2 Limits & Continuity

In this chapter, we'll define how limit of function values are defined and calculated.

Definition: the limit of f(x) as x tends to a is defined as the value of f(x) as x approaches closer and closer to a without actually reaching it and denoted by:

$\lim_{x\to a} f_{(x)} = L$ L is a single finite real number

It's important to know

1. We don't evaluate the limit by actually substituting x = a in f(x) in general, although in some cases its possible.
2. The value of the limit can depend on which side its approach
3. The limit may not exist at all.

Example 13: to explain the concept of limit, take the function f(x) = 2x – 4 if the

$\lim_{x\to 1} f_{(x)} = 2*1-4=-2$

But the following table express many values of x can be expressed close to 1.

x	0.5	0.8	0.9	0.99	0.999	1.001	1.01	1.1	1.2
f(x)	-3	-2.4	-2.2	-2.02	-2.002	-1.998	-1.98	-1.8	-1.6

Question: Why we take values approaches to 2 in example 13 instead we take x = 1 directly?

Solution: the answer about this question can be expressed in the following example:

$$f_{(x)} = 3^{-\frac{1}{x^2}} + 1$$

If x = 0 then 1/0 = ∞

So..

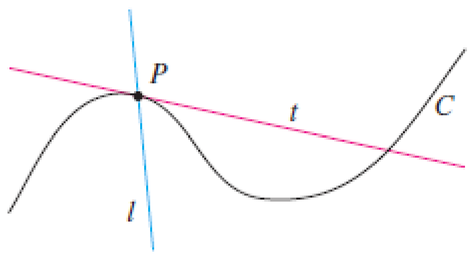
x	±0.2	±0.5	etc..
f(x)	1.00000	1.012345679	

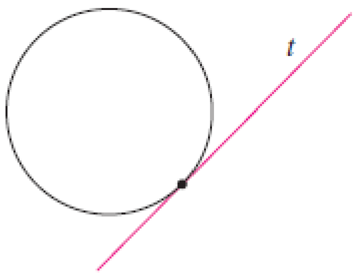
In limits we avoid ∞

THE TANGENT PROBLEM

The word tangent is derived from the Latin word tangens, which means "touching." Thus a tangent to a curve is a line that touches the curve. In other words, a tangent line should have the same direction as the curve at the point of contact. How can this idea be made precise?

For a circle we could simply follow Euclid and say that a tangent is a line that intersects the circle once and only once as in Figure (a). For more complicated curves this definition is inadequate as shown in Figure (b)





Example 14: Find an eq. of the tangent line to the parabola y = x^2 at point (1,1)?

Solution

We will be able to find an equation of the tangent line t as soon as we know its slope m. The difficulty is that we know only one point, P, on t, whereas we need two points to compute the slope. But observe that we can compute an approximation to m by choosing a nearby point $Q(x, x^2)$ on the parabola (as in Figure) and computing the slope m_{PQ} of the secant line PQ.

We choose $x \neq 1$ so that $Q \neq P$. Then

$$m_{PQ} = \frac{x^2 - 1}{x - 1}$$

For instance, for the point $Q(1.5, 2.25)$ we have

$$m_{PQ} = \frac{2.25 - 1}{1.5 - 1} = \frac{1.25}{0.5} = 2.5$$

The tables in the margin show the values of m_{PQ} for several values of x close to 1. The closer Q is to P, the closer x is to 1 and, it appears from the tables, the closer m_{PQ} is to 2. This suggests that the slope of the tangent line t should be $m = 2$.

We say that the slope of the tangent line is the *limit* of the slopes of the secant lines, and we express this symbolically by writing

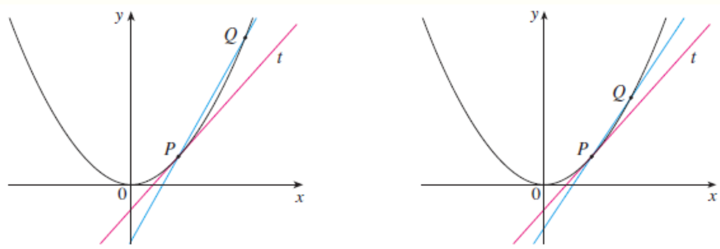
$$\lim_{Q \to P} m_{PQ} = m \qquad \text{and} \qquad \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2$$

