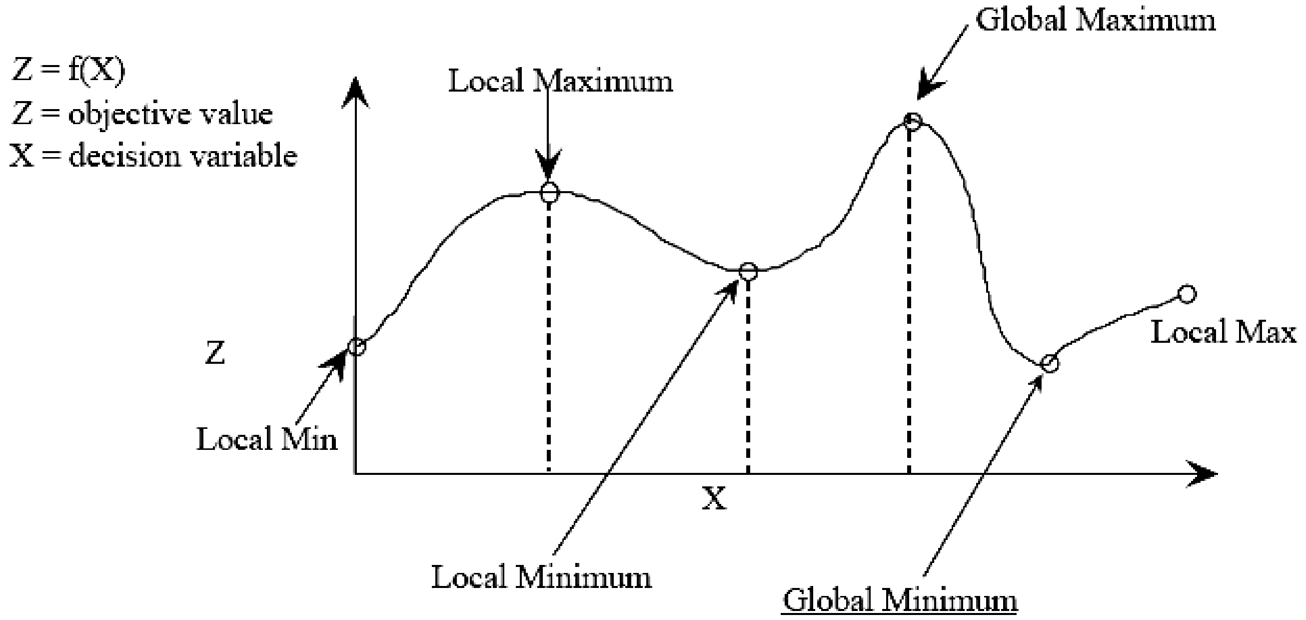
NON –LINEAR PROGRAMMING

Unconstrained Extremal Problems:

An extreme point of a function f (x) defines either a maximum or a minimum point of the function f (x). $x^o = x_1, x_2, \ldots, x_n)$ is max in the value of f (x) @ every point in neighborhood of x^o not exceed $f(x^o)$ thus is :

$$f(x^o + h) \le f(x^o)$$

for h =(h_1, h_2, ………. h_3) such that $|h_j|$ is sufficiently small for all j ,in a similar manner , x^o is a minimum if $f(x^o + h) \ge f(x^o)$.



For a single variable function:
$f(x_3)$ is a global max. (absolute)
$f(x_1)$, $f(x_5)$ are local max (relative)
$f(x_4)$ is a global min. (absolute)
$f(x_0)$, $f(x_2)$ is relative min. (local min.).

Necessary & Sufficient Conditions

Necessary Condition: for x^o to be an extreme point of f (x) is that the gradient vector must be hold that is:

$$\nabla f(x_1, x_2, \ldots, x_n) = 0$$

$z = f(\vec{x})$, $\vec{x} = [x_1, x_2, ---, x_n]$, a vector of n decision variables

$$\nabla f(\vec{x}^o) = \begin{bmatrix} \dfrac{\partial f(\vec{x}^o)}{\partial x_1} \\ \dfrac{\partial f(\vec{x}^o)}{\partial x_2} \\ \dfrac{\partial f(\vec{x}^o)}{\partial x_n} \end{bmatrix} = 0$$

Solve to find x^o

<u>Sufficient Conditions:</u>

For a point x^o to be extreme max. or min. ,that the matrix (***Hessian Matrix***)

$$\mathbf{H}(\vec{x}^o) = \begin{bmatrix} \dfrac{\partial^2 f(\vec{x}^o)}{\partial x_1^2} & \dfrac{\partial^2 f(\vec{x}^o)}{\partial x_1 \ \partial x_2} & \cdots & \dfrac{\partial^2 f(\vec{x}^o)}{\partial x_1 \ \partial x_n} \\ | & & & | \\ | & & & | \\ | & & & | \\ \dfrac{\partial^2 f(\vec{x}^o)}{\partial x_n \ \partial x_1} & \dfrac{\partial^2 f(\vec{x}^o)}{\partial x_n \ \partial x_2} & \cdots & \dfrac{\partial^2 f(\vec{x}^o)}{\partial x_n^2} \end{bmatrix}$$

