3 Differentiation

Introduction

Derivative: it’s a function we use to measure the rates at which things change, like slope and velocity and accelerations.

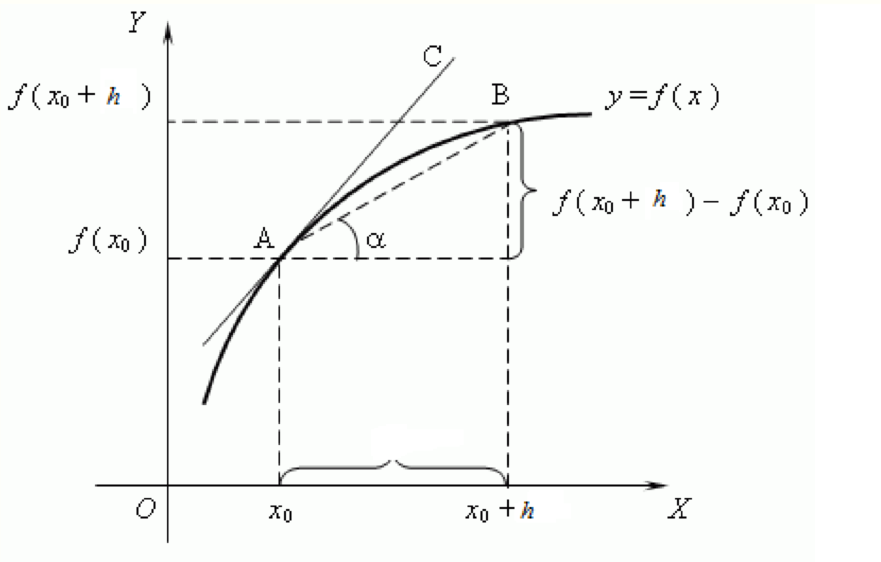
The derivative of a function is a function f′ where value at x is defined in the equation:

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

The function $\frac{f(x+h)-f(x)}{h}$ is the difference quotient for f at x.

h is the difference increment.

f′(x) is the first derivate of the function f at x. See figure below.



The most common notation for the differentiation of a function y = f(x) besides f′(x) or dy/dx and df/dx Dx(f) (Dx of f) .. etc..

Application of differentiation:

- The velocity and acceleration at time t and
- Problems of cost , maxima and minima
- Electrical circuits' problem
- Any other problems related to rate of change.

Example 29: Find the derivative of f(x) = $x^2 - 2x$ using the definition.

Solution

f(x) = $x^2 - 2x$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f(x+h) = (x+h)^2 - 2(x+h) = x^2 + 2xh + h^2 - 2x - 2h$$

$$\frac{f(x+h) - f(x)}{h} = \frac{x^2 + 2xh + h^2 - 2x - 2h - (x^2 - 2x)}{h}$$

$$= \frac{x^2 + 2xh + h^2 - 2x - 2h - x^2 + 2x}{h} = \frac{h^2 + 2hx - 2h}{h} = h + 2x - 2$$

We can take the limit as h $\longrightarrow$ 0: -

$$f'(x) = \lim_{h \to 0} (h + 2x - 2) = 2x - 2$$

Example 30: Show that the derivative of $y = \sqrt{x}$ is $\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$

Solution:

$f(x+h) = \sqrt{x+h}$ and $f(x) = \sqrt{x}$

$$\frac{f(x+h)-f(x)}{h}=\frac{\sqrt{x+h}-\sqrt{x}}{h}$$ ÷ 0 (Not Ok)

$$\frac{f(x+h)-f(x)}{h}=\frac{\sqrt{x+h}-\sqrt{x}}{h}\times\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$$

$$= \;=\frac{(x+h)-x}{h\sqrt{x+h}+\sqrt{x}}=\frac{1}{h\sqrt{x+h}+\sqrt{x}}$$

$$\frac{dy}{dx}=\lim_{h\to-0}\frac{1}{h\sqrt{x+h}+\sqrt{x}}=\frac{1}{2\sqrt{x}}$$ Ok