Assuming that the slope of the tangent line is indeed 2, we use the point-slope form of the equation of a line to write the equation of the tangent line

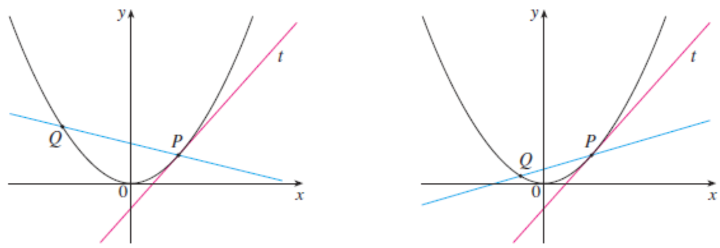
$$y - 1 = 2(x - 1) \qquad \text{or} \qquad y = 2x - 1$$

x	m_{PQ}
2	3
1.5	2.5
1.1	2.1
1.01	2.01
1.001	2.001

x	m_{PQ}
0	1
0.5	1.5
0.9	1.9
0.99	1.99
0.999	1.999



Q approaches P from the right



Q approaches P from the left

THE VELOCITY PROBLEM

If you watch the speedometer of a car as you travel in city traffic, you see that the needle doesn't stay still for very long; that is, the velocity of the car is not constant. We assume from watching the speedometer that the car has a definite velocity at each moment, but how is the "instantaneous" velocity defined? Let's investigate the example of a falling ball.

Suppose that a ball is dropped from the upper observation deck of the CN Tower in Toronto, 450 m above the ground. Find the velocity of the ball after 5 seconds.

Through experiments carried out four centuries ago, Galileo discovered that the distance fallen by any freely falling body is proportional to the square of the time it has been falling. (This model for free fall neglects air resistance.) If the distance fallen after t seconds is denoted by $s(t)$ and measured in meters, then Galileo's law is expressed by the equation

$$s(t) = 4.9t^2$$

The difficulty in finding the velocity after 5 s is that we are dealing with a single instant of time ($t = 5$), so no time interval is involved. However, we can approximate the desired quantity by computing the average velocity over the brief time interval of a tenth of a second from $t = 5$ to $t = 5.1$:

$$\text{average velocity} = \frac{\text{change in position}}{\text{time elapsed}}$$

$$= \frac{s(5.1) - s(5)}{0.1}$$

The following table shows the results of similar calculations of the average velocity over successively smaller time periods.

Time interval	Average velocity (m/s)
$5 \leq t \leq 6$	53.9
$5 \leq t \leq 5.1$	49.49
$5 \leq t \leq 5.05$	49.245
$5 \leq t \leq 5.01$	49.049
$5 \leq t \leq 5.001$	49.0049

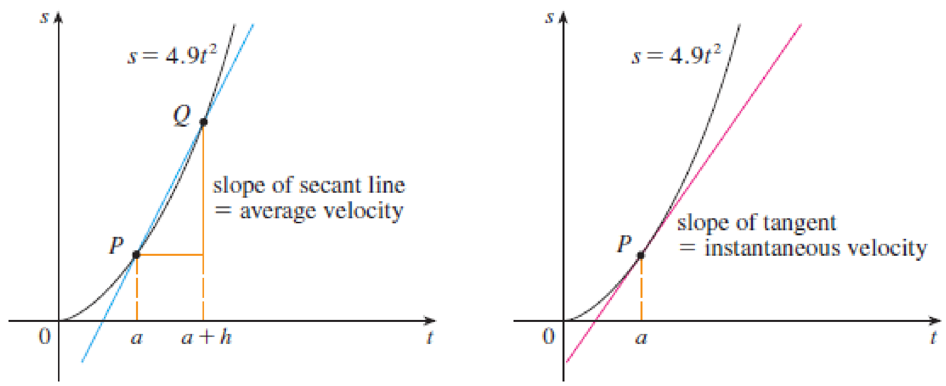
It appears that as we shorten the time period, the average velocity is becoming closer to 49 m/s. The **instantaneous velocity** when $t = 5$ is defined to be the limiting value of these average velocities over shorter and shorter time periods that start at $t = 5$. Thus the (instantaneous) velocity after 5 s is