- **Condition of Optimality – Maxima**

Functions of One Variable

$Z = f(x)$

- Necessary condition for functions of one variable to have a local maxima at $x = x^*$ is

$$\frac{df(x)}{dx} = 0 \text{ at } x = x^*$$

- Sufficient condition for functions of one variable to have a local maxima at $x = x^*$ is

$$\frac{df(x)}{dx} = 0 \text{ at } x = x^* \qquad \frac{d^2 f(x)}{dx^2} \le 0 \text{ at } x = x^*$$

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Condition of Optimality – Minima

Functions of One Variable

$$Z = f(x)$$

Necessary condition for functions of one variable to have a local minima at x = x* is

$$\frac{df(x)}{dx} = 0 \text{ at } x = x^*$$

Sufficient condition for functions of one variable to have a local minima at x = x* is

$$\frac{df(x)}{dx} = 0 \text{ at } x = x^* \qquad \frac{d^2f(x)}{dx^2} \geq 0 \text{ at } x = x^*$$

Condition of Optimality – Maxima

Functions of Two Variable

$$Z = f(X_1, X_2)$$

Necessary condition for bivariate functions to have a local maxima at x = x* is

$$\frac{\partial f(x_1)}{\partial x_1} = \frac{\partial f(x_2)}{\partial x_2} = 0 \text{ at } x_1 = x_1^*, x_2 = x_2^*$$

Sufficient Condition for bivariate functions to have a local Maxima at x = x* is ($f_{11}f_{22}$ - $f_{12}f_{21}$) > 0 and f_{11} and f_{22} < 0,where

$$H_0 = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix} = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}$$

Condition of Optimality – Minima

Functions of Two Variable

$$Z = f(X_1, X_2)$$

Necessary condition for bivariate functions to have a local minima at x = x* is

$$\frac{\partial f(x_1)}{\partial x_1} = \frac{\partial f(x_2)}{\partial x_2} = 0 \text{ at } x_1 = x_1^*, x_2 = x_2^*$$

Sufficient Condition for bivariate functions to have a local Minima at x = x* is ($f_{11}f_{22}$ - $f_{12}f_{21}$) > 0 and f_{11} and f_{22} > 0,where

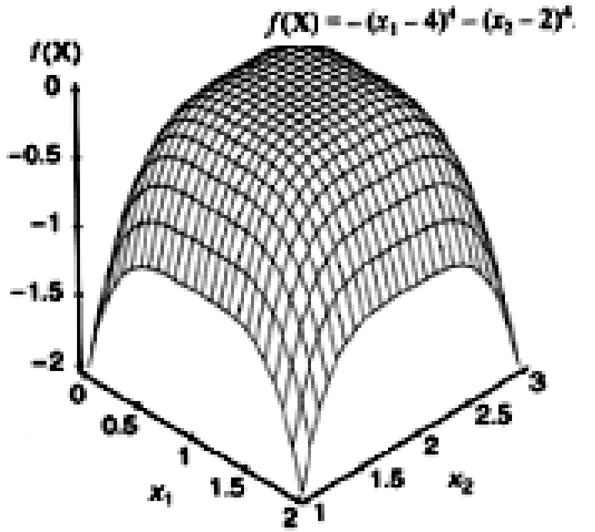
$$H_0 = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix} = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}$$

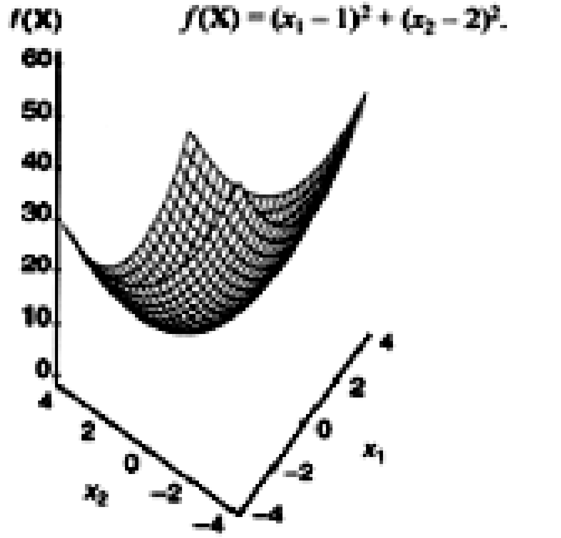
Saddle Point

$$\frac{\partial f(x_1)}{\partial x_1} = \frac{\partial f(x_2)}{\partial x_2} = 0 \text{ at } x_1 = x_1^*,\ x_2 = x_2^*$$

$$H_0 = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix} = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}$$

If $(f_{11}f_{22} - f_{12}f_{21}) < 0$ then it is a Saddle Point at $x_1 = x_1^*$, $x_2 = x_2^*$





- has a positive values of the principal minor determinates than $(x^o, f(x^o))$ is a min. point.
- has the sign of $(-1)^k$, k =1,2,…..,n for the values of the principle minor determinate $(x^o, f(x^o))$ is a max point.