The Power Rule If n is a positive integer, then

$$\frac{d}{dx}(x^n)=nx^{n-1}$$

FIRST PROOF The formula

$$x^n-a^n=(x-a)(x^{n-1}+x^{n-2}a+\cdots+xa^{n-2}+a^{n-1})$$

can be verified simply by multiplying out the right-hand side (or by summing the second factor as a geometric series). If $f(x)=x^n$, we can use Equation 2.7.5 for $f'(a)$ and the equation above to write

$$f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}=\lim_{x\to a}\frac{x^n-a^n}{x-a}$$

$$=\lim_{x\to a}(x^{n-1}+x^{n-2}a+\cdots+xa^{n-2}+a^{n-1})$$

$$=a^{n-1}+a^{n-2}a+\cdots+aa^{n-2}+a^{n-1}$$

$$=na^{n-1}$$

k

Example 31: Differentiate **(a)** $f(x)=\frac{1}{x^2}$ **(b)** $y=\sqrt[3]{x^2}$

Solution:

In each case we rewrite the function as a power of x.

(a) Since $f(x) = x^{-2}$, we use the Power Rule with $n = -2$:

$$f'(x) = \frac{d}{dx}(x^{-2}) = -2x^{-2-1} = -2x^{-3} = -\frac{2}{x^3}$$

(b)
$$\frac{dy}{dx} = \frac{d}{dx}\left(\sqrt[3]{x^2}\right) = \frac{d}{dx}(x^{2/3}) = \tfrac{2}{3}x^{(2/3)-1} = \tfrac{2}{3}x^{-1/3}$$

Slope and tangent lines:

Example 32: Find an eq. for the tangent to the curve y = 2/x at x = 3

Solution:

$$m = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f(x+h) = \frac{2}{x+h}$$

$$\frac{f(x+h)-f(x)}{h} = \frac{\frac{2}{x+h} - \frac{2}{x}}{h} = \frac{\frac{2x-2x-2h}{(x+h)x}}{h} = \frac{-2h}{(x+h)x} = \frac{-2}{x^2}$$

$m = f'(x) = -2/x^2$ at x = 3 m = $-2/(3)^2$

Then y = -2/9

y + 2/3 = -2/9 (x-3)

Rules for differentiation

If f and g are differentiable functions, the following differentiation rules are valid

1. $\frac{d}{dx}\{f(x)+g(x)\} = \frac{d}{dx}f(x) + \frac{d}{dx}g(x) = f'(x) + g'(x)$ (Addition Rule)

2. $\frac{d}{dx}\{f(x)-g(x)\} = \frac{d}{dx}f(x) - \frac{d}{dx}g(x) = f'(x) - g'(x)$

3. $\frac{d}{dx}\{Cf(x)\} = C\frac{d}{dx}f(x) = Cf'(x)$ where C is any constant

4. $\frac{d}{dx}\{f(x)g(x)\} = f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x) = f(x)g'(x) + g(x)f'(x)$ (Product Rule)

5. $\frac{d}{dx}\left\{\frac{f(x)}{g(x)}\right\} = \frac{g(x)\frac{d}{dx}f(x) - f(x)\frac{d}{dx}g(x)}{[g(x)]^2} = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$ if $g(x) \neq 0$ (Quotient Rule)

6. $\frac{d}{dx}(C) = 0$

7. $\frac{d}{dx}(x^n) = nx^{n-1}$

8. $\frac{d}{dx}(\ln x) = \frac{dx}{x}$ or $\frac{1}{x}dx$

9. $\frac{d}{dx}(e^x) = e^x \; dx$

10. $\frac{d}{dx}(a^x) = a^x \; \ln a \; dx$

11. $\frac{d}{dx}(\log_a x) = \frac{1}{x}\log_a .e = \frac{1}{x}.\frac{1}{\ln a}dx$

Example 33: If $f(x) = e^x - x$, find f' and f''. Compare the graphs of f and f'.

