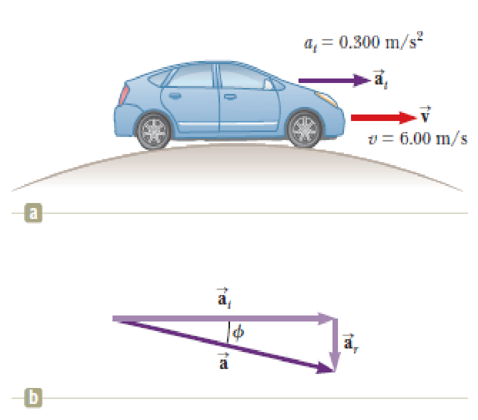
Example (6.2):

A car exhibits a constant acceleration of 0.3 m/s^2 parallel to the roadway. The car passes over a rise in the roadway such that the top of the rise is shaped like a circle of radius 500 m. At the moment the car is at the top of the rise, its velocity vector is horizontal and has a magnitude of

(6 m/s). What are the magnitude and direction of the total acceleration vector for the car at this instant?

Solution:

Because the accelerating car is moving along a curved path, the car has both tangential and radial acceleration.



The radial acceleration vector is directed straight downward, and the tangential acceleration vector has magnitude (0.3m/s^2) and is horizontal.

The radial acceleration:

$$a_r = -\frac{v^2}{r} = -\frac{(6)^2}{500} = -0.072\ m/s^2$$

The total acceleration is:

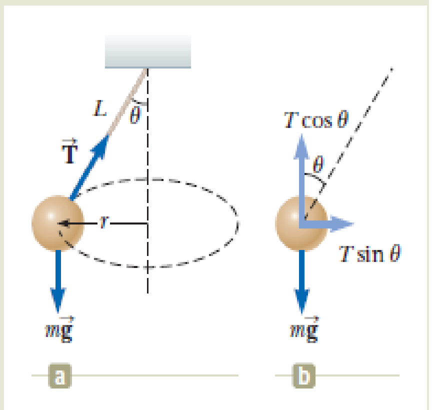
$$a = \sqrt{a_r^2 + a_t^2} = \sqrt{(-0.072)^2 + (0.3)^2} = 0.309\ m/s^2$$

The direction of the total acceleration vector is:

$$\emptyset = \tan^{-1}\frac{a_r}{a_t} = \tan^{-1}\frac{(-0.072)}{(0.3)} = -13.5°$$

Example (6.3):

A small ball of mass *m* is suspended from a string of length *L*. The ball revolves with constant speed *v* in a horizontal circle of radius *r* as shown in the figure. Find an expression for *v*.



Solution:

Let θ represents the angle between the string and the vertical.

The force $\vec{T}$ exerted by the string on the ball is resolved into a vertical component ($T\cos\theta$) and a horizontal component ($T\sin\theta$) acting toward the center of the circular path.

Apply the particle in equilibrium model in the vertical direction:

$$\sum F_y = T\cos\theta - mg = 0$$

$$(1) \quad T\cos\theta = mg$$

Use equation (6.3) in the horizontal direction:

$$(2) \quad \sum F_x = T\sin\theta = ma_c = \frac{mv^2}{r}$$

Divide equation (2) by equation (1) and use $\sin\theta/\cos\theta = \tan\theta$:

$$\tan\theta = \frac{v^2}{rg}$$

Solve for *v*:

$$v = \sqrt{rg\tan\theta}$$

Since (*r* = *L* sin)

$$v = \sqrt{Lg\sin\theta\tan\theta}$$

Notice that the speed is independent of the mass of the ball.

Example (6.4):

A car moving on a flat, horizontal road negotiates a curve as shown in the figure a. If the radius of the curve is 35 m and the coefficient of static friction between the tires and dry pavement is 0.523, find the maximum speed the car can have and still make the turn successfully.

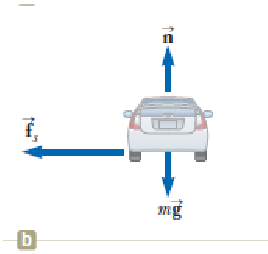
Solution:

Imagine that the curved roadway is part of a large circle so that the car is moving in a circular path.

The force that enables the car to remain in its circular path is the force of static friction. (It is *static* because no slipping occurs at the point of contact between road and tires). The maximum speed v_{max} the car can have around the curve is the speed at which it is on the verge of skidding outward. At this point, the friction force has its maximum value $f_{s,max}=\mu_s n$.

$$(1)\quad f_{s,\max} = \mu_s n = m\frac{v_{\max}^2}{r}$$

$$\sum F_y = 0 \;\rightarrow\; n - mg = 0 \;\rightarrow\; n = mg$$



Solve equation (1) for the maximum speed and substitute for n:

$$(2)\quad v_{\max} = \sqrt{\frac{\mu_s n r}{m}} = \sqrt{\frac{\mu_s m g r}{m}} = \sqrt{\mu_s g r}$$

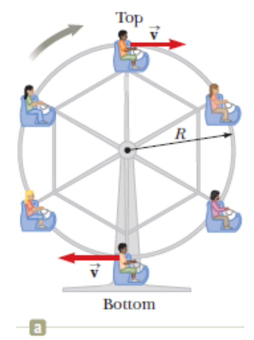
$$v_{\max} = \sqrt{(0.523)(9.80\ \text{m/s}^2)(35.0\ \text{m})} = 13.4\ \text{m/s}$$

Notice that the maximum speed does not depend on the mass of the car.

Example (6.5):

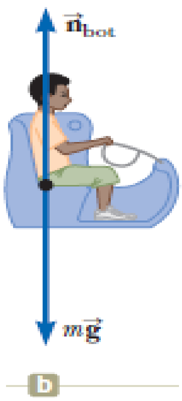
Child of mass m rides on a wheel as shown in the figure a. The child moves in a vertical circle of radius (10 m) at a constant speed of (3 m/s).

(A) Determine the force exerted by the seat on the child at the bottom of the ride.



Solution:

We draw a diagram of forces acting on the child at the bottom of the ride as shown in the figure b. The only forces acting on him are the downward gravitational force $\vec{F}_g = m\vec{g}$ and the upward force $\vec{n}_{bot}$ exerted by the seat. The net upward force on the child that provides his centripetal acceleration has a magnitude ($\vec{n}_{bot} - mg$). Apply Newton's second law to the child in the radial direction:



$$\sum F = n_{bot} - mg = m\frac{v^2}{r}$$

$$n_{bot} = mg + m\frac{v^2}{r} = mg\left(1 + \frac{v^2}{rg}\right)$$

$$n_{bot} = mg\left[1 + \frac{(3.00 \text{ m/s})^2}{(10.0 \text{ m})(9.80 \text{ m/s}^2)}\right]$$

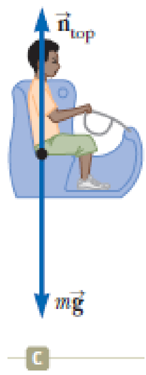
$$= 1.09\, mg$$

Hence, the magnitude of the force $\vec{n}_{bot}$ exerted by the seat on the child is *greater* than the weight of the child by a factor of (1.09).

(B) Determine the force exerted by the seat on the child at the top of the ride.

Solution:

The diagram of forces acting on the child at the top of the ride is shown in the figure c. The net downward force that provides the centripetal acceleration has a magnitude ($mg - \vec{n}_{top}$).



Apply Newton's second law to the child at this position:

$$\sum F = mg - n_{top} = m\frac{v^2}{r}$$

$$n_{top} = mg - m\frac{v^2}{r} = mg\left(1 - \frac{v^2}{rg}\right)$$

$$n_{top} = mg\left[1 - \frac{(3.00 \text{ m/s})^2}{(10.0 \text{ m})(9.80 \text{ m/s}^2)}\right]$$

$$= 0.908\, mg$$

In this case, the magnitude of the force exerted by the seat on the child is *less* than his true weight by a factor of 0.908, and the child feels lighter.

6.3 Gravitation

Newton's Law of Universal Gravitation:

Newton's law of universal gravitation states that:

(Every particle in the Universe attracts every other particle with a force that is directly proportional to the product of their masses and inversely proportional to the square of the distance between them).

$$F_g = G\frac{m_1 m_2}{r^2}$$ Gravitational force (6.7)

Where G is a constant, called the universal gravitational constant:

$$G = 6.67 \times 10^{-11}\ \text{N.m}^2/\ \text{kg}^2$$

Free-Fall Acceleration and the Gravitational Force

The magnitude of the gravitational force on an object near the Earth's surface is called the *weight* of the object:

$$mg = G\frac{M_E m}{{R_E}^2}$$

$$g = G\frac{M_E}{{R_E}^2} \qquad (6.8)$$

Where M_E is the Earth's mass and R_E is its radius.

According to equation (6.8), we see that the free - fall acceleration (g) near the Earth's surface is constant since the other quantities in this equation are also constants.

Example (6.6):

The surface of the Earth is approximately (6400 km) from its center and its mass is (6×10^{24} kg), what is the acceleration due to gravity (g) near the surface?

Solution:

Apply equation (6.8): $\boldsymbol{g} = \frac{6.67\times10^{-11}\times6\times10^{24}}{(6.4\times10^{6})^{2}}$

$$\boldsymbol{g = 9.8\ m/s^2}$$

- Now consider an object of mass m located a distance h above the Earth's surface or a distance r from the Earth's center,

  Where $(\boldsymbol{r = R_E + h})$. The magnitude of the gravitational force acting on this object is:

$$F_g = G\frac{M_E m}{r^2} = G\frac{M_E m}{(R_E + h)^2}$$

The magnitude of the gravitational force acting on the object at this position is also $F_g = mg$, where g is the value of the free-fall acceleration at the altitude h. Substituting this expression for F_g into the last equation shows that g is given by:

$$g = \frac{GM_E}{r^2} = \frac{GM_E}{(R_E + h)^2} \qquad \textbf{Variation of } \boldsymbol{g} \textbf{ with altitude} \qquad (6.9)$$

Therefore, it follows that g *decreases* with *increasing altitude.*

Chapter 7

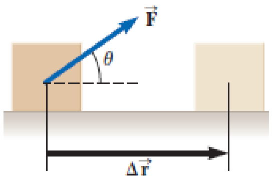
(Work and Energy)

7.1 Work Done by a Constant Force

The **work** W done on a system by exerting a constant force on the system is (the product of the magnitude F of the force, the magnitude Δr of the displacement of the point of application of the force, and $\cos\theta$, where θ is the angle between the force and displacement vectors):

$$W \equiv F \Delta r \cos\theta \quad \text{Work done by a constant force} \quad (7.1)$$

Work is a scalar, even though it is defined in terms of two vectors; a force $\vec{F}$ and a displacement $\Delta\vec{r}$.



- A force does no work on an object if the force does not move through a displacement, that is if $\Delta r = 0$ then W=0.
- If the work done by a force on a moving object is zero when the force applied is perpendicular to the displacement of its point of application. That is, if $\theta = 90°$, then $W = 0$ because $\cos 90° = 0$.
- The sign of the work depends on the direction of $\vec{F}$ relative to $\Delta\vec{r}$.

The work done by the applied force on a system is positive when the projection of $\vec{F}$ onto $\Delta\vec{r}$ is in the same direction as the displacement. For example, when an object is lifted, the work done by the applied force on the object is positive because the direction of that force is upward, in the same direction as the displacement of its point of application.

When the projection of $\vec{F}$ onto $\Delta\vec{r}$ is in the direction opposite the displacement, W is negative. For example, as an object is lifted, the work done by the gravitational force on the object is negative.

If an applied force $\vec{F}$ is in the same direction as the displacement $\Delta\vec{r}$, then θ = 0 and cos 0 = 1. In this case, equation (7.1) gives:

$$W = F\,\Delta r$$

The units of work are those of force multiplied by those of length. Therefore, the SI unit of work is the **newton . meter** (N . m = kg . m^2/s^2). This combination of units is given a name, the **joule** (J).

- Work is an energy transfer. If *W* is the work done on a system and *W* is positive, energy is transferred *to* the system; if *W* is negative, energy is transferred *from* the system.