$$v = 49 \text{ m/s}$$ □

You may have the feeling that the calculations used in solving this problem are very similar to those used earlier in this section to find tangents. In fact, there is a close connection between the tangent problem and the problem of finding velocities. If we draw the graph of the distance function of the ball (as in Figure 5) and we consider the points $P(a, 4.9a^2)$ and $Q(a + h, 4.9(a + h)^2)$ on the graph, then the slope of the secant line PQ is

$$m_{PQ} = \frac{4.9(a + h)^2 - 4.9a^2}{(a + h) - a}$$

which is the same as the average velocity over the time interval $[a, a + h]$. Therefore, the velocity at time $t = a$ (the limit of these average velocities as h approaches 0) must be equal to the slope of the tangent line at P (the limit of the slopes of the secant lines).



Example 15: Discuss the function $f_{(x)} = \dfrac{x^2 - 9}{x - 3}$

If (1) x = 1, x = 2

(2) x = 3

(3) x→1, x→2

(4) x→3

Solution:

$f_{(x)} = \frac{x^2-9}{x-3} = \frac{(x-3)(x+3)}{(x-3)} = x+3$ and x ≠ 3

Its equivalent to g(x) = x+3 and x ≠ 3, then:

f(1) = g(1) = 4

f(2) = g(2) = 5

if x ➞1 then f(x) = 4 and $\lim_{x\to 1} f_{(x)} = 4$

if x = 3 then f(3) = 0/0 = ∞

if x ➞ 3 then $\lim_{x\to 3} f_{(x)} = 6$

note: if f(x) is defined by two different forms before and after x = a then we must discuss the left limit and the right limit.

Properties of limits:

If $\lim_{x\to a} f_{(x)} = b$ $\lim_{x\to a} f_{(x)} = c$

Then:

1. $\lim_{x\to a} k\ f_{(x)} = kb$ for any constant k
2. $\lim_{x\to a}[\ f_{(x)} \pm g_{(x)}] = \lim_{x\to a} f_{(x)} \pm \lim_{x\to a} g_{(x)} = b + c$
3. $\lim_{x\to a}[\ f_{(x)}.g_{(x)}] = \lim_{x\to a} f_{(x)}.\lim_{x\to a} g_{(x)} = b.c$
4. $\lim_{x\to a}[\ f_{(x)} / g_{(x)}] = \lim_{x\to a} f_{(x)} / \lim_{x\to a} g_{(x)} = b/c\ if\ c \neq 0$
5. $\lim_{x\to a}[\ f_{(x)}]^{1/n} == b^{1/n}$ $real\ values\ only\ for\ n$

The limit must exist before applying the above results.

Example 16: find the limits of the following functions:

1. $\lim_{x\to 3} = \frac{x^2-9}{x-3}$ = $\lim_{x\to 3} = \frac{(x-3)(x+3)}{(x-3)} = \lim_{x\to 3}(x+3) = 3+3 = 6$

2. $\lim_{x\to 2} = \frac{\sqrt{x+2}-2}{x-2} = \lim_{x\to 2} = \frac{\sqrt{x+2}-2}{x-2} * \frac{\sqrt{x+2}+2}{\sqrt{x+2}+2} = \lim_{x\to 2} = \frac{(x+2)-4}{x-2\sqrt{x+2}+2}$

$= \frac{1}{\sqrt{2+2}+2} = \frac{1}{2+2} = \frac{1}{4}$

3. $\lim_{x\to\infty} = \frac{3x^2+2x+1}{5x^2+7x+1}$ ÷ x^2

$\lim_{x\to\infty} = \frac{3+\frac{2}{x}+\frac{1}{x^2}}{5+\frac{7}{x}+\frac{4}{x^2}} = \frac{3}{5}$

Note: $\lim_{x\to\infty} = \frac{a_n x^n + a_{n-1}x^{n-1} + ...+ a_o}{b_m x^m + a_{m-1}x^{m-1} + ...+ b_o} = \left\langle \begin{array}{ll} 0 & n<m \\ \frac{a}{b} & n=m \\ \infty & n>m \end{array} \right.$