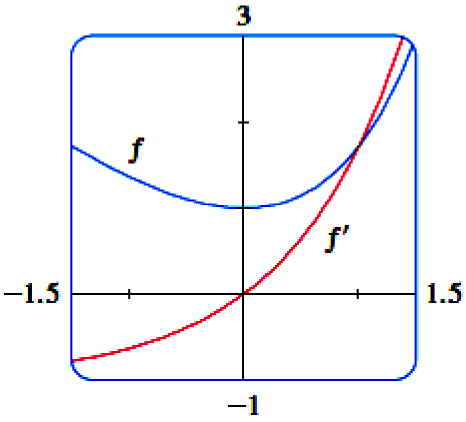
Solution: Using the Difference Rule, we have

$$f'(x) = \frac{d}{dx}(e^x - x) = \frac{d}{dx}(e^x) - \frac{d}{dx}(x) = e^x - 1$$

we defined the second derivative as the derivative of f', so

$$f''(x) = \frac{d}{dx}(e^x - 1) = \frac{d}{dx}(e^x) - \frac{d}{dx}(1) = e^x$$

The function f and its derivative f' are graphed in Figure . Notice that f has a horizontal tangent when $x = 0$; this corresponds to the fact that $f'(0) = 0$. Notice also that, for $x > 0$, $f'(x)$ is positive and f is increasing. When $x < 0$, $f'(x)$ is negative and f is decreasing. ■



Example 34: Let $y = \frac{x^2 + x - 2}{x^3 + 6}$. Then

$$y' = \frac{(x^3 + 6)\frac{d}{dx}(x^2 + x - 2) - (x^2 + x - 2)\frac{d}{dx}(x^3 + 6)}{(x^3 + 6)^2}$$

$$= \frac{(x^3 + 6)(2x + 1) - (x^2 + x - 2)(3x^2)}{(x^3 + 6)^2}$$

$$= \frac{(2x^4 + x^3 + 12x + 6) - (3x^4 + 3x^3 - 6x^2)}{(x^3 + 6)^2}$$

$$= \frac{-x^4 - 2x^3 + 6x^2 + 12x + 6}{(x^3 + 6)^2}$$

Example 35:

Find an equation of the tangent line to the curve $y = e^x/(1 + x^2)$ at the point $\left(1, \frac{1}{2}e\right)$.

According to the Quotient Rule, we have

$$\frac{dy}{dx} = \frac{(1 + x^2)\frac{d}{dx}(e^x) - e^x\frac{d}{dx}(1 + x^2)}{(1 + x^2)^2}$$

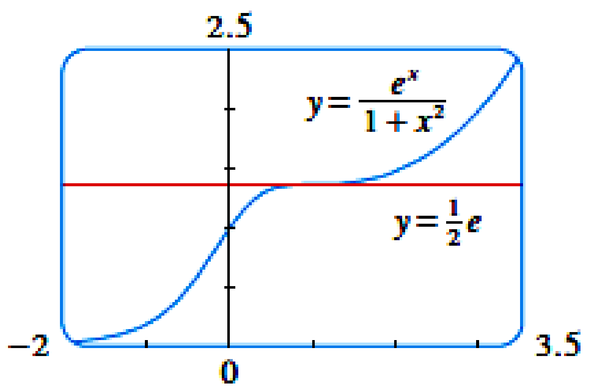
$$= \frac{(1 + x^2)e^x - e^x(2x)}{(1 + x^2)^2} = \frac{e^x(1 - 2x + x^2)}{(1 + x^2)^2}$$

$$= \frac{e^x(1 - x)^2}{(1 + x^2)^2}$$

So the slope of the tangent line at $\left(1, \frac{1}{2}e\right)$ is

$$\left.\frac{dy}{dx}\right|_{x=1} = 0$$

5 This means that the tangent line at $\left(1, \frac{1}{2}e\right)$ is horizontal and its equation is $y = \frac{1}{2}e$.



Derivatives of trigonometric functions

- X is measured in radians

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

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Example 36: Find d/dx {sinx} ?