7.2 Work Done by a Varying Force

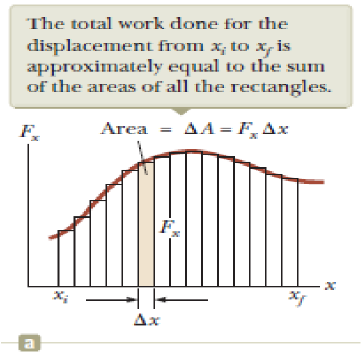
Consider a particle being displaced along the *x* - axis under the action of a force that varies with position. The particle is displaced in the direction of increasing *x* from $x = x_i$ to $x = x_f$. In such a situation, we cannot use ($W = F\,\Delta r\cos\theta$) to calculate the work done by the force because this relationship applies only when $\vec{F}$ is constant in magnitude and direction. If, however, we imagine that the particle undergoes a very small displacement Δx, the *x* component F_x of the force is approximately constant over this small interval; for this small displacement, we can approximate the work done on the particle by the force as:

$$W \approx F_x\,\Delta x$$

Which is the area of the shaded rectangle in the figure a. If we imagine the F_x versus *x* curve divided into a large number of such intervals, the total work done for the displacement from x_i to x_f is approximately equal to the sum of a large number of such terms:

$$W \approx \sum_{x_i}^{x_f} F_x\,\Delta x$$

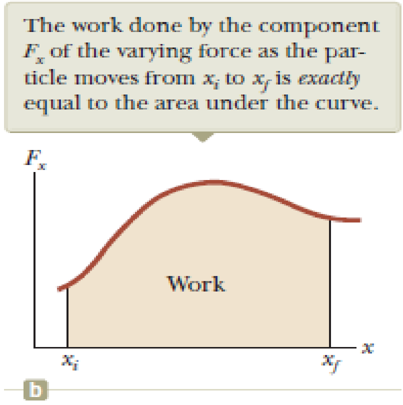
$$\lim_{\Delta x \to 0} \sum_{x_i}^{x_f} F_x\,\Delta x = \int_{x_i}^{x_f} F_x\,dx$$



Therefore, we can express the work done by F_x on the particle as it moves from x_i to x_f as:

$$W = \int_{x_i}^{x_f} F_x \, dx \qquad (7.2)$$

This equation reduces to equation (7.1) when the component $F_x = F \cos \theta$ remains constant.



Work done by a spring

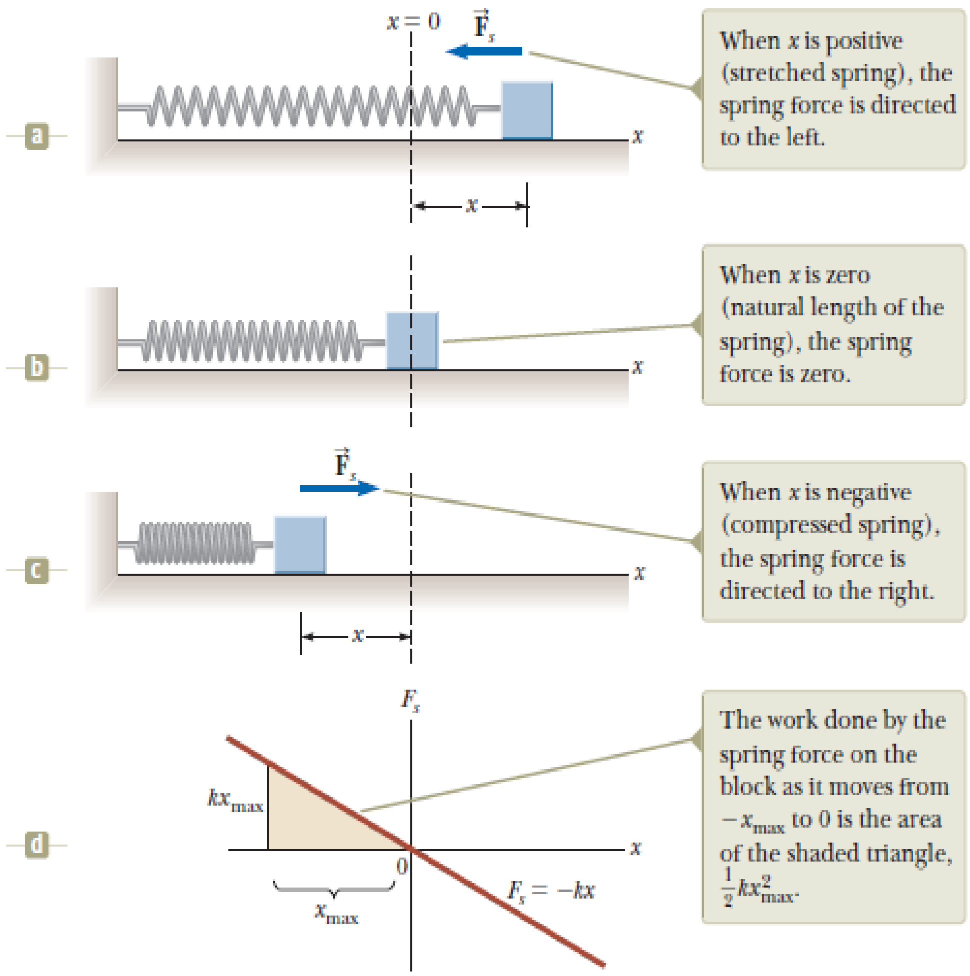
A model of a common physical system on which the force varies with position is shown in the figure.

The system is a block on a frictionless, horizontal surface and connected to a spring. For many springs, if the spring is either stretched or compressed a small distance from its unstretched (equilibrium) configuration, it exerts on the block a force that can be mathematically modeled as:

$$F_s = -kx$$

Spring force (**Hooke's law**) (7.3)

Where x is the position of the block relative to its equilibrium ($x = 0$) position and k is a positive constant called the **force constant** or the **spring constant** of the spring.



- The force required to stretch or compress a spring is proportional to the amount of stretch or compression *x*. This force law for springs is known as **Hooke's law.**

The value of *k* is a measure of the *stiffness* of the spring. Stiff springs have large *k* values, and soft springs have small *k* values. The units of *k* are N/m.

- The negative sign in equations (7.3) signifies that the force exerted by the spring is always directed ***opposite*** the displacement from equilibrium.

- When $x > 0$ as in the (figure a) so that the block is to the right of the equilibrium position, the spring force is directed to the left, in the negative x direction. When $x < 0$ as in the (figure c), the block is to the left of equilibrium and the spring force is directed to the right, in the positive x direction. When $x = 0$ as in the (figure b), the spring is unstretched and $F_s = 0$. Because the spring force always acts toward the equilibrium position ($x = 0$), it is sometimes called a ***restoring force***.
- The work W_s done by the spring force on the block as the block moves from $x_i = - x_{max}$ to $x_f = 0$:

$$W_s = \int_{-x_{\max}}^{0} (-kx)\, dx = \tfrac{1}{2}kx_{\max}^2 \qquad (7.4)$$

The work done by the spring force is positive because the force is in the same direction as its displacement (both are to the right).

- If the block undergoes a displacement from $x = x_i$ to $x = x_f$, the work done by the spring force on the block is:

$$W_s = \int_{x_i}^{x_f} (-kx)\, dx = \tfrac{1}{2}kx_i^2 - \tfrac{1}{2}kx_f^2$$ **Work done by a spring** (7.5)

We see that the work done by the spring force is zero for any motion that ends where it began ($x_i = x_f$).

Example (7.1):

A spring is hung vertically, and an object of mass m is attached to its lower end. The spring stretches a distance d from its equilibrium position.

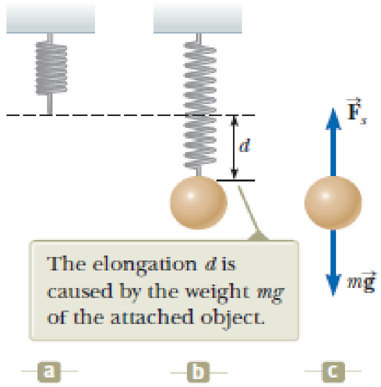
(A) If a spring is stretched 2 cm by a suspended object having a mass of 0.55 kg, what is the force constant of the spring?

(B) How much work is done by the spring on the object as it stretches through this distance?

Solution:

(A) Because the object is in equilibrium, the net force on it is zero and the upward spring force balances the downward gravitational force.

$$\vec{\mathbf{F}}_s + m\vec{\mathbf{g}} = 0 \;\rightarrow\; F_s - mg = 0 \;\rightarrow\; F_s = mg$$



Apply Hooke's law to give $F_s = kd$ and solve for k:

$$k = \frac{mg}{d} = \frac{(0.55\text{ kg})(9.80\text{ m/s}^2)}{2.0\times10^{-2}\text{ m}} = 2.7\times10^{2}\text{ N/m}$$

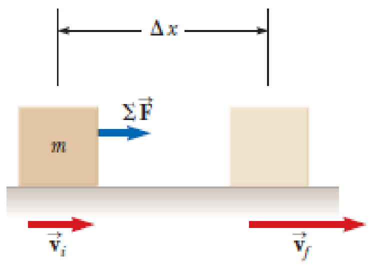
(B) To find the work done by the spring on the object:

$$W_s = 0 - \tfrac{1}{2}kd^2 = -\tfrac{1}{2}(2.7\times10^{2}\text{ N/m})(2.0\times10^{-2}\text{ m})^2$$
$$= -5.4\times10^{-2}\text{ J}$$

7.3 Kinetic Energy and the Work–Kinetic Energy Theorem

Consider a system consisting of a single object. The figure shows a block of mass m moving through a displacement directed to the right under the action of a net force $\vec{F}$, also directed to the right. We know from Newton's second law that the block moves with an acceleration $\vec{a}$. If the block and (therefore the force) moves through a displacement $\Delta\vec{\mathbf{r}} = \Delta x\hat{\mathbf{i}} = (x_f - x_i)\hat{\mathbf{i}}$, the net work done on the block by the external net force $\Sigma\vec{F}$ is:

$$W_{\text{ext}} = \int_{x_i}^{x_f} \sum F\,dx \qquad (7.6)$$



Using Newton's second law, we substitute for the magnitude of the net force $\Sigma\vec{F} = ma$.

$$W_{\text{ext}} = \int_{x_i}^{x_f} ma\,dx = \int_{x_i}^{x_f} m\frac{dv}{dt}\,dx = \int_{x_i}^{x_f} m\frac{dv}{dx}\frac{dx}{dt}\,dx = \int_{v_i}^{v_f} mv\,dv$$

$$W_{\text{ext}} = \tfrac{1}{2}mv_f^{\,2} - \tfrac{1}{2}mv_i^{\,2} \qquad (7.7)$$

Where v_i is the speed of the block when it is at $x = x_i$ and v_f is its speed at x_f.

- The quantity ($\frac{1}{2}mv^2$) represents the energy associated with the motion of the particle and it is called (Kinetic energy).

$K \equiv \frac{1}{2}mv^2$ Kinetic energy (7.8)

- Kinetic energy is a scalar quantity and has the same units as work.
- Equation 7.7 states that the work done on a particle by a net force $\vec{F}$ acting on it equals the change in kinetic energy of the particle. It is often convenient to write equation 7.7 in the form:

$$W_{ext} = K_f - K_i = \Delta K \qquad (7.9)$$

Another way to write it is $K_f = K_i + W_{ext}$ which tells us that the final kinetic energy of an object is equal to its initial kinetic energy plus the change in energy due to the net work done on it.

Equation 7.9 is an important result known as the

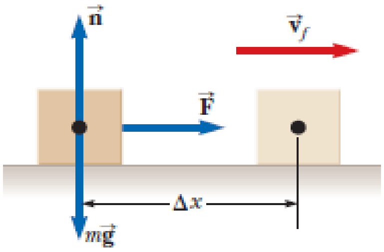
Work–kinetic energy theorem: (When work is done on a system and the only change in the system is in its speed, the net work done on the system equals the change in kinetic energy of the system).

- The work–kinetic energy theorem indicates that the speed of a system *increases* if the net work done on it is *positive* because the final kinetic energy is greater than the initial kinetic energy. The speed *decreases* if the net work is *negative* because the final kinetic energy is less than the initial kinetic energy.

Example (7.2):

A 6.0-kg block initially at rest is pulled to the right along a frictionless, horizontal surface by a constant horizontal force of 12 N. Find the block's speed after it has moved 3.0 m.