Example 17: find

$\lim_{x\to 1} = \frac{x^2-1}{(x-1)\sqrt{x^2+2x+3}}$

Solution:

$= \lim_{x\to 1} = \frac{(x-1)(x+1)}{(x-1)\sqrt{x^2+2x+3}} = \lim_{x\to 1}(x+1) - \lim_{x\to 1}\sqrt{x^2+2x+3} = 2 \div \sqrt{\lim_{x\to 1}(x^2+2x+3)}$

$= 2 \div \sqrt{6} = \frac{2}{\sqrt{6}}$

Theorem I If g(x) ≤ f(x) ≤ h(x) and $\lim_{x\to a} g_{(x)} = \lim_{x\to a} h_{(x)} = L$

Theorem II $\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1$ or $\lim_{x\to a}\frac{\sin(x-a)}{(x-a)}=1$

Example 18:

1. $\lim_{x\to 0}\frac{\sin 5x}{\sin 7x}=\frac{\frac{\sin 5x}{5x}.5x}{\frac{\sin 7x}{7x}.7x}=5/7$

2. $\lim_{x\to 2}\frac{\sin(x-2)}{x^2-4}=\lim_{x\to 2}\frac{\sin(x-2)}{(x-2)(x+2)}=\frac{1}{(x+2)}=1/4$

Left and right – side limits

Example 19: Discuss the $\lim_{x\to 2} f(x)$ if $f(x)=\begin{cases}3x+2 & x<2\\ 4 & x=2\\ 8-x & x>2\end{cases}$

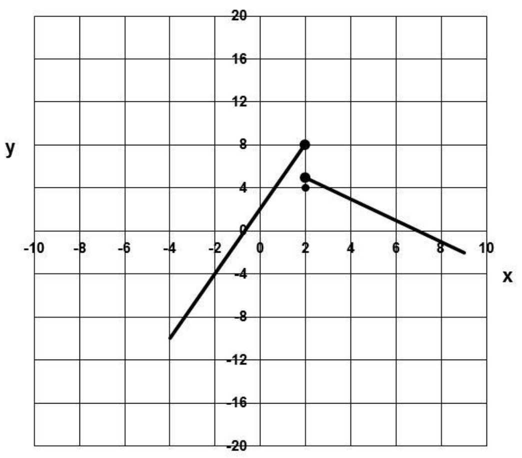
Solution:

If x > 2

Then f (2^+) = $\lim_{x\to 2} f(x)=\lim_{x\to 2}(8-x)=8-2=6$

If x < 2 then

f(2^-) $\lim_{x\to 2} f(x)=\lim_{x\to 2}(3x+2)=8$



Then right limit ≠ left limit at x = 2

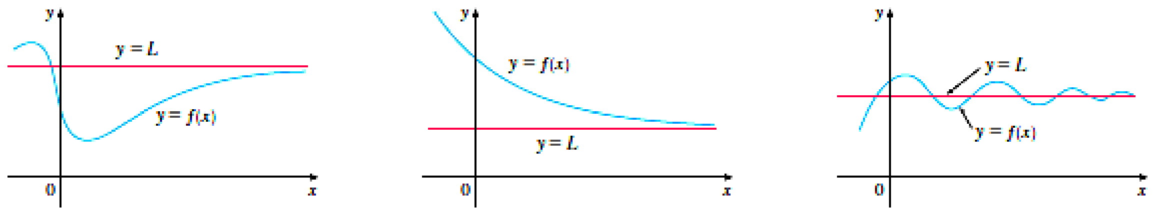
Then, we say that $\lim_{x\to 2} f(x)$ doesn’t exists

Limits at Infinity: Horizontal Asymptote

1 DEFINITION Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x \to \infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large.



The line L is called horizontal asymptote of the graph of the function (f). If the value of f(x) increases without bound as x $\to$ + ∞ or x $\to$ - ∞, then we write:

$$\lim_{x \to +\infty} f(x) = +\infty \quad \text{Or} \quad \lim_{x \to -\infty} f(x) = -\infty$$

If the value of f(x) decreases without bound as x + ∞ or x - ∞, then we write:

$$\lim_{x \to +\infty} f(x) = -\infty \quad \text{Or} \quad \lim_{x \to -\infty} f(x) = +\infty$$

2 DEFINITION Let f be a function defined on some interval $(-\infty, a)$. Then

$$\lim_{x \to -\infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large negative.