Solution:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sin(x+h)-\sin x}{h}$$

$$= \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$

$$= \lim_{h\to 0} \left[\frac{\sin x \cos h - \sin x}{h} + \frac{\cos x \sin h}{h} \right]$$

$$= \lim_{h\to 0} \left[\sin x \left(\frac{\cos h - 1}{h} \right) + \cos x \left(\frac{\sin h}{h} \right) \right]$$

1 $$= \lim_{h\to 0} \sin x \cdot \lim_{h\to 0} \frac{\cos h - 1}{h} + \lim_{h\to 0} \cos x \cdot \lim_{h\to 0} \frac{\sin h}{h}$$

Two of these four limits are easy to evaluate. Since we regard x as a constant when computing a limit as $h \to 0$, we have

$$\lim_{h\to 0} \sin x = \sin x \qquad \text{and} \qquad \lim_{h\to 0} \cos x = \cos x$$

But :

$$\lim_{\theta\to 0} \frac{\sin \theta}{\theta} = 1 \qquad\qquad \lim_{\theta\to 0} \frac{\cos \theta - 1}{\theta} = 0$$

Then

$$f'(x) = \lim_{h\to 0} \sin x \cdot \lim_{h\to 0} \frac{\cos h - 1}{h} + \lim_{h\to 0} \cos x \cdot \lim_{h\to 0} \frac{\sin h}{h}$$

$$= (\sin x) \cdot 0 + (\cos x) \cdot 1 = \cos x$$

So we have proved the formula for the derivative of the sine function:

4 $$\frac{d}{dx}(\sin x) = \cos x$$

Derivatives of Trigonometric Functions

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

$$\frac{d}{dx}(\cos x) = -\sin x \qquad \frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

Example 37: Differentiate $y = x^2 \sin x$.

Solution: Using the Product Rule and Formula 4, we have

$$\frac{dy}{dx} = x^2 \frac{d}{dx}(\sin x) + \sin x \frac{d}{dx}(x^2)$$
$$= x^2 \cos x + 2x \sin x$$

Example 38:

Differentiate $f(x) = \dfrac{\sec x}{1 + \tan x}$. For what values of x does the graph of f have a horizontal tangent?

Solution:

$$f'(x) = \frac{(1 + \tan x)\dfrac{d}{dx}(\sec x) - \sec x \dfrac{d}{dx}(1 + \tan x)}{(1 + \tan x)^2}$$

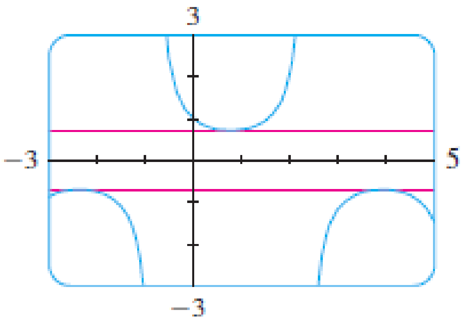
$$= \frac{(1 + \tan x)\sec x \tan x - \sec x \cdot \sec^2 x}{(1 + \tan x)^2}$$

$$= \frac{\sec x\,(\tan x + \tan^2 x - \sec^2 x)}{(1 + \tan x)^2}$$

$$= \frac{\sec x\,(\tan x - 1)}{(1 + \tan x)^2}$$

In simplifying the answer we have used the identity $\tan^2 x + 1 = \sec^2 x$.

Since $\sec x$ is never 0, we see that $f'(x) = 0$ when $\tan x = 1$, and this occurs when $x = n\pi + \pi/4$, where n is an integer (see Figure). ■



Example 39:

$$\frac{d}{dx}(x^8 + 12x^5 - 4x^4 + 10x^3 - 6x + 5)$$

$$= \frac{d}{dx}(x^8) + 12\frac{d}{dx}(x^5) - 4\frac{d}{dx}(x^4) + 10\frac{d}{dx}(x^3) - 6\frac{d}{dx}(x) + \frac{d}{dx}(5)$$

$$= 8x^7 + 12(5x^4) - 4(4x^3) + 10(3x^2) - 6(1) + 0$$

$$= 8x^7 + 60x^4 - 16x^3 + 30x^2 - 6$$

Example 40:

The equation of motion of a particle is $s = 2t^3 - 5t^2 + 3t + 4$, where s is measured in centimeters and t in seconds. Find the acceleration as a function of time. What is the acceleration after 2 seconds?

