INTEGER AND INTEGER-LINEAR PROGRAMMING: A BRANCH-AND-BOUND METHOD

The following algorithm can be used to solve a mixed integer-linear program using linear programming solution methods. Consider the problem:

Maximize z = 5 X1 + 3 X2
Subject to: 4 X1 + 2 X2 ≤ 25
X1 ≤ 5
X2 ≤ 8
Both X1 and X2 are integers.

Step 1: Find the optimal continuous solution. Relax the integer requirements for the decision variables and solve the resulting non-integer linear program. This gives an optimal continuous solution (OCS). If all originally integer variables have integer solutions in the OCS, then STOP; this is the optimal solution. Otherwise, continue with Step 2.

For the problem above, the solution is X1 = 2.25, X2 = 8, z = 35.25, so this is not the optimal solution to the integer programming problem.

Step 2: Branch. Divide the problem into two sub-problems, searching for good integer-valued solutions. For our problem, X1 has a fractional value in Step 1 of 2.25. We produce the two sub problems by adding new constraints on the non-integer value which keep it from its current non-integer value. Here, since X1 = 2.25 above, we add the constraints X1 ≤ 2 and X1 ≥ 3 to give the following two sub-problems. This is called branching.

Sub-problem 1:

Maximize z = 5 X1 + 3 X2
Subject to: 4 X1 + 2 X2 ≤ 25
X1 ≤ 5 (now redundant)
X2 ≤8
X1 ≤2

Sub-problem 2:

Maximize z = 5 X1 + 3 X2
Subject to: 4 X1 + 2 X2 ≤ 25
X1 ≤ 5
X2 ≤ 8
X1 ≥ 3

Step 3: Solve the branched problems. When these sub-problems are solved by linear programming, the solutions are compared.

Sub-problem 1: X1 = 2, X2 = 8, Z_1 = 34

Sub-problem 2: X1 = 3, X2 = 6.5, Z_2 = 34.5

Step 4: Set the bound. The best solution to the sub-problem which satisfies all integer requirements is called the upper (or lower) bound. In this case, the sub-problem 1 solution, z = 34, is the lower bound of the optimal solution.

If no sub-problem solution satisfies the integer conditions, then further branching and solution is required until a bound can be set. (If a bound has already been set, any new bound must have a better value than the existing bound.)

Step 5: Continued branching. Branch on sub-problems that both do not satisfy integer conditions and have superior solutions. In this case the solution to sub-problem 2 both does not satisfy the integer conditions and has a superior solution to the bound (bound = 34, Z2 = 34.5). If the solutions to the all other sub-problems are inferior to the bound, then STOP and the solution that sets the bound is the optimal solution which satisfies all constraints, including the integer conditions. Otherwise, GO TO Step 2.

Continuing with the solution to the example:

Following Step 2, continued branching for this problem (Sub-problem 2) gives the following two sub-problems:

Sub problem 2a:

Maximize z = 5 X1 + 3 X2

Subject to: 4 X1 + 2 X2 ≤ 25

X1 ≤5

X2 ≤ 8 (now redundant)

X1 ≥3

X2 ≤ 6 (new constraint)

Sub problem 2b:

Maximize z = 5 X1 + 3 X2

Subject to: 4 X1 + 2 X2 ≤ 25

X1 ≤ 5

X2 ≤ 8

X1 ≥ 3

X2 ≥7 (new constraint)

Repeating Step 3, the solutions to these new sub-problems are: *Sub-problem 2a:* X1 = 3.25, X2 = 6, Z2a = 4.25

Sub-problem 2b: infeasible

Step 4: Since neither of the new solutions both satisfies the integer conditions and has a better objective function value than the existing bound (existing bound = 34), the old bound remains.

Step 5: Since the objective function value for the solution to sub-problem 2a is superior to the bound, there remains a possibility that there is an integer solution better than the current bound (= 34). So branching on sub-problem 2a continues, using Step 2.