3 DEFINITION The line $y = L$ is called a **horizontal asymptote** of the curve $y = f(x)$ if either

$$\lim_{x \to \infty} f(x) = L \qquad \text{or} \qquad \lim_{x \to -\infty} f(x) = L$$

Example 20: Find $\lim_{x\to\infty} \frac{1}{x}$ and $\lim_{x\to-\infty} \frac{1}{x}$

Solution:

Observe that when x is large, $1/x$ is small. For instance,

$$\frac{1}{100} = 0.01 \qquad \frac{1}{10{,}000} = 0.0001 \qquad \frac{1}{1{,}000{,}000} = 0.000001$$

In fact, by taking x large enough, we can make $1/x$ as close to 0 as we please. Therefore, according to Definition 1, we have

$$\lim_{x\to\infty} \frac{1}{x} = 0$$

Similar reasoning shows that when x is large negative, $1/x$ is small negative, so we also have

$$\lim_{x\to-\infty} \frac{1}{x} = 0$$

It follows that the line $y = 0$ (the x-axis) is a horizontal asymptote of the curve $y = 1/x$. (This is an equilateral hyperbola; see Figure 6.) □

5 THEOREM If $r > 0$ is a rational number, then

$$\lim_{x\to\infty} \frac{1}{x^r} = 0$$

If $r > 0$ is a rational number such that x^r is defined for all x, then

$$\lim_{x\to-\infty} \frac{1}{x^r} = 0$$

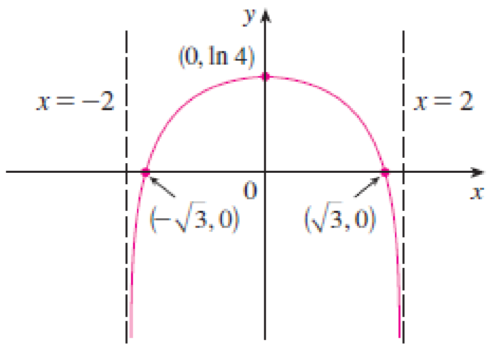
Infinite limits and Vertical Asymptotes

As the line x = a is a vertical asymptote if at least one of the following statements is true:

$$\lim_{x \to a^+} f(x) = \infty \qquad \lim_{x \to a^-} f(x) = \infty$$

$$\lim_{x \to a^+} f(x) = -\infty \qquad \lim_{x \to a^-} f(x) = -\infty$$

Example 21: $\lim_{x \to 2^-} \ln(4 - x^2) = -\infty \qquad \lim_{x \to -2^+} \ln(4 - x^2) = -\infty$



Continuity

If the limit of a function as approaches can often be found simply by calculating the value of the function at . Functions with this property are called continuous at a.

1 DEFINITION A function f is continuous at a number a if

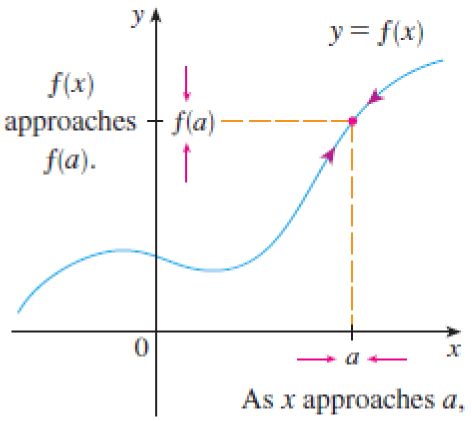
$$\lim_{x \to a} f(x) = f(a)$$

Notice that Definition 1 implicitly requires three things if f is continuous at a:

1. $f(a)$ is defined (that is, a is in the domain of f)
2. $\lim_{x \to a} f(x)$ exists
3. $\lim_{x \to a} f(x) = f(a)$

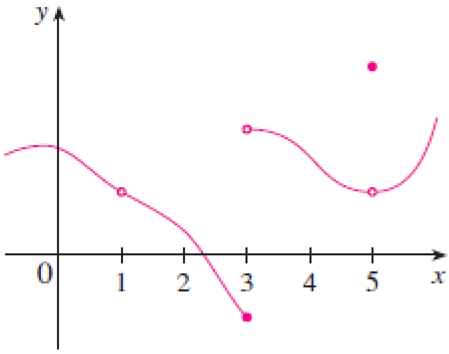
Physical phenomena are usually continuous. For instance, the displacement or velocity of a vehicle varies continuously with time, as does a person's height. But discontinuities do occur in such situations as electric currents.