Solution:



The normal force balances the gravitational force on the block, and neither of these vertically acting forces does work on the block because their points of application are horizontally displaced.

The net external force acting on the block is the horizontal 12-N force.

Use the work–kinetic energy theorem for the block, noting that its initial kinetic energy is zero:

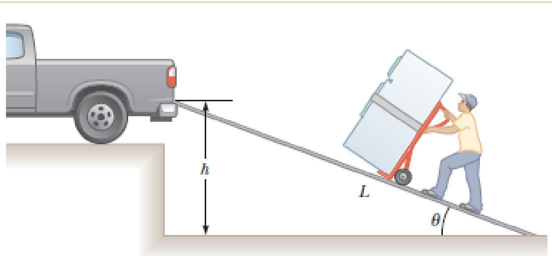
$$W_{ext} = K_f - K_i = \tfrac{1}{2}mv_f^2 - 0 = \tfrac{1}{2}mv_f^2$$

$$v_f = \sqrt{\frac{2W_{ext}}{m}} = \sqrt{\frac{2F\Delta x}{m}}$$

$$v_f = \sqrt{\frac{2(12\text{ N})(3.0\text{ m})}{6.0\text{ kg}}} = \boxed{3.5\text{ m/s}}$$

Example (7.3):

A man wishes to load a refrigerator onto a truck using a ramp at angle θ as shown in the figure. He claims that less work would be required to load the truck if the length L of the ramp were increased. Is his claim valid?



Solution:

The normal force exerted by the ramp on the system is directed at 90° to the displacement of its point of application and so does no work on the system. Because $\Delta K = 0$, the work–kinetic energy theorem gives:

$$W_{ext} = W_{by\ man} + W_{by\ gravity} = 0$$

The work done by the gravitational force equals the product of [the weight (mg) of the system, the distance (L) through which the refrigerator is displaced, and cos (θ + 90°)]. Therefore,

$$W_{\text{by man}} = -W_{\text{by gravity}} = -(mg)(L)[\cos(\theta + 90°)]$$
$$= mgL\sin\theta = mgh$$

Where ($h = L \sin\theta$) is the height of the ramp. Therefore, the man must do the same amount of work (*mgh*) on the system *regardless* of the length of the ramp. The work depends only on the height of the ramp.

7.4 Power

The time rate at which work is done by a force is said to be the **power** due to the force. If a force does an amount of work W in an amount of time Δt, the **average power** due to the force during that time interval is:

$$P_{\text{avg}} = \frac{W}{\Delta t} \quad \text{(average power).}$$

The SI unit of power is joules per second (J/s), also called the **watt** (W)

$$1\ \text{W} = 1\ \text{J/s} = 1\ \text{kg} \cdot \text{m}^2/\text{s}^3$$

Another unit of power is **horsepower** (hp): 1 hp = 746 W

Chapter 8

(Conservation of Energy)

8.1 Potential Energy of a System

We call the energy storage mechanism before the object is released **potential energy.** The amount of potential energy in the system is determined by the *configuration* of the system. The work represents a transfer of energy into the system and the system energy appears in a different form, which we have called potential energy. Therefore, we can identify the quantity (mgy) as the **gravitational potential energy** U_g:

$$U_g = mgy \quad \textbf{Gravitational potential energy} \qquad (8.1)$$

Where (y) is the height above the ground.

The units of gravitational potential energy are joules, the same as the units of work and kinetic energy. Potential energy, like work and kinetic energy, is a scalar quantity.

8.2 Conservative and Non-conservative Forces

Conservative Forces

Conservative forces have these two equivalent properties:

1. The work done by a conservative force on a particle moving between any two points is independent of the path taken by the particle.

2. The work done by a conservative force on a particle moving through any closed path is zero. (A closed path is one for which the beginning point and the endpoint are identical.)

The gravitational force is one example of a conservative force; the force that an ideal spring exerts on any object attached to the spring is another.

Non conservative Forces

A force is **non-conservative** if it does not satisfy properties 1 and 2 for conservative forces. We define the sum of the kinetic and potential energies of a system as the **mechanical energy** of the system:

$$E_{mech} = K + U \qquad (8.2)$$

Where *K* is the kinetic energy of the system and *U* is the potential energy in the system.

- The force of kinetic friction is a non-conservative force.

8.3 Relationship between Conservative Forces and Potential Energy

A **potential energy function *U*** is defined as (the work done within the system by the conservative force equals the decrease in the potential energy of the system).

The work done by the force **F** as the particle moves along the *x* axis is:

$$W_{int} = \int_{x_i}^{x_f} F_x\,dx = -\Delta U \qquad (8.3)$$

Where F_x is the component of $\vec{F}$ in the direction of the displacement.

We can also express equation (8,3) as:

$$\Delta U = U_f - U_i = -\int_{x_i}^{x_f} F_x\,dx$$

We can then define the potential energy function as:

$$U_f(x) = -\int_{x_i}^{x_f} F_x\,dx + U_i$$

The value of U_i is often taken to be zero.

$$dU = -F_x\,dx$$

Therefore, the conservative force is related to the potential energy function through the relationship:

$$F_x = -\frac{dU}{dx}$$ **Relationship between Conservative Forces and Potential Energy** 8.4)

- The potential energy for a spring is:

$$U_s \equiv \tfrac{1}{2}kx^2$$ Elastic potential energy (8.5)

The elastic potential energy of the system can be thought of as the energy stored in the deformed spring (one that is either compressed or stretched from its equilibrium position). The elastic potential energy stored in a spring is zero whenever the spring is undeformed ($x = 0$). Energy is stored in the spring only when the spring is either stretched or compressed. Because the elastic potential energy is proportional to x^2, we see that U_s is always positive in a deformed spring.

- In the case of the deformed spring:

$$F_s = -\frac{dU_s}{dx} = -\frac{d}{dx}\left(\tfrac{1}{2}kx^2\right) = -kx$$

This is corresponding to the restoring force in the spring (Hooke's law).

$F_s = - k\,x$ Hooke's law

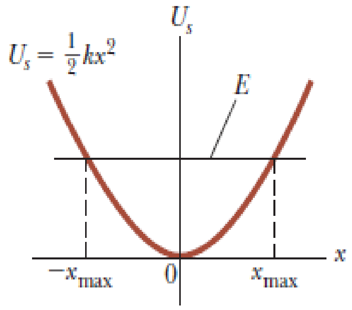
8.4 Potential Energy Diagram

Consider the potential energy function for a block–spring system, given by: $U_s \equiv \tfrac{1}{2}kx^2$

This function is plotted versus x in the figure. The force F_s exerted by the spring on the block is related to U_s through equation:

$$F_s = -\frac{dU_s}{dx} = -kx$$ (8.6)

This means that the x component of the force is equal to the negative of the slope of the U_s versus x curve.



8.5 Conservation of Energy

The general statement of the principle of **conservation of energy** can be described mathematically with the **conservation of energy equation** as follows:

$$\Delta E_{system} = \sum T \qquad (8.7)$$

Where E_{system} is the total energy of the system, including all methods of energy storage (kinetic, potential, and internal), and T (for *transfer*) is the amount of energy transferred across the system boundary by some mechanism.

We can express the conservation of energy of the system as:

$$\Delta E_{system} = 0 \qquad (8.8)$$

Therefore, $\Delta K + \Delta U = 0$

Or $$(K_f - K_i) + (U_f - U_i) = 0$$

$$K_f + U_f = K_i + U_i \qquad (8.9)$$

Example (8.1):

A ball of mass m is dropped from a height h above the ground as shown in the figure.

(A) Neglecting air resistance, determine the speed of the ball when it is at a height y above the ground.

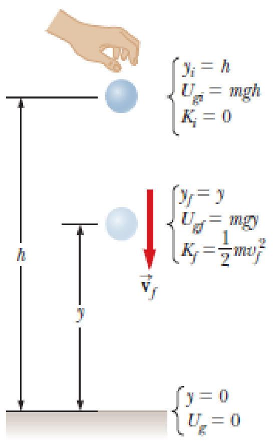
Solution: (A) At the instant the ball is released, its kinetic energy is

$K_i = 0$ and the gravitational potential energy of the system is $U_{gi} = mgh$. When the ball is at a position y above the ground, its kinetic energy is

$$K_f = 1/2\, mv_f^{\,2}$$

and the potential energy relative to the ground is

$$U_{gf} = mgy.$$



Apply equation (8.9):

$$K_f + U_{gf} = K_i + U_{gi}$$

$$\tfrac{1}{2}mv_f^{\,2} + mgy = 0 + mgh$$

$$v_f^2 = 2g(h - y) \quad \rightarrow \quad v_f = \boxed{\sqrt{2g(h - y)}}$$

(B): Determine the speed of the ball at y if at the instant of release it already has an initial upward speed v_i at the initial altitude h.

Solution:

In this case, the initial energy includes kinetic energy equal to $1/2\, mv_i^{\,2}$.

$$\tfrac{1}{2}mv_f^{\,2} + mgy = \tfrac{1}{2}mv_i^{\,2} + mgh$$

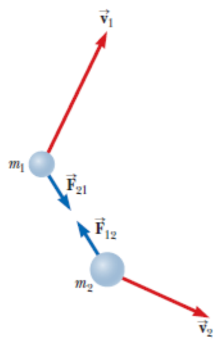
$$v_f^{\,2} = v_i^{\,2} + 2g(h - y) \quad \rightarrow \quad v_f = \boxed{\sqrt{v_i^{\,2} + 2g(h - y)}}$$

Chapter 9

(Linear Momentum and Collisions)

9.1 Linear Momentum

Consider an isolated system of two particles as in the figure, with masses m_1 and m_2 moving with velocities $\vec{v}_1$ and $\vec{v}_2$ at an instant of time. Because the system is isolated, the only force on one particle is that from the other particle. If a force from particle 1 (for example, a gravitational force) acts on particle 2, there must be a second force—equal in magnitude but opposite in direction—that particle 2 exerts on particle 1. That is, the forces on the particles form



Newton's third law action–reaction pair, and $\vec{F}_{12} = -\vec{F}_{21}$.

We can express this condition as:

$$\vec{F}_{12} + \vec{F}_{21} = 0$$

The interacting particles in the system have accelerations corresponding to the forces on them.

Therefore, replacing the force on each particle with $m_1\vec{a}$ for the particle gives:

$$m_1\vec{a}_1 + m_2\vec{a}_2 = 0$$

Now we replace each acceleration with its definition:

$$m_1\frac{d\vec{\mathbf{v}}_1}{dt} + m_2\frac{d\vec{\mathbf{v}}_2}{dt} = 0$$

If the masses m_1 and m_2 are constant, we can bring them inside the derivative operation, which gives:

$$\frac{d(m_1\vec{\mathbf{v}}_1)}{dt} + \frac{d(m_2\vec{\mathbf{v}}_2)}{dt} = 0$$

$$\frac{d}{dt}(m_1\vec{\mathbf{v}}_1 + m_2\vec{\mathbf{v}}_2) = 0 \qquad (9.1)$$

Notice that the derivative of the sum $m_1\vec{v}_1 + m_2\vec{v}_2$ with respect to time is zero. Consequently, this sum must be constant.

We call the quantity $m\vec{v}$ of a particle as (*linear momentum*).

- The **linear momentum** of a particle or an object that can be modeled as a particle of mass *m* moving with a velocity $\vec{v}$ is defined to be the product of the mass and velocity of the particle:

$$\vec{p} = m\,\vec{v} \quad \textbf{linear momentum} \qquad (9.2)$$

- Linear momentum is a vector quantity because it equals the product of a scalar quantity *m* and a vector quantity $\vec{v}$. Its direction is along $\vec{v}$, and its SI unit is kg . m/s.
- Using Newton's second law of motion, we can relate the linear momentum of a particle to the resultant force acting on the particle. We start with Newton's second law and substitute the definition of acceleration:

$$\sum \vec{F} = m\vec{a} = m\frac{d\vec{v}}{dt}$$

In Newton's second law, the mass *m* is assumed to be constant. Therefore, we can bring *m* inside the derivative operation to give us:

$$\sum \vec{F} = \frac{d(m\vec{v})}{dt} = \frac{d\vec{p}}{dt} \qquad \text{Newton's second law for a particle} \quad (9.3)$$

This equation shows that **the time rate of change of the linear momentum of a particle is equal to the net force acting on the particle**.