Step 2: Branching on sub-problem 2a gives the following new sub-problems:

Sub problem 2a1:

Maximize z = 5 X1 + 3 X2

Subject to: 4 X1 + 2 X2 ≤ 25

X1 ≤5

X2 ≤ 8 (now redundant)

X1 ≥3

X2 ≤6

X1 ≤ 3 (new constraint)

Sub problem 2a2:

Maximize z = 5 X1 + 3 X2

Subject to: 4 X1 + 2 X2 ≤ 25

X1 ≤ 5

X2 ≤ 8 (now redundant)

X1 ≥ 3 (now redundant)

X2 ≤ 6

X1 ≥ 4 (new constraint)

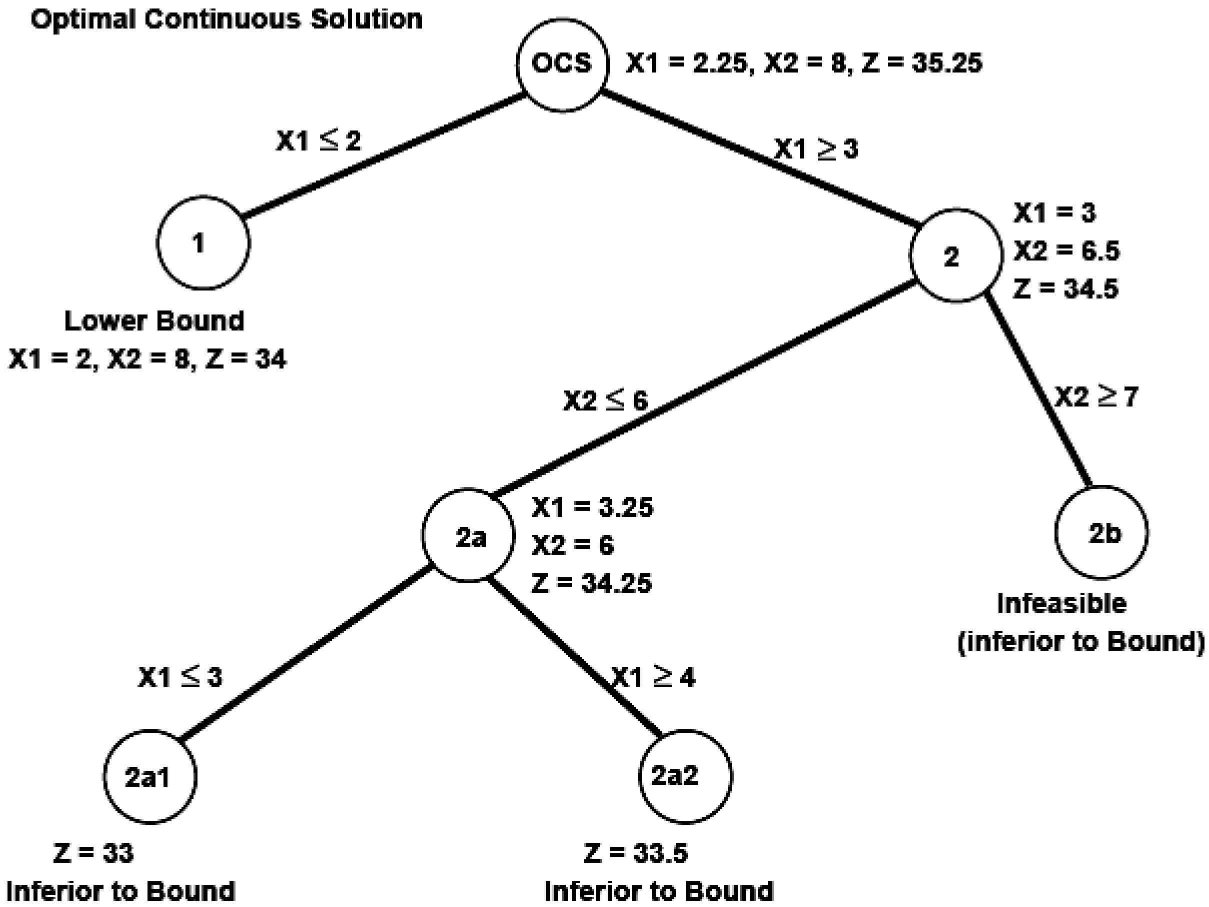
Repeating Step 3, the solutions to these new sub-problems are: *Sub-problem 2a1:* X1 = 3, X2 = 6, Z2a1 = 33

Sub-problem 2a2: X1 = 4, X2 = 4.5, Z2a2 = 33.5

Steps 4 and 5: Since neither solution to the new sub-problem is superior to the bound, there is no hope for an integer solution better than the bounded solution (Z = 34). (Since further branching ads constraints, and a solution is never improved by adding constraints, new sub-problems will never produce better solutions than their parent problems.) Therefore, we can STOP, knowing the optimal solution is that of the current bound, Z = 34, X1 = 2, X2 = 8.

Graphically, in solving the example we have created the following tree from the branches, with each branch terminating when we reach an inferior solution or an upper bound.