■ As illustrated in Figure 1, if f is continuous, then the points $(x, f(x))$ on the graph of f approach the point $(a, f(a))$ on the graph. So there is no gap in the curve.



Example 22: In figure below, at which numbers the function f is discontinuous? Why?

Solution:



It looks as if there is a discontinuity when $a = 1$ because the graph has a break there. The official reason that f is discontinuous at 1 is that $f(1)$ is not defined.

The graph also has a break when $a = 3$, but the reason for the discontinuity is different. Here, $f(3)$ is defined, but $\lim_{x\to 3} f(x)$ does not exist (because the left and right limits are different). So f is discontinuous at 3.

What about $a = 5$? Here, $f(5)$ is defined and $\lim_{x\to 5} f(x)$ exists (because the left and right limits are the same). But

$$\lim_{x\to 5} f(x) \neq f(5)$$

So f is discontinuous at 5. □

Example 23: Where are each of the following functions discontinuous?

(a) $f(x) = \dfrac{x^2 - x - 2}{x - 2}$

(b) $f(x) = \begin{cases} \dfrac{1}{x^2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

(c) $f(x) = \begin{cases} \dfrac{x^2 - x - 2}{x - 2} & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$

(d) $f(x) = [\![x]\!]$

Solution:

(a) Notice that $f(2)$ is not defined, so f is discontinuous at 2. Later we'll see why f is continuous at all other numbers.

(b) Here $f(0) = 1$ is defined but

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{1}{x^2}$$

does not exist. (See Example 8 in Section 2.2.) So f is discontinuous at 0.

(c) Here $f(2) = 1$ is defined and

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{x^2 - x - 2}{x - 2} = \lim_{x \to 2} \frac{(x - 2)(x + 1)}{x - 2} = \lim_{x \to 2} (x + 1) = 3$$

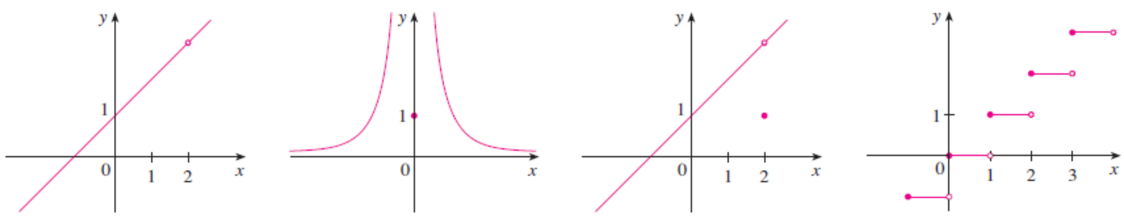
exists. But

$$\lim_{x \to 2} f(x) \neq f(2)$$

so f is not continuous at 2.

(d) The greatest integer function $f(x) = [\![x]\!]$ has discontinuities at all of the integers because $\lim_{x \to n} [\![x]\!]$ does not exist if n is an integer.

Figure shows the graphs of the functions in Example 23. In each case the graph can't be drawn without lifting the pen from the paper because a hole or break or jump occurs in the graph. The kind of discontinuity illustrated in parts (a) and (c) is called **removable** because we could remove the discontinuity by redefining f at just the single number 2. [The function $g(x) = x + 1$ is continuous.] The discontinuity in part (b) is called an **infinite discontinuity**. The discontinuities in part (d) are called **jump discontinuities** because the function "jumps" from one value to another.