- Using the definition of momentum, equation (9.1) can be written:

$$\frac{d}{dt}(\vec{p}_1 + \vec{p}_2) = 0$$

Because the time derivative of the total momentum $\vec{p}_{tot} = \vec{p}_1 + \vec{p}_2$ is *zero*, we conclude that the *total* momentum must remain constant:

$$\vec{p}_{tot} = \text{constant} \qquad (9.4)$$

or, equivalently, $\vec{p}_{1i} + \vec{p}_{2i} = \vec{p}_{1f} + \vec{p}_{2f}$ (9.5)

9.2 Impulse

Let us assume a net force $\Sigma\vec{F}$ acts on a particle and this force may vary with time. According to Newton's second law, $\Sigma\vec{F} = d\vec{p}/dt$, or

$$d\vec{p} = \Sigma\vec{F}\, dt \qquad (9.6)$$

We can integrate this expression to find the change in the momentum of a particle when the force acts over some time interval.

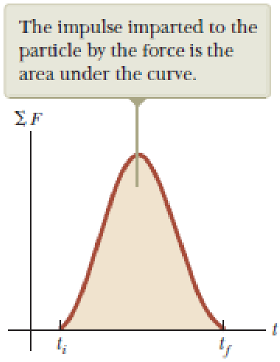
If the momentum of the particle changes from $\vec{p}_i$ at time t_i to $\vec{p}_f$ at time t_f, integrating equation (9.6) gives:

$$\Delta\vec{p} = \vec{p}_f - \vec{p}_i = \int_{ti}^{tf} \Sigma\vec{F}\, dt \qquad (9.7)$$

- The quantity on the right side of this equation is a vector called the **impulse** of the net force $\Sigma\vec{F}$ acting on a particle over the time interval $\Delta t = t_f - t_i$:

$$\vec{I} = \int_{ti}^{tf} \Sigma\vec{F}\, dt \qquad \text{Impulse of a force} \qquad (9.8)$$

From its definition, we see that impulse $\vec{I}$ is a vector quantity having a magnitude equal to the area under the force–time curve as described in the figure.



- The direction of the impulse vector is the same as the direction of the change in momentum.
- Impulse has the dimensions of momentum.
- Impulse is *not* a property of a particle;

it is a measure of the degree to which an external force changes the particle's momentum.

- Combining equations (9.7) and (9.8) gives us an important statement known as the

Impulse–momentum theorem:

(The change in the momentum of a particle is equal to the impulse of the net force acting on the particle):

$\Delta\vec{p} = \vec{I}$ Impulse–momentum theorem for a particle (9.9)

This statement is equivalent to Newton's second law. When we say that an impulse is given to a particle, we mean that momentum is transferred from an external agent to that particle.

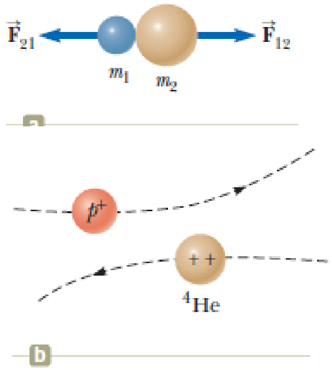
- Equation (9.9) is the most general statement of the principle of **conservation of momentum** and is called the **conservation of momentum equation**. The conservation of momentum equation is often identified as the special case of equation (9.5).

9.3 Collisions

Collisions in One Dimension

The term **collision** represents an event during which two particles come close to each other and interact by means of forces.

A collision may involve physical contact between two macroscopic objects as described in figure (a).



To understand this concept, consider a collision on an atomic scale as in the figure (b) such as the collision of a proton with an alpha particle (the nucleus of a helium atom). Because the particles are both positively charged, they repel each other due to the strong electrostatic force between them at close separations and never come into "physical contact."

- Collisions are categorized as being either *elastic* or *inelastic* depending on whether or not kinetic energy is conserved.
- An **elastic collision** between two objects is one in which the total kinetic energy (as well as total momentum) of the system is the

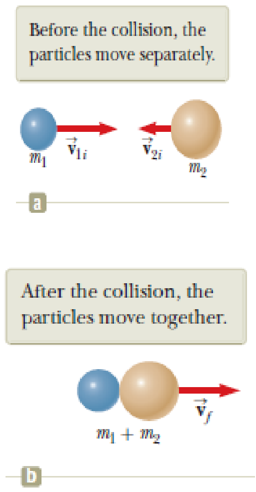
same before and after the collision. Elastic collisions occur between atomic and subatomic particles. There must be no transformation of kinetic energy into other types of energy within the system.

- An **inelastic collision** is one in which the total kinetic energy of the system is not the same before and after the collision (even though the momentum of the system is conserved).

Inelastic collisions are of two types. When the objects stick together after they collide, as happens when a meteorite collides with the Earth, the collision is called **perfectly inelastic.** When the colliding objects do not stick together but some kinetic energy is transformed or transferred away, as in the case of a rubber ball colliding with a hard surface, the collision is called **inelastic.**

Perfectly Inelastic Collisions

Consider two particles of masses m_1 and m_2 moving with initial velocities $\vec{v}_{1i}$ and $\vec{v}_{2i}$ along the same straight line as shown in the figure. The two particles collide head-on, stick together, and then move with some common velocity $\vec{v}_f$ after the collision.



Because the momentum of an isolated system is conserved in *any* collision, we can say that the total momentum before the collision equals the total momentum of the composite system after the collision:

$$m_1\vec{\mathbf{v}}_{1i} + m_2\vec{\mathbf{v}}_{2i} = (m_1 + m_2)\vec{\mathbf{v}}_f \qquad (9.10)$$

$$\vec{\mathbf{v}}_f = \frac{m_1\vec{\mathbf{v}}_{1i} + m_2\vec{\mathbf{v}}_{2i}}{m_1 + m_2} \qquad (9.11)$$

Elastic Collisions

Consider two particles of masses m_1 and m_2 moving with initial velocities $\vec{v}_{1i}$ and $\vec{v}_{2i}$ along the same straight line as shown in the figure.

The two particles collide head-on and then leave the collision site with different velocities, $\vec{v}_{1f}$ and $\vec{v}_{2f}$. In an elastic collision, both the momentum and kinetic energy of the system are conserved.

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f} \tag{9.12}$$

$$\tfrac{1}{2} m_1 {v_{1i}}^2 + \tfrac{1}{2} m_2 {v_{2i}}^2 = \tfrac{1}{2} m_1 {v_{1f}}^2 + \tfrac{1}{2} m_2 {v_{2f}}^2 \tag{9.13}$$

$$m_1({v_{1i}}^2 - {v_{1f}}^2) = m_2({v_{2f}}^2 - {v_{2i}}^2)$$

$$m_1(v_{1i} - v_{1f})\,(v_{1i} + v_{1f}) = m_2(v_{2f} - v_{2i})(v_{2f} + v_{2i}) \tag{9.14}$$

Let us separate the terms containing m_1 and m_2 in equation (9.12) to obtain:

$$m_1(v_{1i} - v_{1f}) = m_2(v_{2f} - v_{2i}) \tag{9.15}$$

To obtain final result, we divide equation 9.14 by equation 9.15 and obtain:

$$v_{1i} + v_{1f} = v_{2f} + v_{2i}$$

$$v_{1i} - v_{2i} = -(v_{1f} - v_{2f}) \tag{9.16}$$

According to equation (9.16), the *relative* velocity of the two particles before the collision, v_{1i} - v_{2i}, equals the negative of their relative velocity after the collision, - (v_{1f} - v_{2f}).

Suppose the masses and initial velocities of both particles are known. Equations (9.12) and (9.16) can be solved for the final velocities in terms of the initial velocities because there are two equations and two unknowns:

$$v_{1f} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right) v_{1i} + \left(\frac{2m_2}{m_1 + m_2}\right) v_{2i} \tag{9.17}$$

$$v_{2f} = \left(\frac{2m_1}{m_1 + m_2}\right) v_{1i} + \left(\frac{m_2 - m_1}{m_1 + m_2}\right) v_{2i} \tag{9.18}$$

Let us consider some special cases. If $m_1 = m_2$, equations 9.17 and 9.18 show that $v_{1f} = v_{2i}$ and $v_{2f} = v_{1i}$, which means that the particles exchange velocities if they have equal masses. That is approximately what one observes in head-on billiard ball collisions: the cue ball stops and the struck ball moves away from the collision with the same velocity the cue ball had.

If particle 2 is initially at rest, then v_{2i} = 0, and equations 9.17 and 9.18 become:

$$v_{1f} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right) v_{1i} \tag{9.19}$$

$$v_{2f} = \left(\frac{2m_1}{m_1 + m_2}\right) v_{1i} \tag{9.20}$$

Example (9.1):

A 1 800-kg car stopped at a traffic light is struck from the rear by a 900-kg car. The two cars become entangled, moving along the same path as that of the originally moving car. If the smaller car were moving at 20.0 m/s before the collision, what is the velocity of the entangled cars after the collision?

Solution:

The phrase "become entangled" tell us that the collision is perfectly inelastic. The magnitude of the total momentum of the system before the collision is equal to that of the smaller car because the larger car is initially at rest.

Set the initial momentum of the system equal to the final momentum of the system:

$$p_i = p_f \quad \rightarrow \quad m_1 v_i = (m_1 + m_2) v_f$$

$$v_f = \frac{m_1 v_i}{m_1 + m_2} = \frac{(900\ \text{kg})(20.0\ \text{m/s})}{900\ \text{kg} + 1\ 800\ \text{kg}} = \boxed{6.67\ \text{m/s}}$$

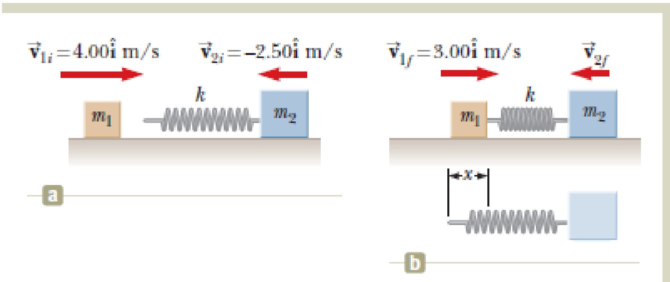
Because the final velocity is positive, the direction of the final velocity of the combination is the same as the velocity of the initially moving car. The speed of the combination is also much lower than the initial speed of the moving car.

Example (9.2):

A block of mass m_1 = 1.60 kg initially moving to the right with a speed of 4m/s on a frictionless, horizontal track collides with a light spring attached to a second block of mass m_2 = 2.10 kg initially moving to the left with a speed of 2.50 m/s as shown in the figure a. The spring constant is 600 N/m.

(A) Find the velocities of the two blocks after the collision.

Solution:



Because the spring force is conservative, kinetic energy in the system of two blocks and the spring is not transformed to internal energy during the compression of the spring. The collision is elastic.

Because momentum of the system is conserved, apply equation 9.12:

$$m_1v_{1i} + m_2v_{2i} = m_1v_{1f} + m_2v_{2f} \qquad (1)$$

Because the collision is elastic, apply equation 9.16:

$$v_{1i} - v_{2i} = -(v_{1f} - v_{2f}) \qquad (2)$$

Multiply equation (2) by m_1: $m_1v_{1i} - m_1v_{2i} = -m_1v_{1f} + m_1v_{2f}$ (3)

Add equations (1) and (3):

$$2m_1v_{1i} + (m_2 - m_1)v_{2i} = (m_1 + m_2)v_{2f}$$

$$v_{2f} = \frac{2m_1v_{1i} + (m_2 - m_1)v_{2i}}{m_1 + m_2}$$

$$v_{2f} = \frac{2(1.60\text{ kg})(4.00\text{ m/s}) + (2.10\text{ kg} - 1.60\text{ kg})(-2.50\text{ m/s})}{1.60\text{ kg} + 2.10\text{ kg}} = \boxed{3.12\text{ m/s}}$$

$$v_{1f} = v_{2f} - v_{1i} + v_{2i} = 3.12\text{ m/s} - 4.00\text{ m/s} + (-2.50\text{ m/s}) = \boxed{-3.38\text{ m/s}}$$

(C) Determine the velocity of block 2 during the collision, at the instant block 1 is moving to the right with a velocity of 13m/s as in figure b.