For a single variable function the necessary condition is:

$f'(x^o)=0$

And $f''(x^o) \leq 0$ is a sufficient condition for $f(x^o)$ to be **max.**

$f''(x^o) \geq 0$ is a sufficient condition for $f(x^o)$ to be **min.**

if $f''(x^o) = 0$ higher derivatives must be evaluated and follow the theorem.

Theorem: if @ a point $(x^o, f(x^o))$, the first (n-1) derivative =0 and $f^n(x^o) \neq 0$ then $f(x^o)$ has:

1- an **inflection** point if n is **odd**.

2- an **extreme** point if n is **even.**

$f(x^o)$ = max. if $f^n(x^o) < 0$, and = min. if $f^n(x^o) > 0$

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Ex1: y= f (x) = x^4

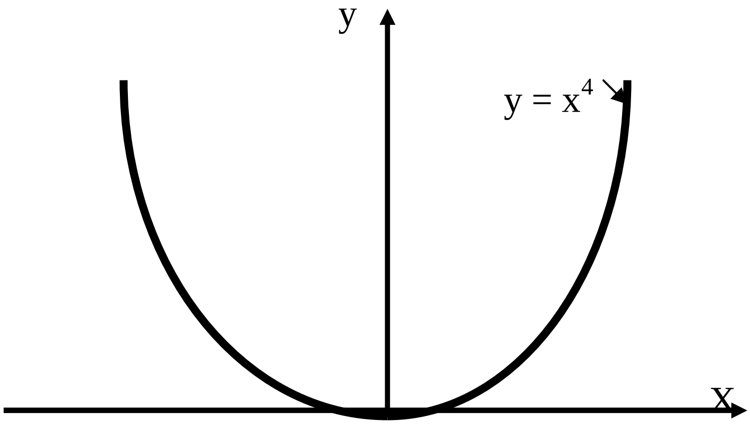
Solution: $f'(x^o) = 4x^3 = 0$ » x = 0
(0 , 0) Extreme point or inflection point

$f''(x) = 12x^2$, $f''(x) = 0$

$f'''(x) = 24x$, $f'''(x) = 0$
$f^{(4)}(x) = 24$, $f^{(4)}(x^o) = 24$

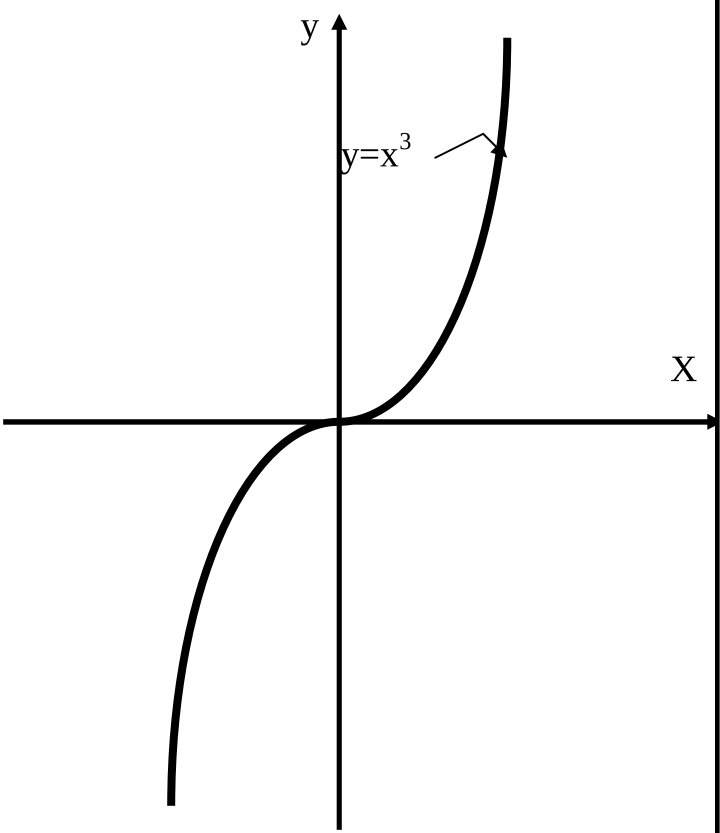
n = 4 (even)
(0,0) is Extreme point

Since $f^4(x^o) > 0$, (0,0) is a min point.

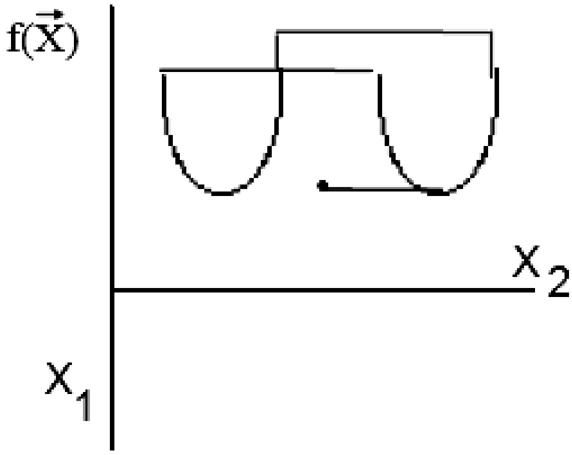


Ex2: y= f (x) = x^3
Solution: $f'(x^o) = 3x^2 = 0$ » x = 0
(0 , 0) Extreme point or inflection point

$f''(x) = 6x$, $f''(x) = 0$
$f'''(x) = 6$, $f'''(x) = 6$
n = 3 (odd)
(0,0) is inflection point



Note: in three dimension inflection point called Saddle point.