(a) $f(x) = \dfrac{x^2 - x - 2}{x - 2}$

(b) $f(x) = \begin{cases} \dfrac{1}{x^2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

(c) $f(x) = \begin{cases} \dfrac{x^2 - x - 2}{x - 2} & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$

(d) $f(x) = [\![x]\!]$

5 THEOREM
(a) Any polynomial is continuous everywhere; that is, it is continuous on $\mathbb{R} = (-\infty, \infty)$.
(b) Any rational function is continuous wherever it is defined; that is, it is continuous on its domain.

7 THEOREM The following types of functions are continuous at every number in their domains:

polynomials, rational functions, root functions
trigonometric functions, inverse trigonometric functions
exponential functions, logarithmic functions

Example 24: $\lim_{x \to -2} \frac{x^3 + 2x^2 - 1}{5 - 3x}$

Solution:

The function

$$f(x) = \frac{x^3 + 2x^2 - 1}{5 - 3x}$$

is rational, so by Theorem 5 it is continuous on its domain, which is $\{x \mid x \neq \frac{5}{3}\}$. Therefore

$$\lim_{x \to -2} \frac{x^3 + 2x^2 - 1}{5 - 3x} = \lim_{x \to -2} f(x) = f(-2)$$

$$= \frac{(-2)^3 + 2(-2)^2 - 1}{5 - 3(-2)} = -\frac{1}{11}$$

□

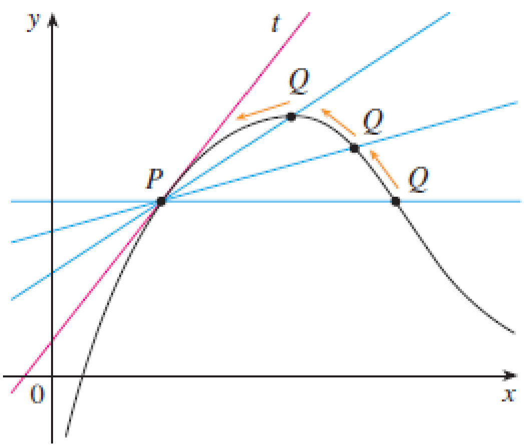
Tangent line, Derivatives and Rates of Change

The problem of finding the tangent line to a curve and the problem of finding the velocity of an object both involve finding the same type of limit, as we saw in previous section. This special type of limit is called a derivative and we will see that it can be interpreted as a rate of change in any of the sciences or engineering.

1 DEFINITION The **tangent line** to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with slope

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

provided that this limit exists.



Example 25: Find an equation of the tangent line to the parabola $y = x^2$ at point P(1,1).

Solution:

Here we have $a = 1$ and $f(x) = x^2$, so the slope is

$$m = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1} \frac{x^2 - 1}{x - 1}$$

$$= \lim_{x \to 1} \frac{(x - 1)(x + 1)}{x - 1}$$

$$= \lim_{x \to 1} (x + 1) = 1 + 1 = 2$$

Using the point-slope form of the equation of a line, we find that an equation of the tangent line at (1,1) is

$$y - 1 = 2(x - 1) \qquad \text{or} \qquad y = 2x - 1$$

Note:

There is another expression for the slope of a tangent line that is sometimes easier to use. If $h = x - a$, then $x = a + h$ and so the slope of the secant line PQ is

$$m_{PQ} = \frac{f(a+h) - f(a)}{h}$$

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Example 26: Find an equation of the tangent line to the hyperbola y = 3/x at point (3,1).

Solution:

Let $f(x) = 3/x$. Then the slope of the tangent at $(3, 1)$ is

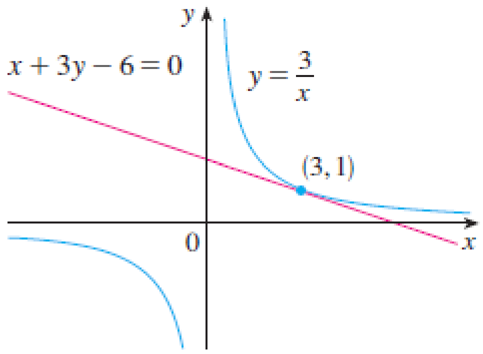
$$m = \lim_{h \to 0} \frac{f(3+h) - f(3)}{h} = \lim_{h \to 0} \frac{\dfrac{3}{3+h} - 1}{h} = \lim_{h \to 0} \frac{\dfrac{3 - (3+h)}{3+h}}{h}$$

$$= \lim_{h \to 0} \frac{-h}{h(3+h)} = \lim_{h \to 0} -\frac{1}{3+h} = -\frac{1}{3}$$

Therefore an equation of the tangent at the point $(3, 1)$ is

$$y - 1 = -\tfrac{1}{3}(x - 3)$$

which simplifies to $$x + 3y - 6 = 0$$



4 DEFINITION **The derivative of a function f at a number a**, denoted by $f'(a)$, is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

if this limit exists.