Solution:

Apply equation 9.12: $m_1v_{1i} + m_2v_{2i} = m_1v_{1f} + m_2v_{2f}$

Solve for v_{2f} :

$$v_{2f} = \frac{m_1v_{1i} + m_2v_{2i} - m_1v_{1f}}{m_2}$$

$$v_{2f} = \frac{(1.60\text{ kg})(4.00\text{ m/s}) + (2.10\text{ kg})(-2.50\text{ m/s}) - (1.60\text{ kg})(3.00\text{ m/s})}{2.10\text{ kg}}$$

$$= -1.74\text{ m/s}$$

The negative value for v_{2f} means that block 2 is still moving to the left.

(D) Determine the distance the spring is compressed at that instant.

Solution:

Write a conservation of mechanical energy equation for the system:

$$K_i + U_i = K_f + U_f$$

Evaluate the energies, recognizing that two objects in the system have kinetic energy and that the potential energy is elastic:

$$\tfrac{1}{2}m_1{v_{1i}}^2 + \tfrac{1}{2}m_2{v_{2i}}^2 + 0 = \tfrac{1}{2}m_1{v_{1f}}^2 + \tfrac{1}{2}m_2{v_{2f}}^2 + \tfrac{1}{2}kx^2$$

$$\tfrac{1}{2}(1.60\text{ kg})(4.00\text{ m/s})^2 + \tfrac{1}{2}(2.10\text{ kg})(2.50\text{ m/s})^2 + 0$$
$$= \tfrac{1}{2}(1.60\text{ kg})(3.00\text{ m/s})^2 + \tfrac{1}{2}(2.10\text{ kg})(1.74\text{ m/s})^2 + \tfrac{1}{2}(600\text{ N/m})x^2$$

$$x = 0.173\text{ m}$$

Collisions in Two Dimensions

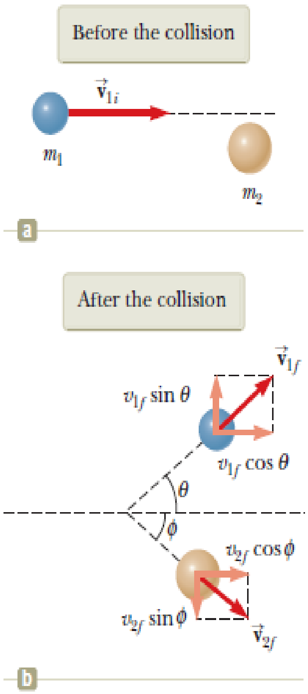
The game of billiards is a familiar example involving multiple collisions of objects moving on a two-dimensional surface. For such two-dimensional collisions, we obtain two component equations for conservation of momentum:

$$m_1 v_{1ix} + m_2 v_{2ix} = m_1 v_{1fx} + m_2 v_{2fx}$$
$$m_1 v_{1iy} + m_2 v_{2iy} = m_1 v_{1fy} + m_2 v_{2fy}$$

Where the three subscripts on the velocity components in these equations represent, respectively, the identification of the object (1, 2), initial and final values (i, f), and the velocity component (x, y).

Let us consider a specific two-dimensional problem in which particle 1 of mass m_1 collides with particle 2 of mass m_2 initially at rest as in the figure. After the collision (Fig. b), particle 1 moves at an angle θ with respect to the horizontal and particle 2 moves at an angle $\emptyset$ with respect to the horizontal. This event is called a *glancing* collision.

Applying the law of conservation of momentum in component form and noting that the initial y component of the momentum of the two-particle system is zero gives:



$$m_1 v_{1i} = m_1 v_{1f}\cos\theta + m_2 v_{2f}\cos\phi \qquad (9.21)$$
$$0 = m_1 v_{1f}\sin\theta - m_2 v_{2f}\sin\phi \qquad (9.22)$$

Where the minus sign in equation 9.22 is included because after the collision particle 2 has a y component of velocity that is downward.

If the collision is elastic, we can also use equation 9.13 (conservation of kinetic energy) with $v_{2i} = 0$:

$$\tfrac{1}{2}m_1 v_{1i}^2 = \tfrac{1}{2}m_1 v_{1f}^2 + \tfrac{1}{2}m_2 v_{2f}^2 \qquad (9.23)$$

If the collision is inelastic, kinetic energy is *not* conserved and equation (9.23) does *not* apply.

Example (9.3):

A proton collides elastically with another proton that is initially at rest. The incoming proton has an initial speed of (3.50×10^5) m/s and makes a glancing collision with the second proton as in the figure. After the collision, one proton moves off at an angle θ =37° to the original direction of motion and the second deflect at an angle ∅ to the same axis. Find the final speeds of the two protons and the angle ∅.

Solution:

Both momentum and kinetic energy of the system are conserved in this glancing elastic collision.

Use equation (9.21) through equation (9.23) gives:

$$v_{1f}\cos\theta + v_{2f}\cos\phi = v_{1i} \qquad (1)$$

$$v_{1f}\sin\theta - v_{2f}\sin\phi = 0 \qquad (2)$$

$$v_{1f}^2 + v_{2f}^2 = v_{1i}^2 \qquad (3)$$

Rearrange equations (1) and (2):

$$v_{2f}\cos\phi = v_{1i} - v_{1f}\cos\theta$$

$$v_{2f}\sin\phi = v_{1f}\sin\theta$$

Square these two equations and add them:

$$v_{2f}^2\cos^2\phi + v_{2f}^2\sin^2\phi = v_{1i}^2 - 2v_{1i}v_{1f}\cos\theta + v_{1f}^2\cos^2\theta + v_{1f}^2\sin^2\theta$$

Since $sin^2\theta + cos^2\theta$ =1

$$v_{2f}^2 = v_{1i}^2 - 2v_{1i}v_{1f}\cos\theta + v_{1f}^2 \qquad (4)$$

Substitute equation (4) into equation (3):

$$v_{1f}^2 + (v_{1i}^2 - 2v_{1i}v_{1f}\cos\theta + v_{1f}^2) = v_{1i}^2$$

$$v_{1f}^2 - v_{1i}v_{1f}\cos\theta = 0 \qquad (5)$$

One possible solution of equation (5) is v_{1f} = 0, which corresponds to a head-on, one-dimensional collision in which the first proton stops and the second continues with the same speed in the same direction. That is not the solution we want.

Divide both sides of equation (5) by v_{1f} and solve for the remaining factor of v_{1f} :

$$v_{1f} = v_{1i} \cos\theta = (3.50 \times 10^5 \text{ m/s}) \cos 37.0° = 2.80 \times 10^5 \text{ m/s}$$

Use equation (3) to find v_{2f} :

$$v_{2f} = \sqrt{v_{1i}{}^2 - v_{1f}{}^2} = \sqrt{(3.50 \times 10^5 \text{ m/s})^2 - (2.80 \times 10^5 \text{ m/s})^2}$$
$$= 2.11 \times 10^5 \text{ m/s}$$

Use equation (2) to find Ø :

$$\phi = \sin^{-1}\left(\frac{v_{1f} \sin\theta}{v_{2f}}\right) = \sin^{-1}\left[\frac{(2.80 \times 10^5 \text{ m/s}) \sin 37.0°}{(2.11 \times 10^5 \text{ m/s})}\right]$$
$$= 53.0°$$

It is interesting that (θ + Ø = 90°). This result is *not* accidental. Whenever two objects of equal mass collide elastically in a glancing collision and one of them is initially at rest, their final velocities are perpendicular to each other.

9.3 The Center of Mass

We describe the overall motion of a system in terms of a special point called the **center of mass** of the system. The system can be a group of particles, such as a collection of atoms in a container.

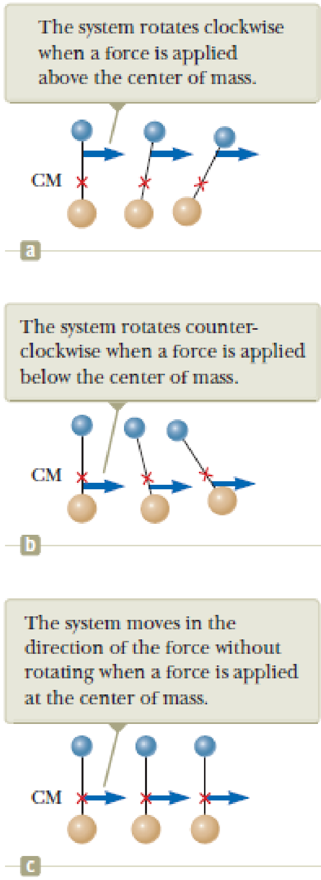
Consider a system consisting of a pair of particles that have different masses and are connected by a light, rigid rod as shown in the figure.

The position of the center of mass of a system can be described as being the *average position* of the system's mass. The center of mass of the system is located somewhere on the line joining the two particles and is closer to the particle having the larger mass.

If a single force is applied at a point on the rod above the center of mass, the system rotates clockwise as shown in the figure (a).

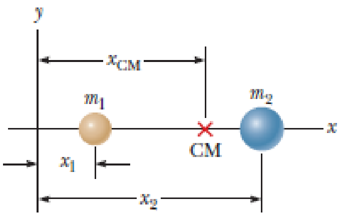
If the force is applied at a point on the rod below the center of mass, the system rotates counterclockwise as shown in the figure (b).

If the force is applied at the center of mass, the system moves in the direction of the force without rotating as shown in the figure (c).



The center of mass of the pair of particles described in the figure below is located on the *x* axis and lies somewhere between the particles. Its *x* coordinate is given by:

$$x_{CM} \equiv \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} \qquad (9.24)$$



- For example, if $x_1 = 0$, $x_2 = d$, and $m_2 = 2m_1$, we find that $x_{CM} = 2/3\ d$. That is, the center of mass lies closer to the more massive particle. If the two masses are equal, the center of mass lies midway between the particles.
- We can extend this concept to a system of many particles with masses m_i in three dimensions.

The x - coordinate of the center of mass of (n) particles is defined to be:

$$x_{CM} = \frac{\sum_i m_i x_i}{M} = \frac{1}{M}\sum_i m_i x_i \qquad (9.25)$$

Where x_i is the x coordinate of the i^{th} particle and the total mass is

$$M = \sum_i m_i$$

The center of mass can be located in three dimensions by its position vector $\vec{r}_{CM}$.

$$\vec{\mathbf{r}}_{CM} \equiv \frac{1}{M}\sum_i m_i \vec{\mathbf{r}}_i \qquad (9.26)$$

Where $\vec{r}_i$ is the position vector of the i^{th} particle, defined by:

$$\vec{\mathbf{r}}_i \equiv x_i\hat{\mathbf{i}} + y_i\hat{\mathbf{j}} + z_i\hat{\mathbf{k}}$$

- We replace the sum by an integral and (Δm_i) by the differential (dm):

$$x_{CM} = \lim_{\Delta m_i \to 0} \frac{1}{M}\sum_i x_i \,\Delta m_i = \frac{1}{M}\int x\,dm$$

$$x_{CM} = \frac{1}{M}\int x\,dm \qquad (9.27)$$

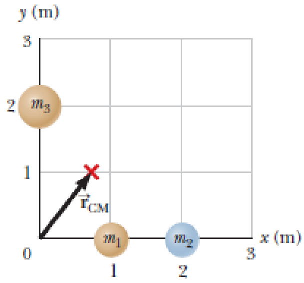
- We can express the vector position of the center of mass:

$$\vec{\mathbf{r}}_{CM} = \frac{1}{M}\int \vec{\mathbf{r}}\,dm \qquad (9.28)$$

- The center of mass of a uniform rod lies in the rod, midway between its ends. The center of mass of a sphere or a cube lies at its geometric center.

Example (9.4):

A system consists of three particles located as shown in the figure. Find the center of mass of the system. The masses of the particles are $m_1 = m_2 = 1.0$ kg and $m_3 = 2.0$ kg.