Ex3: $f\ (x_1,x_2,x_3) = x_1 + 2x_3 + x_2x_3 - (x_1^{\ 2} + x_2^{\ 2} + x_3^{\ 2})$

$\partial f/\partial x_1 = 1 - 2\ x_1 = 0 \quad \rightarrow x_1 = 1/2$

$\partial f/\partial x_2 = x_3 - 2\ x_2 = 0$ ………………. 1 }

$\partial f/\partial x_3 = 2 + x_2 - 2\ x_3 = 0$ …………….. 2} $\rightarrow$ $x_2 = 2/3$, $x_3 = 4/3$

$x^o = \{1/2\ , 1/3\ , 4/3\}$

$$H(\vec{x}^o) = \begin{bmatrix} \frac{\partial^2 f(\vec{x}^o)}{\partial x_1^2} & \frac{\partial^2 f(\vec{x}^o)}{\partial x_1\ \partial x_2} & \cdots & \frac{\partial^2 f(\vec{x}^o)}{\partial x_1\ \partial x_n} \\ | & & & | \\ | & & & | \\ \frac{\partial^2 f(\vec{x}^o)}{\partial x_n\ \partial x_1} & \frac{\partial^2 f(\vec{x}^o)}{\partial x_n\ \partial x_2} & \cdots & \frac{\partial^2 f(\vec{x}^o)}{\partial x_n^2} \end{bmatrix}$$

$$= \begin{vmatrix} fx_1x_1 & fx_1x_2 & fx_1x_3 \\ fx_2x_1 & fx_2x_2 & fx_2x_3 \\ fx_3x_1 & fx_3x_2 & fx_3x_3 \end{vmatrix} = \begin{vmatrix} -2 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 1 & -2 \end{vmatrix}$$

The values of the principal minor determinates are (-2,4,-6)

It has a sign $(-1)^k$, k =1,2,…..,n

(1/2 , 2/3, 4/3, 1.58) is a max point.

Ex4:

$$\text{MIN } f(\vec{X}) = (X_1-1)^2 + X_2^2 + 1$$

$$\nabla f(\vec{X}) = \begin{bmatrix} \vec{0} \\ 0 \end{bmatrix} = \begin{bmatrix} \dfrac{\partial f(\vec{X}^o)}{\partial X_1} \\ \dfrac{\partial f(\vec{X}^o)}{\partial X_2} \end{bmatrix} = \begin{bmatrix} 2(X_1-1) \\ 2X_2 \end{bmatrix}$$

$X_1 = 1, X_2 = 0.$

$$\mathbf{H}(\vec{X}^o) \begin{bmatrix} \dfrac{\partial^2 f(\vec{X}^o)}{\partial X_1^2} & \dfrac{\partial^2 f(\vec{X}^o)}{\partial X_1 \partial X_2} \\ \dfrac{\partial^2 f(\vec{X}^o)}{\partial X_2 \partial X_1} & \dfrac{\partial^2 f(\vec{X}^o)}{\partial X_2^2} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

Is <u>H</u> positive definite?

$$\Delta\vec{X}_i^T \mathbf{H}\ \Delta\vec{X}_i > 0 \text{ for all } \Delta X \neq 0$$

$$[\Delta X_1, \Delta X_2] \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \Delta X_1 \\ \Delta X_2 \end{bmatrix} \overset{?}{>} 0$$

$$[2\Delta X_1, 2\Delta X_2] \begin{bmatrix} \Delta X_1 \\ \Delta X_2 \end{bmatrix} \overset{?}{>} 0$$

$$2\Delta X_1^2 + 2\Delta X_2^2 > 0, \text{ for any } \overrightarrow{\Delta X} \neq 0$$

H is positive definite.

Ex5: Optimal Container Dimension

-) A cylinder must contain a given volume of oil, V.
-) Find the height and radius of the cylinder that will minimize the cylinder's surface area.

Min $S = 2\pi rh + 2\pi r^2$

S.T. $V = \pi r^2 h \rightarrow h = \frac{V}{\pi r^2}$

Min $S = 2\pi r \frac{V}{\pi r^2} + 2\pi r^2$

or

Min $S = \frac{2V}{r} + 2\pi r^2$.

1st-order condition:

$$\frac{dS}{dr} = 0 \rightarrow 0 = \frac{-2V}{r^2} + 4\pi r$$

$$0 = 4\pi r^3 - 2V \rightarrow \sqrt[3]{\frac{V}{2\pi}} = r^*$$

2nd-order condition:

$$\frac{d^2S}{dr^2} = \frac{4V}{r^3} + 4\pi > 0 \rightarrow \frac{V}{r^3} > -\pi$$

If $V = 10ft^3$, $r^* = 1.167$ ft from 1st-order condition, and $\frac{V}{r^3} > -\pi$, satisfying 2nd-order condition for a minimum of S.