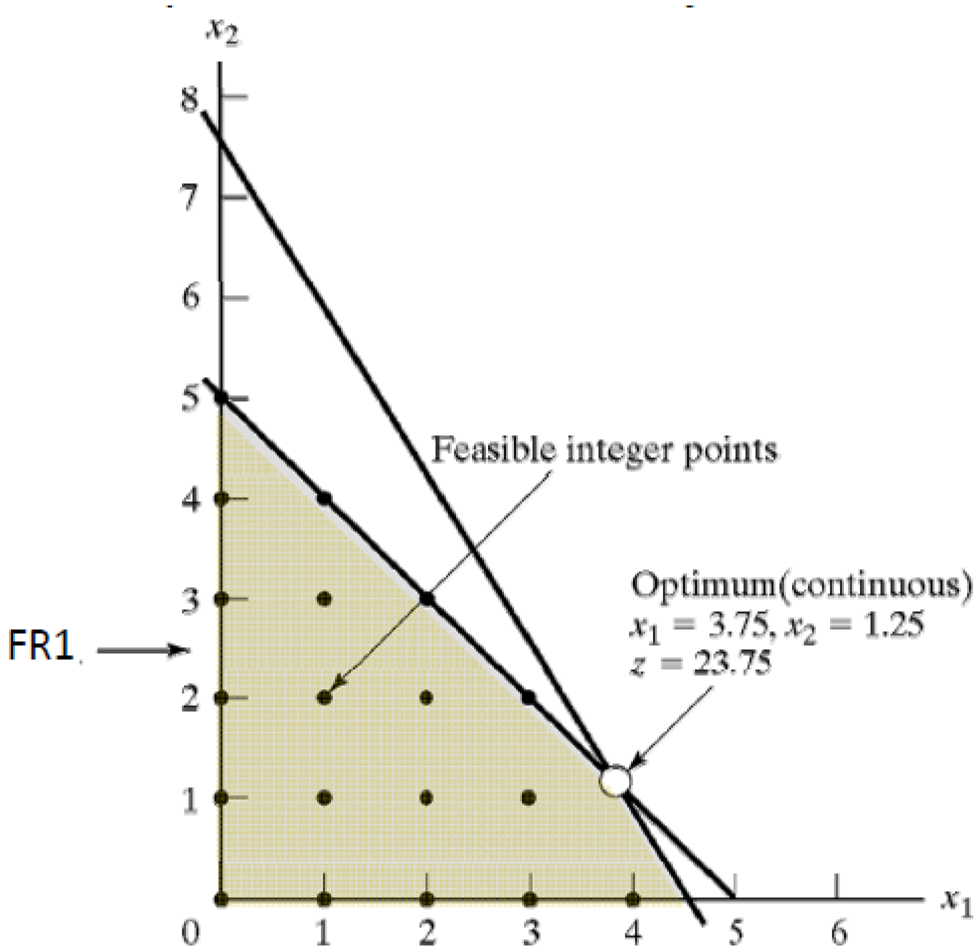
Optimal Continuous Solution



Example: Consider the MIP model

$$\begin{aligned} \max \quad & z = 5x_1 + 4x_2 \\ \text{s.t.} \quad & x_1 + x_2 \le 5 \\ & 10x_1 + 6x_2 \le 45 \\ & x_1, x_2 \ge 0 \text{ and integer} \end{aligned}$$