Solution:

$$x_{CM} = \frac{1}{M}\sum_i m_i x_i = \frac{m_1x_1 + m_2x_2 + m_3x_3}{m_1 + m_2 + m_3}$$

$$= \frac{(1.0\text{ kg})(1.0\text{ m}) + (1.0\text{ kg})(2.0\text{ m}) + (2.0\text{ kg})(0)}{1.0\text{ kg} + 1.0\text{ kg} + 2.0\text{ kg}} = \frac{3.0\text{ kg}\cdot\text{m}}{4.0\text{ kg}} = 0.75\text{ m}$$

$$y_{CM} = \frac{1}{M}\sum_i m_i y_i = \frac{m_1y_1 + m_2y_2 + m_3y_3}{m_1 + m_2 + m_3}$$

$$= \frac{(1.0\text{ kg})(0) + (1.0\text{ kg})(0) + (2.0\text{ kg})(2.0\text{ m})}{4.0\text{ kg}} = \frac{4.0\text{ kg}\cdot\text{m}}{4.0\text{ kg}} = 1.0\text{ m}$$

$$\vec{r}_{CM} \equiv x_{CM}\hat{i} + y_{CM}\hat{j} = (0.75\hat{i} + 1.0\hat{j})\text{ m}$$

Example (9.5):

(A) Show that the center of mass of a rod of mass (M) and length (*L*) lies midway between its ends, assuming the rod has a uniform mass per unit length.

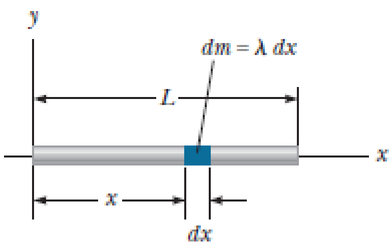
Solution:

The mass per unit length (this quantity is called the *linear mass density*) can be written as $\lambda = M/L$ for the uniform rod. If the rod is divided into elements of length dx, the mass of each element is $dm = \lambda\, dx$.

Use equation (9.27) to find an expression for x_{CM}:

$$x_{CM} = \frac{1}{M}\int x\,dm = \frac{1}{M}\int_0^L x\lambda\,dx = \frac{\lambda}{M}\left.\frac{x^2}{2}\right|_0^L = \frac{\lambda L^2}{2M}$$

$$x_{CM} = \frac{L^2}{2M}\left(\frac{M}{L}\right) = \tfrac{1}{2}L$$



(B) Suppose a rod is *non-uniform* such that its mass per unit length varies linearly with x according to the expression $\lambda = \alpha x$, where α is a constant. Find the x coordinate of the center of mass as a fraction of L.

Solution: In this case, we replace (dm) in equation (9.27) by ($\lambda\, dx$), where $\lambda = \alpha\, x$.

$$x_{CM} = \frac{1}{M}\int x\,dm = \frac{1}{M}\int_0^L x\lambda\,dx = \frac{1}{M}\int_0^L x\alpha x\,dx$$

$$= \frac{\alpha}{M}\int_0^L x^2\,dx = \frac{\alpha L^3}{3M}$$

$$M = \int dm = \int_0^L \lambda\,dx = \int_0^L \alpha x\,dx = \frac{\alpha L^2}{2}$$

$$x_{CM} = \frac{\alpha L^3}{3\alpha L^2/2} = \frac{2}{3}L$$

Notice that the center of mass in part (B) is farther to the right than that in part (A).

Chapter 10

(Rotational motion)

10.1 Angular Position, Velocity, and Acceleration

Figure (10.1) illustrates a rotating compact disc, or CD. The disc rotates about a fixed axis perpendicular to the plane of the figure and passing through the center of the disc at *O*. A small element of the disc modeled as a particle at *P* is at a fixed distance (*r*) from the origin and rotates about it in a circle of radius *r*. (In fact, every element of the disc undergoes circular motion about *O*). It is convenient to represent the position of (*P*) with its polar coordinates (*r*, θ), where *r* is the distance from the origin to *P* and θ is measured *counterclockwise* from some reference line fixed in space as shown in figure (10.1a).

In this representation, the angle θ changes in time while *r* remains constant. As the particle moves along the circle from the reference line, which is at angle θ = 0, it moves through an arc of length *s* as in figure (10.1b).

The arc length *s* is related to the angle θ through the relationship:

$$s = r\,\theta \qquad (10.1)$$

$$\theta = \frac{s}{r}$$

Because θ is the ratio of an arc length and the radius of the circle, it is a pure number.

We give θ the artificial unit **radian** (rad).

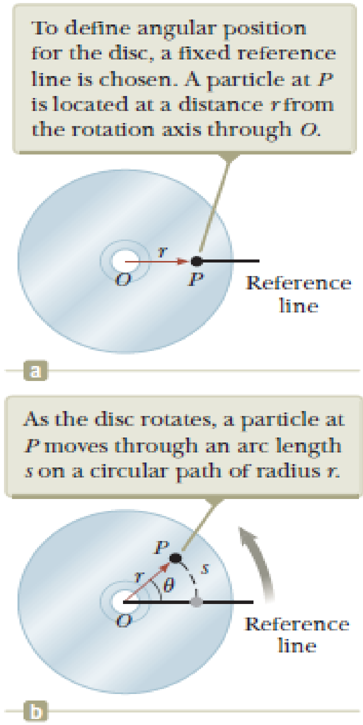


Figure 10.1 A compact disc rotating about a fixed axis through *O* perpendicular to the plane of the figure.

- Because the circumference of a circle is 2π*r*,

it follows from equation (10.1) that:

360° corresponds to an angle of (2π*r*/*r*) rad= 2π rad.

Hence, 1 rad = 360°/2π = 57.3°.

- To convert an angle in degrees to an angle in radians, we use :

$$\pi \text{ rad} = 180°$$

so, $\theta \text{ (rad)} = \pi / 180° \text{ (degree)}$

For example 60° = (π / 3) rad and 45° = (π / 4) rad.

- We choose a reference line on the object, such as a line connecting *O* and a chosen particle on the object.

  The **angular position** of the rigid object is (the angle θ between this reference line on the object and the fixed reference line in space), which is often chosen as the *x* - axis.

- As the particle travels from position (A) to position (B) in a time interval Δ*t* as in figure (10.2), the reference line fixed to the object sweeps out an angle $\Delta\theta = \theta_f - \theta_i$. This quantity Δθ is defined as the **angular displacement** of the rigid object:

$$\Delta\theta = \theta_f - \theta_i$$

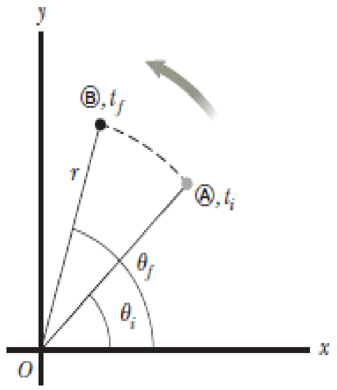


Figure 10.2 A particle on a rotating rigid object moves from Ⓐ to Ⓑ along the arc of a circle. In the time interval $\Delta t = t_f - t_i$, the radial line of length *r* moves through an angular displacement $\Delta\theta = \theta_f - \theta_i$.

- The **average angular speed** (ω_{avg}) (Greek letter omega) as (the ratio of the angular displacement of a rigid object to the time interval Δ*t* during which the displacement occurs):

$$\omega_{avg} \equiv \frac{\theta_f - \theta_i}{t_f - t_i} = \frac{\Delta\theta}{\Delta t} \qquad \textbf{Average angular speed} \qquad (10.2)$$

- The **instantaneous angular speed** ω is defined as (the limit of the average angular speed as Δt approaches zero):

$$\omega \equiv \lim_{\Delta t \to 0} \frac{\Delta \theta}{\Delta t} = \frac{d\theta}{dt}$$ **Instantaneous angular speed** (10.3)

Angular speed has units of radians per second (rad/s), which can be written as (s^{-1}) because radians are not dimensional.

- We take (ω) to be positive when θ is increasing (counterclockwise motion in figure 10.2) and negative when θ is decreasing (clockwise motion in figure 10.2).
- If the instantaneous angular speed of an object changes from ω_i to ω_f in the time interval Δt, the object has an angular acceleration. The **average angular acceleration** α_{avg} (Greek letter alpha) of a rotating rigid object is defined as (the ratio of the change in the angular speed to the time interval Δt during which the change in the angular speed occurs):

$$\alpha_{avg} \equiv \frac{\omega_f - \omega_i}{t_f - t_i} = \frac{\Delta \omega}{\Delta t}$$ **Average angular acceleration** (10.4)

The **instantaneous angular acceleration** is defined as the limit of the average angular acceleration as Δt approaches zero:

$$\alpha \equiv \lim_{\Delta t \to 0} \frac{\Delta \omega}{\Delta t} = \frac{d\omega}{dt}$$ **Instantaneous angular acceleration** (10.5)

- Angular acceleration has units of radians per second squared (rad/s^2), or simply (s^{-2}).
- Notice that (α) is positive when a rigid object rotating counterclockwise is speeding up or when a rigid object rotating clockwise is slowing down during some time interval.

- It is convenient to use the *right-hand rule* demonstrated in figure (10.3) to determine the direction of $\vec{\omega}$:

  (When the four fingers of the right hand are wrapped in the direction of rotation, the extended right thumb points in the direction of $\vec{\omega}$).

- The direction of $\vec{\alpha}$ follows from its definition ($\vec{\alpha} \equiv d\vec{\omega}/dt$). It is in the same direction as $\vec{\omega}$ if the angular speed is increasing in time, and it is antiparallel to $\vec{\omega}$ if the angular speed is decreasing in time.

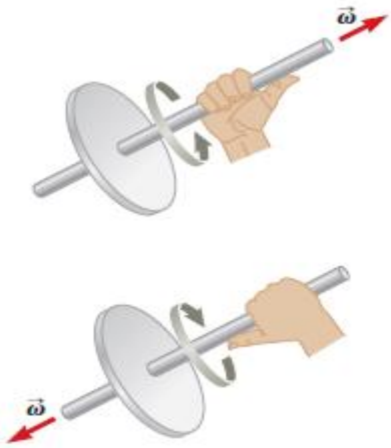


Figure 10.3 The right-hand rule for determining the direction of the angular velocity vector.

10.2 Rigid Object under Constant Angular Acceleration

Writing equation (10.5) in the form: ($d\omega = \alpha dt$) and integrating from $t_i = 0$ to $t_f = t$ gives:

$$\omega_f = \omega_i + \alpha t \qquad \text{for constant} \qquad (10.6)$$

Where ω_i is the angular speed of the rigid object at time $t = 0$. Equation (10.6) allows us to find the angular speed ω_f of the object at any later time t. Substituting equation (10.6) into equation (10.3) and integrating once more, we obtain:

$$\theta_f = \theta_i + \omega_i t + \frac{1}{2}\alpha t^2 \quad \text{(for constant } \alpha\text{)} \qquad (10.7)$$

Where θ_i is the angular position of the rigid object at time $t = 0$.

Equation (10.7) allows us to find the angular position θ_f of the object at any later time t.

Eliminating (t) from equations (10.6) and (10.7) gives:

$$\omega_f^2 = \omega_i^2 + 2\alpha(\theta_f - \theta_i) \quad \text{(for constant } \alpha\text{)} \qquad (10.8)$$

This equation allows us to find the angular speed ω_f of the rigid object for any value of its angular position θ_f .

If we eliminate (α) between equations (10.6) and (10.7), we obtain:

$$\theta_f = \theta_i + \frac{1}{2}(\omega_i + \omega_f)t \quad \text{(for constant } \alpha\text{)} \qquad (10.9)$$

Notice that these kinematic expressions for the rigid object under constant angular acceleration are of the same mathematical form as those for a particle under constant acceleration (Chapter 2).

They can be generated from the equations for translational motion by making the substitutions $x \to \theta$, $v \to \omega$, and $a \to \alpha$.

Table (10.1) compares the kinematic equations for rotational and translational motion.

TABLE 10.1 *Kinematic Equations for Rotational and Translational Motion*

Rigid Body Under Constant Angular Acceleration	Particle Under Constant Acceleration
$\omega_f = \omega_i + \alpha t$	$v_f = v_i + at$
$\theta_f = \theta_i + \omega_i t + \frac{1}{2}\alpha t^2$	$x_f = x_i + v_i t + \frac{1}{2}at^2$
$\omega_f^2 = \omega_i^2 + 2\alpha(\theta_f - \theta_i)$	$v_f^2 = v_i^2 + 2a(x_f - x_i)$
$\theta_f = \theta_i + \frac{1}{2}(\omega_i + \omega_f)t$	$x_f = x_i + \frac{1}{2}(v_i + v_f)t$

Example (10.1):

A wheel rotates with a constant angular acceleration of (3.50 rad/s^2).