Ex6: Optimal scheduling of water meter Maintenance

A water meter is used to charge a large industrial customer for water use. The water utility must decide how often to repair the meter, given that the accuracy deteriorates with time. The meter always deteriorates such that it registers less than the actual flow amount.

The average yearly volume used is 10,000 x 100 cu.ft. (ccf). The price of water is 50¢/ccf. The cost of repairing the meter (which restores accuracy) is $500. The meter loses 1% of its remaining accuracy each year.

Find how often the utility should repair the meter to minimize present value of total repair and revenue-loss costs. The interest rate is 5%.

The present value of costs and losses over one period between repairs of length T is

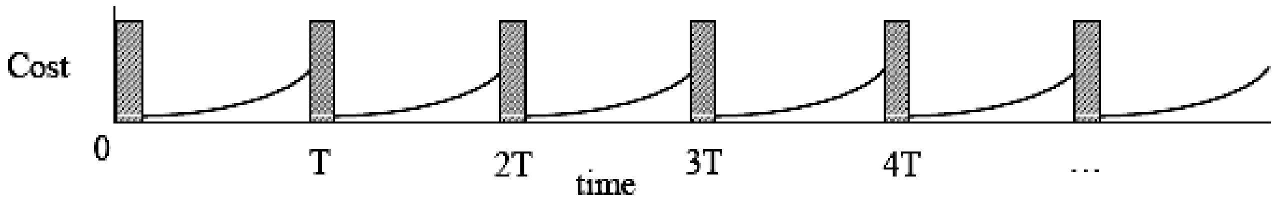
$$C(T) = 500 + 0.5 \int_0^T 10{,}000\,(1 - e^{-0.01t})\; e^{-0.05t}\,dt$$

$$= 500 + 5{,}000\left(\frac{1}{0.05}(1 - e^{-0.05T}) - \frac{1}{0.06}(1 - e^{-0.06T})\right)$$

$$C(T) = 500 + 5{,}000\left(3.333 - \frac{e^{-0.05T}}{0.05} + \frac{e^{-0.06T}}{0.06}\right)$$

An infinite series of these repairs, occurring at intervals T, has a present value cost of:

$$PV(T) = \sum_{k=0}^{\infty} C(T) e^{-0.05(kT)}$$



This infinite series simplifies to $PV(T) = C(T) + PV(T)e^{-0.05T}$, which further simplifies to:

$$PV(T) = \frac{C(T)}{1 - e^{-0.05T}}$$

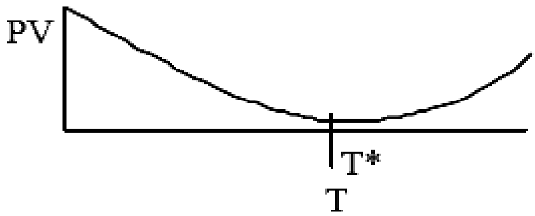
The objective is to minimize this present value cost.

$$\underline{\text{Min}}\;\; PV(T) = \frac{C(T)}{1 - e^{-0.05T}}$$

$$PV(T) = \frac{500}{1-e^{-0.05T}} + 100{,}000 - 83{,}333\;\frac{1-e^{-0.06T}}{1-e^{-0.05T}}$$

Solution by search/enumeration:

Plot



and select T.*

Change-out Period	Present Value Cost
A	B
1	10742.8
2	6229.3
3	5036.6
3.5	4796.0
4	4665.69
4.5*	4614.94*
5	4616.98
5.5	4656.7
6	4723.9
6.5	4811.9
7	4915.4
8	5155.8
9	5425.5
10	5712.6
11	6009.8
12	6312.3

Solution by Calculus:

1st order condition: Solve: $\frac{dPV(T)}{dT} = 0$

$$\rightarrow \frac{0.01}{0.06} e^{-0.06T} - e^{-0.01T} = \frac{0.05(500)}{5,000} - \frac{0.05}{0.06}$$

$$0.167\, e^{-0.06T} - e^{-0.01T} = -0.8283$$

Here, the first-order condition needs a numerical solution.

T	LHS	
10	-0.8132	
9	-0.8166	
7	-0.8227	
5	-0.8275	
4.5	-0.8285	T* ≈ 4.5 years
4	-0.8294	

The steepest ascent method:

Ex1: Max. $f(x_1,x_2) = 4x_1 + 6x_2 - 2(x_1^2 + x_1 x_2 + x_2^2)$, start with $x_1 = x_2 = 1$

Solution: 1st iteration

We will find the optimal step size (r) that max $f(x_1^{i+1})$.

Where $x_1^{i+1} = x_1^i + r \nabla f(x_1^i)$

$\nabla f(x_1^i) = \nabla f(x_1, x_2) = (4 - 4x_1 - 2x_2), (6 - 2x_1 - 4x_2)$

$\nabla f(1,1) = (-2,0)$

$x_1^{i+1} = (1,1) + r(-2,0) = \{(1-2r), 1\}$

$f(x_1^{i+1}) = f((1-2r), 1) = 4(1-2r) + 6 - 2\{(1-2r)^2 + (1-2r) + 1\}$

$f(x_1^{i+1}) = f((1-2r), 1) = -2(1-2r)^2 + 2(1-2r) + 4$

solve for (r) for max. $f((1-2r), 1)$

$f'(x_1^{i+1}) = f'((1-2r), 1) = -4(1-2r)(-2) - 4 = 0$

$8 - 16r - 4 = 0 \rightarrow r = 1/4$

$x_1^{i+1} = (1/2, 1)$

x_1	x_2	r	∇f	
1	1	1/4		
1/2	1	1/4	(0,0)	Relative max.
1/2	5/4	1/4		
3/8	5/4	1/4	$(0,1/16) \approx (0,0)$	Absolute max.