Example 27: Find the derivative of the function $f(x) = x^2 - 8x + 9$ at the number a.

Solution: From Definition 4 we have

$$\begin{aligned} f'(a) &= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \\ &= \lim_{h \to 0} \frac{[(a+h)^2 - 8(a+h) + 9] - [a^2 - 8a + 9]}{h} \\ &= \lim_{h \to 0} \frac{a^2 + 2ah + h^2 - 8a - 8h + 9 - a^2 + 8a - 9}{h} \\ &= \lim_{h \to 0} \frac{2ah + h^2 - 8h}{h} = \lim_{h \to 0} (2a + h - 8) \\ &= 2a - 8 \end{aligned}$$

RATES OF CHANGE

Suppose y is a quantity that depends on another quantity x. Thus y is a function of x and we write $y = f(x)$. If x changes from x_1 to x_2, then the change in x (also called the **increment of** x) is

$$\Delta x = x_2 - x_1$$

and the corresponding change in y is

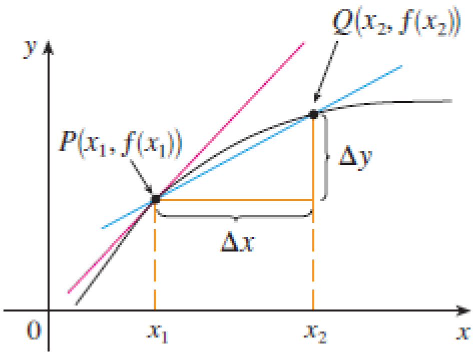
$$\Delta y = f(x_2) - f(x_1)$$

The difference quotient

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

is called the **average rate of change of y with respect to x** over an interval

6 $$\text{instantaneous rate of change} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$



average rate of change $= m_{PQ}$
instantaneous rate of change $=$ slope of tangent at P

Example 28: A manufacturer produces bolts of a fabric with a fixed width. The cost of producing x yards of this fabric is C= f(x) dollars,

(a) What is the meaning of the derivative f'(x), what are its units?

(b) In practical terms, what does it mean to say that f'(1000) = 9?

(c) Which do you think is greater f'(50) or f'(500), what about f'(5000)?

Solution:

(a) The derivative $f'(x)$ is the instantaneous rate of change of C with respect to x; that is, $f'(x)$ means the rate of change of the production cost with respect to the number of yards produced. (Economists call this rate of change the *marginal cost.*

Because

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta C}{\Delta x}$$

the units for $f'(x)$ are the same as the units for the difference quotient $\Delta C/\Delta x$. Since ΔC is measured in dollars and Δx in yards, it follows that the units for $f'(x)$ are dollars per yard.

(b) The statement that $f'(1000) = 9$ means that, after 1000 yards of fabric have been manufactured, the rate at which the production cost is increasing is \$9/yard. (When $x = 1000$, C is increasing 9 times as fast as x.)

Since $\Delta x = 1$ is small compared with $x = 1000$, we could use the approximation

$$f'(1000) \approx \frac{\Delta C}{\Delta x} = \frac{\Delta C}{1} = \Delta C$$

and say that the cost of manufacturing the 1000th yard (or the 1001st) is about \$9.

(c) The rate at which the production cost is increasing (per yard) is probably lower when $x = 500$ than when $x = 50$ (the cost of making the 500th yard is less than the cost of the 50th yard) because of economies of scale. (The manufacturer makes more efficient use of the fixed costs of production.) So

$$f'(50) > f'(500)$$

But, as production expands, the resulting large-scale operation might become inefficient and there might be overtime costs. Thus it is possible that the rate of increase of costs will eventually start to rise. So it may happen that

$$f'(5000) > f'(500)$$ □