(A) If the angular speed of the wheel is (2 rad/s) at $t_i = 0$, through what angular displacement does the wheel rotate in 2 s?

Solution: Arrange equation (10.7) so that it expresses the angular displacement of the object: $\Delta\theta = \theta_f - \theta_i = \omega_i t + \frac{1}{2}\alpha t^2$

Substitute the known values to find the angular displacement at t =2 s:

$$\Delta\,\theta = (2\text{ rad/s})(2\text{ s}) + 1/2\,(3.5\text{ rad/s}^2)\,(2\text{ s})^2 = 11\text{ rad}$$

$$= (11\text{ rad})(180°/\pi\text{ rad}) = 630°$$

(B): Through how many revolutions has the wheel turned during this time interval?

$$\Delta\theta = 630°\left(\frac{1\text{ rev}}{360°}\right) = 1.75\text{ rev}$$

(C): What is the angular speed of the wheel at t =2 s?

$$\omega_f = \omega_i + \alpha t = 2.00\text{ rad/s} + (3.50\text{ rad/s}^2)(2.00\text{ s}) = 9.00\text{ rad/s}$$

10.3 Angular and Translational Quantities

Point P in figure (10.4) moves in a circle, the translational velocity vector $\vec{v}$ is always tangent to the circular path and hence is called *tangential velocity*. The magnitude of the tangential velocity of the point P is by definition the tangential speed ($v = ds/dt$), where (s) is the distance traveled by this point measured along the circular path.

Recalling that $s = r\,\theta$ and noting that r is constant,

$$v = \frac{ds}{dt} = r\frac{d\theta}{dt}$$

Because $d\,\theta/dt = \omega$

Then $v = r\,\omega$ (10.10)

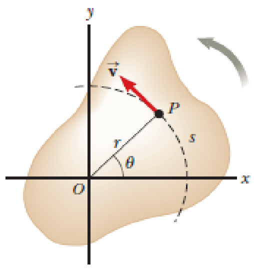


Figure (10.4)

We can relate the angular acceleration of the rotating rigid object to the tangential acceleration of the point P by taking the time derivative of v:

$$a_t = \frac{dv}{dt} = r\frac{d\omega}{dt}$$

$$a_t = r\alpha$$ Relation between tangential acceleration and angular acceleration (10.11)

We can express the centripetal acceleration at that point in terms of angular speed as:

$$a_c = \frac{v^2}{r} = r\omega^2 \qquad (10.12)$$

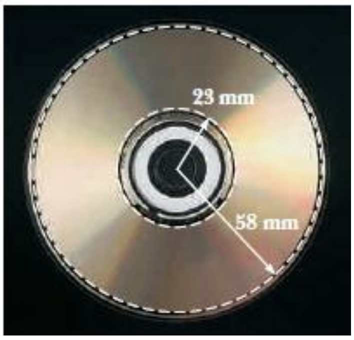
The total acceleration vector at the point P is $\vec{\mathbf{a}} = \vec{\mathbf{a}}_t + \vec{\mathbf{a}}_r$, where the magnitude of $\vec{\mathbf{a}}_r$ is the centripetal acceleration a_c. Because $\vec{\mathbf{a}}$ is a vector having a radial and a tangential component, the magnitude of $\vec{\mathbf{a}}$ at the point P on the rotating rigid object is:

$$a = \sqrt{a_t^2 + a_r^2} = \sqrt{r^2\alpha^2 + r^2\omega^4} = r\sqrt{\alpha^2 + \omega^4}$$ Total Acceleration) (10.13)

Example (10.2):

(A) Find the angular speed of the disc in revolutions per minute when information is being read from the innermost first track (r = 23 mm) and the outermost final track (r =58 mm)? The constant speed of the CD player is 1.3 m/s.

Solution:



The angular speed that gives the required tangential speed at the position of the inner track:

$$\omega_i = \frac{v}{r_i} = \frac{1.3\ \text{m/s}}{2.3 \times 10^{-2}\ \text{m}} = 57\ \text{rad/s}$$

$$= (57\ \text{rad/s})\left(\frac{1\ \text{rev}}{2\pi\ \text{rad}}\right)\left(\frac{60\ \text{s}}{1\ \text{min}}\right) = 5.4 \times 10^2\ \text{rev/min}$$

The same for the outer track:

$$\omega_f = \frac{v}{r_f} = \frac{1.3\ \text{m/s}}{5.8 \times 10^{-2}\ \text{m}} = 22\ \text{rad/s} = \boxed{2.1 \times 10^2\ \text{rev/min}}$$

(B) The maximum playing time of a standard music disc is 74 min and 33 s. How many revolutions does the disc make during that time?

Solution:

If $t = 0$ is the instant the disc begins rotating, with angular speed of 57 rad/s, the final value of the time t is:

[(74 min)(60 s/min) + 33 s] =4473 s.

The angular displacement $\Delta\theta$ during this time interval is:

$$\Delta\theta = \theta_f - \theta_i = \tfrac{1}{2}(\omega_i + \omega_f)t$$
$$= \tfrac{1}{2}(57\ \text{rad/s} + 22\ \text{rad/s})(4\,473\ \text{s}) = 1.8 \times 10^5\ \text{rad}$$

Convert this angular displacement to revolutions:

$$\Delta\theta = (1.8 \times 10^5\ \text{rad})\left(\frac{1\ \text{rev}}{2\pi\ \text{rad}}\right) = \boxed{2.8 \times 10^4\ \text{rev}}$$

(C) What is the angular acceleration of the compact disc over the 4473 s time interval?

Solution:

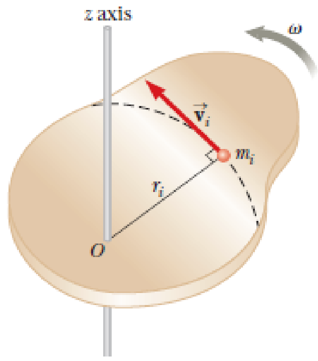
$$\alpha = \frac{\omega_f - \omega_i}{t} = \frac{22\ \text{rad/s} - 57\ \text{rad/s}}{4\,473\ \text{s}} = \boxed{-7.6 \times 10^{-3}\ \text{rad/s}^2}$$

10.4 Rotational Kinetic Energy

Let us consider an object as a system of particles and assume it rotates about a fixed z- axis with an angular speed ω. Figure (10.5) shows the rotating object and identifies one particle on the object located at a distance r_i from the rotation axis. If the mass of the i^{th} particle is m_i and its tangential speed is (v_i), its kinetic energy is:

$$K_i = \frac{1}{2} m_i v_i^2$$

The *total* kinetic energy of the rotating rigid object is the sum of the kinetic energies of the individual particles:



$$K_R = \sum_i K_i = \sum_i \frac{1}{2} m_i v_i^2 = \frac{1}{2} \sum_i m_i r_i^2 \omega^2$$

We can write this expression in the form:

$$K_R = \frac{1}{2}\left(\sum_i m_i r_i^2\right)\omega^2 \qquad (10.14)$$

Figure (10.5)

Where we have factored (ω^2) from the sum because it is common to every particle. We simplify this expression by defining the quantity in parentheses as the **moment of inertia *I*** of the rigid object:

$$I \equiv \sum_i m_i r_i^2$$ **Moment of inertia** (10.15)

It has dimensions (Kg.m^2).

Equation (10.15) becomes:

$$K_R = \frac{1}{2} I \omega^2$$ **Rotational kinetic energy** (10.16)

The analogy between kinetic energy ($\frac{1}{2}mv^2$) associated with translational motion and rotational kinetic energy ($\frac{1}{2}I\omega^2$). The quantities *I* and ω in rotational motion are analogous to *m* and *v* in translational motion, respectively.

- Moment of inertia is (a measure of the resistance of an object to changes in its rotational motion), just as mass is (a measure of the of the tendency of an object to resist changes in its translational Motion).

10.5 Calculation of Moments of Inertia

We use the definition ($I = \Sigma_i\ r_i^2 \Delta m_i$) and take the limit of this sum as $\Delta m_i \rightarrow 0$. In this limit, the sum becomes an integral over the volume of the object:

Moment of inertia of a rigid object:

$$I = \lim_{\Delta m_i \rightarrow 0} \sum_i r_i^2\ \Delta m_i = \int r^2\ dm \qquad (10.17)$$

It is usually easier to calculate moments of inertia in terms of the volume of the elements rather than their mass, and we can easily make that change by using:

$\rho = \frac{m}{V}$ where ρ is the density of the object and V is its volume.

The mass of small element is: $dm = \rho\ dV$

Substituting this result into equation (10.17) gives:

$$I = \int \rho\ r^2\ dV$$

If the object is homogeneous, ρ is constant.

The density given by ($\rho = m/V$) sometimes is referred to as (*volumetric mass density*) because it represents (*mass per unit volume*).

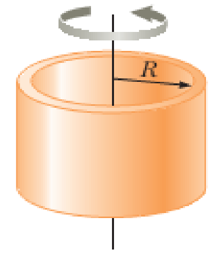
For instance, when dealing with a sheet of uniform thickness t, we can define a (*surface mass density*) ($\sigma = \rho t$), which represents (*mass per unit area*).

Finally, when mass is distributed along a rod of uniform cross-sectional area A, we sometimes use (*linear mass density*) ($\lambda = M / L = \rho\ A$), which is the (*mass per unit length*).

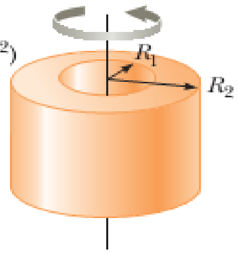
Table (10.2) gives the moments of inertia for a number of objects about specific axes.

TABLE 10.2 *Moments of Inertia of Homogeneous Rigid Objects with Different Geometries*

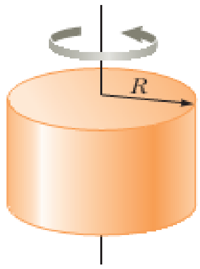
Hoop or thin cylindrical shell
$I_{CM} = MR^2$



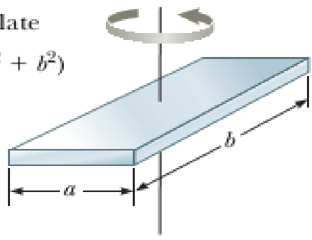
Hollow cylinder
$I_{CM} = \frac{1}{2}M(R_1^2 + R_2^2)$



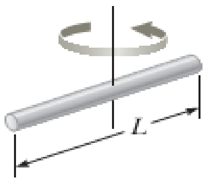
Solid cylinder or disk
$I_{CM} = \frac{1}{2}MR^2$



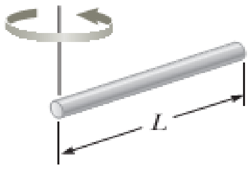
Rectangular plate
$I_{CM} = \frac{1}{12}M(a^2 + b^2)$



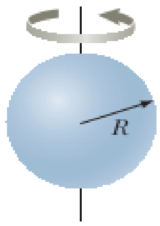
Long, thin rod with rotation axis through center
$I_{CM} = \frac{1}{12}ML^2$



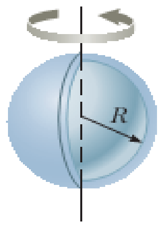
Long, thin rod with rotation axis through end
$I = \frac{1}{3}ML^2$



Solid sphere
$I_{CM} = \frac{2}{5}MR^2$



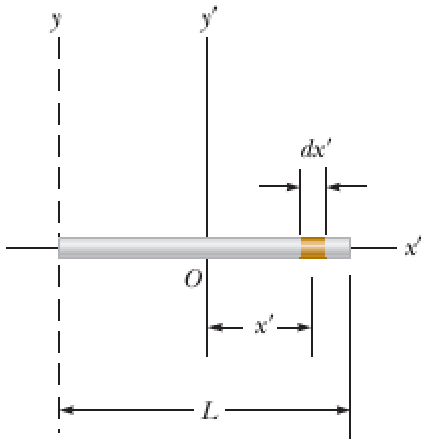
Thin spherical shell
$I_{CM} = \frac{2}{3}MR^2$



Example (10.3):

Calculate the moment of inertia of a uniform rigid rod of length L and mass M about an axis perpendicular to the rod and passing through its center of mass?