Constrained extermal problems:

Equality Constraint:

Ex: Min $Z = f(x_1, x_2, \ldots, x_n)$

Subject to: $g_1(x_1, x_2, \ldots, x_n) = 0$

$g_2(x_1, x_2, \ldots, x_n) = 0$

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$g_m(x_1, x_2, \ldots, x_n) = 0$

Lagrangean Method **(Indirect Search Method)**

$L(x, \lambda) = f(x_1, x_2, \ldots, x_n) - \lambda\, g_1(x_1, x_2, \ldots, x_n) - \lambda\, g_m(x_1, x_2, \ldots, x_n)$

Where L = Lagrangean function

λ = Lagrangean multiplier

The Lagrangean function can be used directly to generate the **necessary condition**, this means that the optimization of $f(x)$ subjected to $g(x)$ is equivalent to the optimization of Lagrangean $L(x, \lambda)$.

The **sufficiency condition for** the Lagrangean Function is defined as follows:

$$H^B = \left[\begin{array}{c|c} O & P \\ \hline P^T & Q \end{array}\right]_{(m+n)\times(m+n)}$$

$$P = \begin{bmatrix} \nabla g_1 \\ - \\ - \\ .- \\ \nabla g_m \end{bmatrix}_{(m \times n)}$$

$$Q = \left[\frac{\partial^2 L}{\partial x_i \partial x_j}\right]$$

Where :

H^B : Bordered Hessian matrix

Depending on H^B , $(x^o, f(x^o))$

1- A **max. point** if :

Starting with the principal major determinate of order (2m +1) , the last (n-m) principal minor determinates **has the sign of $(-1)^k$.**

2- A **min. point** if :

Starting with the principal major determinate of order (2m +1) , the last (n-m) principal minor determinates **has the sign of $(-1)^m$.**

Note: $(x^o, f(x^o))$ may be an extreme point without satisfying these conditions.

Ex1: Min $f(x_1, x_2, x_3) = (x_1^2 + x_2^2 + x_3^2)$

Subject to: $x_1 + x_2 + 3x_3 - 2 = 0$

$5x_1 + 2x_2 + 3x_3 - 5 = 0$

Solution: $L(x, \lambda) = f(x_1, x_2, x_3, \lambda_1, \lambda_2)$

$L(x, \lambda) = x_1^2 + x_2^2 + x_3^2 - \lambda_1(x_1 + x_2 + 3x_3 - 2) - \lambda_2(5x_1 + 2x_2 + 3x_3 - 5)$

$\partial L/\partial x_1 = 2x_1 - \lambda_1 - 5\lambda_2 = 0$

$\partial L/\partial x_2 = 2x_2 - \lambda_1 - 2\lambda_2 = 0$

$\partial L/\partial x_3 = 2x_3 - 3\lambda_1 - \lambda_2 = 0$

$\partial L/\partial \lambda_1 = -(x_1 + x_2 + 3x_3 - 2) = 0$

$\partial L/\partial \lambda_2 = -(5x_1 + 2x_2 + 3x_3 - 5)$

Solve for $x_1, x_2, x_3, \lambda_1, \lambda_2$

$x^o = \{x_1, x_2, x_3\}$ {0.81 , 0.35 ,0.28}

$\lambda^o = \{\lambda_1, \lambda_2\}$ {0.0867 , 0.3067}

$$P = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 1 \end{bmatrix} ,$$

$$Q = \begin{bmatrix} fx_1x_1 & fx_1x_2 & fx_1x_3 \\ fx_2x_1 & fx_2x_2 & fx_2x_3 \\ fx_3x_1 & fx_3x_2 & fx_3x_3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$H^B = \left[\begin{array}{cc:ccc} 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 5 & 2 & 1 \\ \hdashline 1 & 5 & 2 & 0 & 0 \\ 1 & 2 & 0 & 2 & 0 \\ 3 & 1 & 0 & 0 & 2 \end{array}\right]$$