The feasible space is illustrated by



- The optimal solution of the relaxed MIP model is

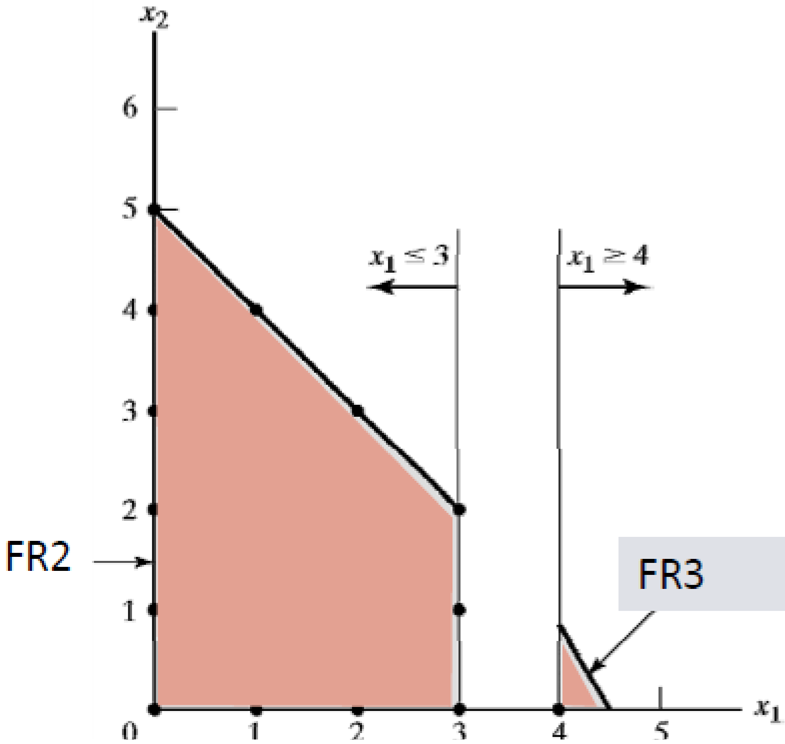
  $x_1 = 3.75$
  $x_2 = 1.25$
  $z = 23.75$
- This solution is not integer.
- Consider first x_1. The optimal integer solution is either
  for $x_1 \ge 4$ or $x_1 \le 3$.
- Hence, the area of the feasible region FR1 that has
  $3 < x_1 < 4$ can be eliminated.

• This breakup in FR1 leads to two sub feasible regions.

The two sub regions can be labeled as:

$FR_2 = FR_1 \cap \{x_1 \le 3\}$

$FR_3 = FR_1 \cap \{x_1 \ge 4\}$



Notice that the feasible region defined by FR_2 **U** FR_3 contains the same integer solutions as FR1.

• Continuing removing the regions that do not include integer solutions will lead to an LP model whose optimum solution is integer.

• Hence, the solution of MIP is reached by solving a sequence of LP problems.

• Choosing $x_1 \ge 4$ and $x_1 \le 3$ is called branching, and x_1 is called the branching variable.

• The optimal integer solution lies either in FR_2 or FR_3.

• Hence, z is optimized with respect to FR_2 and FR_3, and when an integer solution is found, the procedure is terminated. Otherwise, the process of branching continues.

• **Solve the sub problem LP2**:

$$\begin{aligned} \max \quad & z = 5x_1 + 4x_2 \\ \text{s.t.} \quad & x_1 + x_2 \le 5 \\ & 10x_1 + 6x_2 \le 45 \\ & x_1 \le 3 \\ & x_1, x_2 \ge 0 \end{aligned}$$

The solution is $x_1 = 3$, $x_2 = 2$, $z = 23$

The solution of LP2 is integer, hence, no further branching is required.

- This solution cannot be said to be also the optimal solution to the original problem, because the sub problem for $x_1 \geq 4$ might give a better solution.
- It can only be said that = 23 is a lower bound on the optimum solution.
- This means that any sub problem that cannot yield a better solution than the lower bound must be discarded.

Solve the second sub problem LP3:

$$\begin{aligned} \max \quad & z = 5x_1 + 4x_2 \\ \text{s.t.} \quad & x_1 + x_2 \leq 5 \\ & 10x_1 + 6x_2 \leq 45 \\ & x_1 \geq 4 \\ & x_1, x_2 \geq 0 \end{aligned}$$

The solution is $x_1 = 4$, $x_2 = 0.83$, $z = 23.33$

Since z of LP3 is greater than 23 (lower bound), FR3 has to be examined further.

- The branching variable now is x_2, hence, $x_2 \leq 0$ or ≥ 1.
- **Solve LP4:**

$$\begin{aligned} \max \quad & z = 5x_1 + 4x_2 \\ \text{s.t.} \quad & x_1 + x_2 \leq 5 \\ & 10x_1 + 6x_2 \leq 45 \\ & x_1 \geq 4 \\ & x_2 \leq 0 \\ & x_1, x_2 \geq 0 \end{aligned}$$

The optimal solution is: $x_1 = 4.5$, $x_2 = 0$, $z = 22.5$.

Since z of LP4 is lower than the current lower bound, branching from LP4 is stopped.

Solve LP5:

$$\begin{aligned} \max \quad & z = 5x_1 + 4x_2 \\ \text{s.t.} \quad & x_1 + x_2 \leq 5 \\ & 10x_1 + 6x_2 \leq 45 \\ & x_1 \geq 4 \\ & x_2 \geq 1 \\ & x_1, x_2 \geq 0 \end{aligned}$$

LP5 has no feasible solution.

- The branch and bound method can now be terminated.