Solution:



The shaded length element dx' in the figure has a mass dm equal to the mass per unit length λ multiplied by dx'.

Express dm in terms of dx':

$$dm = \lambda\, dx' = \frac{M}{L}\, dx'$$

Substitute this expression into equation (10.17), with $r^2 = (x')^2$

$$I_y = \int r^2\, dm = \int_{-L/2}^{L/2} (x')^2 \frac{M}{L}\, dx' = \frac{M}{L}\int_{-L/2}^{L/2} (x')^2\, dx'$$

$$= \frac{M}{L}\left[\frac{(x')^3}{3}\right]_{-L/2}^{L/2} = \frac{1}{12} ML^2$$

Moment of inertia for thin rod

Check this result in Table (10.2).

Example (10.4):

A uniform solid cylinder has a radius R, mass M, and length L. Calculate its moment of inertia about its central axis?

Solution:

It is convenient to divide the cylinder into many cylindrical shells, each having radius r, thickness dr, and length L as shown in the figure.

The density of the cylinder is ρ.

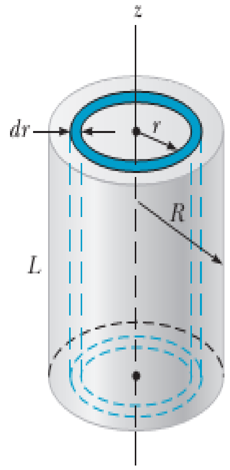
The volume dV of each shell is its cross-sectional area multiplied by its length:

$$dV = L\,dA = L(2\pi r)\,dr$$

Express dm in terms of dr:

$$dm = \rho\,dV = \rho\,L(2\pi r)\,dr$$

Substitute this expression into equation (10.17):



$$I_z = \int r^2\,dm = \int r^2[\rho L(2\pi r)\,dr] = 2\pi\rho L\int_0^R r^3\,dr = \tfrac{1}{2}\pi\rho L R^4$$

Use the total volume ($\pi R^2 L$) of the cylinder to express its density:

$$\rho = \frac{M}{V} = \frac{M}{\pi R^2 L}$$

Substitute this value into the expression for I_z :

$$I_z = \tfrac{1}{2}\pi\left(\frac{M}{\pi R^2 L}\right)LR^4 = \tfrac{1}{2}MR^2$$

Moment of inertia for a cylinder

Check this result in Table (10.2).

Notice that the result for the moment of inertia of a cylinder does not depend on L, the length of the cylinder. Therefore, the moment of inertia of the cylinder would not be affected by changing its length.

Parallel-axis theorem:

$$I = I_{CM} + MD^2$$ **Parallel-axis theorem** (10.18)

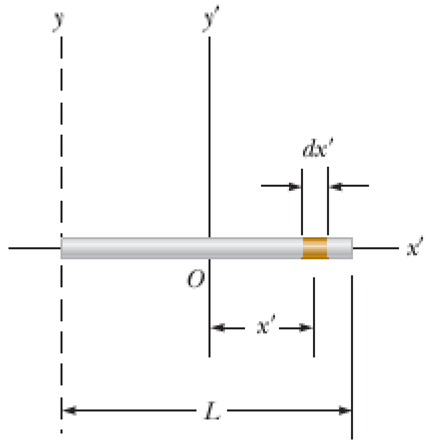
I_{CM} : The moment of inertia about an axis that is parallel to the z - axis and passes through the center of mass.

D is the distance between the center of mass axis and an axis parallel to that axis.

Example (10.5):

Find the moment of inertia of uniform rigid rod of mass M and length L about an axis perpendicular to the rod through one end (the y axis in the figure)?

Solution:



The distance between the center of mass axis and the y - axis is $D = L/2$.

Use the parallel-axis theorem:

$$I = I_{CM} + MD^2 = \tfrac{1}{12}ML^2 + M\left(\frac{L}{2}\right)^2 = \tfrac{1}{3}ML^2$$

Check this result in Table (10.2).

10.6 Torque

When a force is exerted on a rigid object pivoted about an axis, the object tends to rotate about that axis.

- (The tendency of a force to rotate an object about some axis is measured by a quantity called **torque** $\vec{\tau}$) (Greek letter tau). Torque is a vector.

  Consider the wrench in the figure (10.6) that we wish to rotate around an axis that is perpendicular to the page and passes through the center of the bolt.

  The applied force $\vec{F}$ acts at an angle ϕ to the horizontal.

  We define the magnitude of the torque associated with the force $\vec{F}$ around the axis passing through O by the expression:

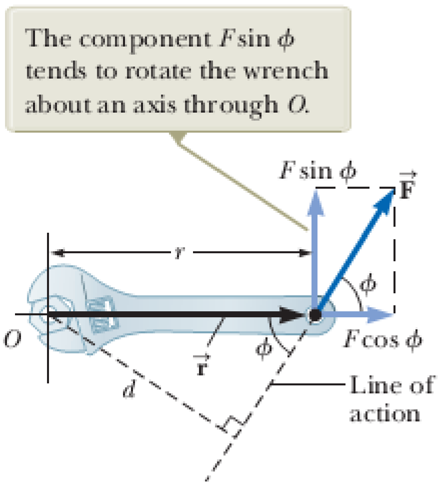


Figure (10.6)

$$\tau \equiv rF\sin\phi = Fd \qquad (10.19)$$

where r is the distance between the rotation axis and the point of application of $\vec{F}$ and d is the perpendicular distance from the rotation axis to the line of action of $\vec{F}$.

(The *line of action* of a force is an imaginary line extending out both ends of the vector representing the force). The dashed line extending from the tail of $\vec{F}$ in figure (10.6) is part of the line of action of $\vec{F}$. From the right triangle in figure (10.6) that has the wrench as its hypotenuse, we see that

$d = r \sin \phi$ The quantity d is called the **moment arm** of $\vec{F}$.

- The only component of $\vec{F}$ that tends to cause rotation of the wrench around an axis through O is ($F \sin \phi$) (the component perpendicular to a line drawn from the rotation axis to the point of application of the force).
- The horizontal component ($F\cos \phi$), because its line of action passes through O, has no tendency to produce rotation about an axis passing through O.
- From the definition of torque, the rotating tendency increases as F increases and as d increases.
- Torque has units of force times length (Newton.meter).
- If the torque is positive, the object begins to rotate in the counterclockwise direction and if it is negative, the rotation is clockwise.

10.7 Rigid Object under a Net Torque

The rotational analog of Newton's second law: (The angular acceleration of a rigid object rotating about a fixed axis is proportional to the net torque acting about that axis).

Consider a particle of mass *m* rotating in a circle of radius *r* under the influence of a tangential net force $\Sigma \vec{F}_t$ and a radial net force $\Sigma \vec{F}_r$ as shown in figure (10.7).

The radial net force causes the particle to move in the circular path with a centripetal acceleration. The tangential force provides a tangential acceleration $\vec{a}_t$, and

$$\Sigma \vec{F}_t = m\vec{a}_t$$

The magnitude of the net torque due to $\Sigma \vec{F}_t$ on the particle about an axis perpendicular to the page through the center of the circle is:

$$\Sigma \tau = \Sigma F_t r = (ma_t)\, r$$

Because the tangential acceleration is related to the angular acceleration through the relationship : $a_t = r\alpha$

The net torque can be expressed as:

$$\Sigma \tau = (mr\alpha)\, r = (mr^2)\, \alpha$$

Since (mr^2) is the moment of inertia of the particle about the *z*-axis passing through the origin, so

$$\Sigma \tau = I \alpha \qquad (10.20)$$

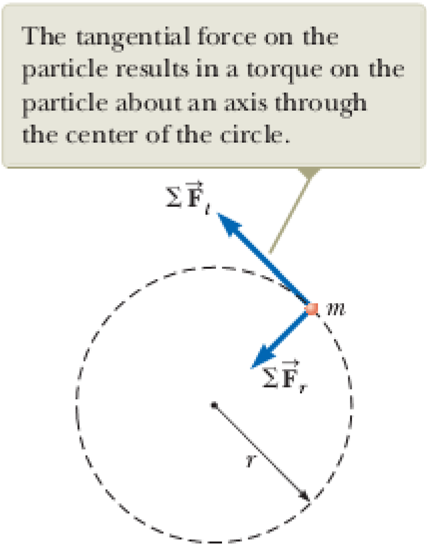


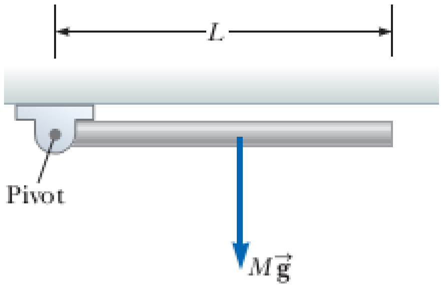
Figure (10.7)

That is, the net torque acting on the particle is proportional to its angular acceleration, and the proportionality constant is the moment of inertia.

Notice that ($\Sigma \tau = I \alpha$) has the same mathematical form as Newton's second law of motion, ($\Sigma F = ma$).

Example (10.6):

A uniform rod of length L and mass M is attached at one end to a frictionless pivot and is free to rotate about the pivot in the vertical plane as in the figure. The rod is released from rest in the horizontal position. What are the initial angular acceleration of the rod and the initial translational acceleration of its right end?



Solution:

When the rod released, it rotates clockwise around the pivot at the left end. The only force contributing to the torque about an axis through the pivot is the gravitational force ($M\vec{g}$) exerted on the rod.

$$\tau = Mg\,(L/2)$$

To obtain the angular acceleration of the rod:

$$\alpha = \frac{\tau}{I} = \frac{Mg(L/2)}{\frac{1}{3}ML^2} = \frac{3g}{2L}$$

To find the initial translational acceleration of the right end of the rod:

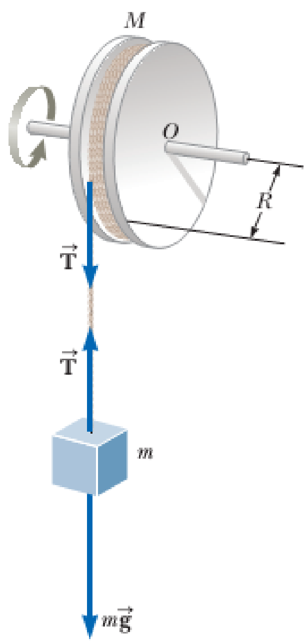
$$a_t = r\,\alpha \qquad \text{with } r = L$$

$$a_t = \tfrac{3}{2}g$$

Example (10.7):

A wheel of radius R, mass M, and moment of inertia I is mounted on a frictionless, horizontal axle as in the figure. A light cord wrapped around the wheel supports an object of mass m. When the wheel is released, the object accelerates downward, the cord unwraps off the wheel, and the wheel rotates with an angular acceleration. Calculate the angular acceleration of the wheel, the translational acceleration of the object, and the tension in the cord?

Solution:



The magnitude of the torque acting on the wheel about its axis of rotation is:

$$\tau = TR$$

Where T is the force exerted by the cord on the rim of the wheel.

Use equation (10.20): $\Sigma \tau = I \alpha$

Then $\alpha = \frac{\Sigma \tau}{I} = \frac{TR}{I}$ (1)

Apply Newton's second law to the motion of the object, taking the downward direction to be positive:

$$\Sigma F_y = mg - T = ma$$

The acceleration is: $a = \frac{mg - T}{m}$ (2)

Therefore, the angular acceleration (α) of the wheel and the translational acceleration of the object are related by: $a = R\,\alpha$

Use this fact together with equations (1) and (2):

$$a = R\alpha = \frac{TR^2}{I} = \frac{mg - T}{m} \quad (3)$$

The tension T is: $T = \frac{mg}{1 + (mR^2/I)}$ (4)

Substitute equation (4) into equation (2) and solve for a:

$$a = \frac{g}{1 + (I/mR^2)} \quad (5)$$

Use ($a = R\,\alpha$) and equation (5) to solve for a:

$$\alpha = \frac{a}{R} = \frac{g}{R + (I/mR)}$$