$n = 3$, $m = 2$, $n – m = 1$, $2m +1 = 5$

we need to check the $|H^B|$ sign

$$|H^B| = \begin{vmatrix} 0 & 0 & 2 & 1 \\ 1 & 5 & 0 & 0 \\ 1 & 2 & 2 & 0 \\ 3 & 1 & 0 & 2 \end{vmatrix} - \begin{vmatrix} 0 & 0 & 5 & 1 \\ 1 & 5 & 2 & 0 \\ 1 & 2 & 0 & 0 \\ 3 & 1 & 0 & 2 \end{vmatrix} + 3 \begin{vmatrix} 0 & 0 & 5 & 2 \\ 1 & 5 & 2 & 0 \\ 1 & 2 & 0 & 2 \\ 3 & 1 & 0 & 0 \end{vmatrix}$$

$$= 2\begin{vmatrix} 1 & 5 & 0 \\ 1 & 2 & 0 \\ 3 & 1 & 2 \end{vmatrix} - \begin{vmatrix} 1 & 5 & 0 \\ 1 & 2 & 2 \\ 3 & 1 & 0 \end{vmatrix} - 5\begin{vmatrix} 1 & 5 & 0 \\ 1 & 2 & 0 \\ 3 & 1 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 5 & 2 \\ 1 & 2 & 0 \\ 3 & 1 & 0 \end{vmatrix} + 3*5\begin{vmatrix} 1 & 5 & 0 \\ 1 & 2 & 2 \\ 3 & 1 & 0 \end{vmatrix} - 3*2\begin{vmatrix} 1 & 5 & 2 \\ 1 & 2 & 0 \\ 3 & 1 & 0 \end{vmatrix}$$

$$= 2 (4- 5*2) – (-2 -5 (-6)) -5 (4 – 5 (2)) + 2 (1- 6) + 15 ((- 2) – 5 (-6)) – 2 (1- 6))$$
$$= - 12 – 28 + 30 – 10 + 15* 28 + 60$$
$$= 460 > 0$$

The values of the principal minor determinates It has a sign $(-1)^m$, $(-1)^2 > 0$

$(x^o , f (x^o))$ is a min point.

Ex2: Solve Min $Z = (x_1^2 + x_2^2 +x_3^2)$

Subject to: $4 x_1 + x_2^2 + 2x_3 – 14 = 0$

Solution: $L(x , \lambda) = f (x_1, x_2, x_3, \lambda)$

$L(x , \lambda) = x_1^2 + x_2^2 +x_3^2 - \lambda (4 x_1 + x_2^2 + 2x_3 – 14)$

$\partial L/\partial x_1 = 2 x_1 - 4 \lambda = 0 \rightarrow x_1= 2 \lambda$

$\partial L/\partial x_2 = 2 x_2 - 2\lambda x_2 = 0 \quad x_2 (2 -2 \lambda) = 0$ --------------- 1

$\partial L/\partial x_3 = 2 x_3 - 2 \lambda = 0 \rightarrow x_3= \lambda$

$\partial L/\partial \lambda = - (4x_1+ x_2^2 + 2x_3 – 14) = 0$, $- \{4 (2 \lambda) + x_2^2 + 2 \lambda – 14) \} = 0$ -------------- 2

$x_2 = \pm \sqrt{14 - 10 \lambda}$ Substituting in eq.(1) find x_1, x_2, x_3, λ

$(x^o , \lambda)_1= (2 , 2 , 1, 1)$

$(x^o , \lambda)_2= (2 , -2 , 1, 1)$

$(x^o, \lambda)_3 = (2.8, 0, 1.4, 1.4)$

$$H^B = \left[\begin{array}{c|ccc} 0 & 4 & 2x_2 & 2 \\ \hline 4 & 2 & 0 & 0 \\ 2x_2 & 0 & 2(1-\lambda) & 0 \\ 2 & 0 & 0 & 0 \end{array}\right]$$

m=1, n =3, n-m=2

The values of the principal minor 2m +1 = 3 determinates must be has a sign $(-1)^m$, $(-1)^1$ = -1> 0

For(2 , 2 , 1 , 1)

$$\begin{vmatrix} 0 & 4 & 4 \\ 4 & 2 & 0 \\ 4 & 0 & 0 \end{vmatrix} = -32 < 0 \qquad \begin{vmatrix} 0 & 4 & 4 & 2 \\ 4 & 2 & 0 & 0 \\ 4 & 0 & 0 & 0 \\ 2 & 0 & 0 & 2 \end{vmatrix} = -64 < 0$$

For(2 , -2 , 1 , 1)

$$\begin{vmatrix} 0 & 4 & -4 \\ 4 & 2 & 0 \\ -4 & 0 & 0 \end{vmatrix} = -32 < 0 \qquad \begin{vmatrix} 0 & 4 & -4 & 2 \\ 4 & 2 & 0 & 0 \\ -4 & 0 & 0 & 0 \\ 2 & 0 & 0 & 2 \end{vmatrix} = -64 < 0$$

For(2.8 , 0 , 1.4 , 1.4)

$$\begin{vmatrix} 0 & 4 & 0 \\ 4 & 2 & 0 \\ 4 & 0 & -0.8 \end{vmatrix} = 12.8 > 0 \qquad \begin{vmatrix} 0 & 4 & 0 & 2 \\ 4 & 2 & 0 & 0 \\ 4 & 0 & -0.8 & 0 \\ 2 & 0 & 0 & 2 \end{vmatrix} = 32 > 0$$

$(x^o, \lambda)_1$&$(x^o, \lambda)_2$ are absolute min points.

Ex3:

Step 1 Max $f = X_1 + X_2$

S.T. $2X_1^2 + X_2^2 = 1$

Step 2

$$L = X_1 + X_2 + \lambda (1 - 2X_1^2 - X_2^2)$$

Step 3

$$\frac{\partial L}{\partial X_1} = 1 - 4\lambda X_1 = 0 \qquad (1)$$

$$\frac{\partial L}{\partial X_2} = 1 - 2\lambda X_2 = 0 \qquad (2)$$

$$\frac{\partial L}{\partial \lambda} = 1 - 2X_1^2 - X_2^2 = 0 \qquad (3)$$

$$(1) \rightarrow \lambda = \frac{1}{4X_1}$$

$$(2) \rightarrow \lambda = \frac{1}{2X_2}$$

$$(1)\ \&\ (2) \rightarrow 4X_1 = 2X_2 \rightarrow 2X_1 = X_2$$

$$\text{into } (3) \rightarrow 1 = 2X_1^2 + (2X_1)^2 \rightarrow \boxed{X_1 = \frac{1}{\sqrt{6}}} \quad \boxed{f = \frac{1}{\sqrt{6}} + \frac{2}{\sqrt{6}} = \frac{3}{\sqrt{6}}}$$

$$\boxed{X_2 = \frac{2}{\sqrt{6}}} \quad \boxed{\lambda = \frac{3}{2\sqrt{6}}}$$

Step 4:

$$\lambda = \frac{\partial f_{max}}{\partial b} = \frac{3}{2\sqrt{6}}$$

$\Delta f_{max} = \lambda\, \Delta b$, for small Δb.

Raising b from 1 to 2 should raise f_{max} by $\frac{3}{2\sqrt{6}}$.

> λ is called a Lagrange multiplier, shadow price, or dual variable.

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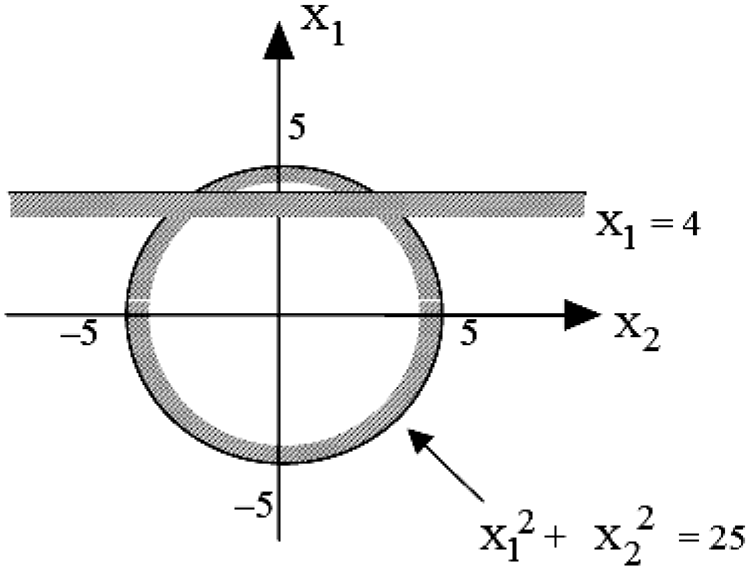
EX4:

Step 1: Max $Z = X_1 + X_2$

S.T. $X_1^2 + X_2^2 \leq 25$

$X_1 \leq 4$



Step 2: Form the Lagrangian.

$$L = X_1 + X_2 + \lambda_1 (25 - X_1^2 - X_2^2) + \lambda_2 (4 - X_1)$$

Step 3: Solve the Lagrangian for the first-order conditions, both for the X_is and the λ_js.

Solve: (1) $\frac{\partial L}{\partial X_1} = 0 = 1 - 2\lambda_1 X_1 - \lambda_2$

(2) $\frac{\partial L}{\partial X_2} = 0 = 1 - 2\lambda_1 X_2$

(3) $\frac{\partial L}{\partial \lambda_1} = 0$, and

(4) $\frac{\partial L}{\partial \lambda_2} = 0.$

May go away, depending on value assumed for λ_1 and λ_2

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FOUR cases exist for the possible sets of Lagrange Multiplier values. For n inequality constraints, there are 2n cases.

1) $\lambda_1 = 0$, $\lambda_2 = 0$ Solution: $Z \to \infty$, $X_1 \to \infty$, $X_2 \to \infty$, but violates both constraints.

2) $\lambda_1 = 0$, $\lambda_2 \neq 0$: Solution: $Z \to \infty$, $X_1 = 4$, $X_2 \to \infty$, but violates 1st constraint.

3) $\lambda_1 \neq 0$, $\lambda_2 = 0$: 1st constraint binds; 2nd constraint is non-binding. There are two solutions for this case:

Solution 1: $Z = 7.07$, $X_1 = X_2 = \sqrt{\frac{25}{2}} \approx 3.54$. OK with both constraints;

Solution 2: $Z = -7.07$, $X_1 = X_2 = -\sqrt{\frac{25}{2}}$. OK with both constraints, but solution is a minimum.

4) $\lambda_1 \neq 0$, $\lambda_2 \neq 0$: Both constraints bind, OK with both constraints; Two solutions:
Solution 1: $Z = 7$, $X_1 = 4$, $X_2 = 3$,
Solution 2: $Z = 1$; $X_1 = 4$, $X_2 = -3$.