The velocity and acceleration are

$$v(t) = \frac{ds}{dt} = 6t^2 - 10t + 3$$

$$a(t) = \frac{dv}{dt} = 12t - 10$$

The acceleration after 2 s is $a(2) = 14 \text{ cm/s}^2$. ■

NOTE Don't use the Quotient Rule *every* time you see a quotient. Sometimes it's easier to rewrite a quotient first to put it in a form that is simpler for the purpose of differentiation. For instance, although it is possible to differentiate the function

$$F(x) = \frac{3x^2 + 2\sqrt{x}}{x}$$

Higher order derivatives

Example 43: Find y′, y″ , and y‴ for the following functions:

1. $y = 2x^3 + x - 5$
2. $y = \frac{2x}{1-2x}$

Solution

$y = 2x^3 + x - 5$

$y' = 6x^2 + 1$

$y'' = 12x$

$y''' = 12$

$y = \frac{2x}{1-2x}$

$y' = \frac{(1-2x)*2 - 2x*(-2)}{(1-2x)^2} = \frac{2-4x+4x}{(1-2x)^2} = \frac{2}{(1-2x)^2}$

$y'' = \frac{-2*2(1-2x)*(-2)}{(1-2x)^4} = \frac{8}{(1-2x)^3}$

$$y'' = \frac{-8 * 3(1-2x)^2 * (-2)}{(1-2x)^6} = \frac{48}{(1-2x)^4}$$

The Chain Rule

The Chain Rule If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $F = f \circ g$ defined by $F(x) = f(g(x))$ is differentiable at x and F' is given by the product

$$F'(x) = f'(g(x)) \cdot g'(x)$$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$ are both differentiable functions, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

COMMENTS ON THE PROOF OF THE CHAIN RULE Let Δu be the change in u corresponding to a change of Δx in x, that is,

$$\Delta u = g(x + \Delta x) - g(x)$$

Then the corresponding change in y is

$$\Delta y = f(u + \Delta u) - f(u)$$

It is tempting to write

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

1 $$= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u} \cdot \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x}$$

$$= \lim_{\Delta u \to 0} \frac{\Delta y}{\Delta u} \cdot \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x}$$

(Note that $\Delta u \to 0$ as $\Delta x \to 0$ since g is continuous.)

$$= \frac{dy}{du}\frac{du}{dx}$$

The Chain Rule can be written either in the prime notation

2 $$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$

or, if $y = f(u)$ and $u = g(x)$, in Leibniz notation:

3 $$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

Example 44: Find F(x)′ if $F'_{(x)} = \sqrt{x^2 + 1}$

SOLUTION 1 (using Equation 2): At the beginning of this section we expressed F as $F(x) = (f \circ g)(x) = f(g(x))$ where $f(u) = \sqrt{u}$ and $g(x) = x^2 + 1$. Since

$$f'(u) = \tfrac{1}{2}u^{-1/2} = \frac{1}{2\sqrt{u}} \qquad \text{and} \qquad g'(x) = 2x$$

we have
$$F'(x) = f'(g(x)) \cdot g'(x)$$
$$= \frac{1}{2\sqrt{x^2 + 1}} \cdot 2x = \frac{x}{\sqrt{x^2 + 1}}$$

SOLUTION 2 (using Equation 3): If we let $u = x^2 + 1$ and $y = \sqrt{u}$, then

$$F'(x) = \frac{dy}{du}\frac{du}{dx} = \frac{1}{2\sqrt{u}}(2x) = \frac{1}{2\sqrt{x^2 + 1}}(2x) = \frac{x}{\sqrt{x^2 + 1}}$$ ■

NOTE In using the Chain Rule we work from the outside to the inside. Formula 2 says that *we differentiate the outer function f [at the inner function g(x)] and then we multiply by the derivative of the inner function.*

$$\frac{d}{dx}\underbrace{f}_{\text{outer function}}\underbrace{(g(x))}_{\text{evaluated at inner function}} = \underbrace{f'}_{\text{derivative of outer function}}\underbrace{(g(x))}_{\text{evaluated at inner function}} \cdot \underbrace{g'(x)}_{\text{derivative of inner function}}$$

Example 45 Differentiate (a) $y = \sin(x^2)$ and (b) $y = \sin^2 x$.

SOLUTION

(a) If $y = \sin(x^2)$, then the outer function is the sine function and the inner function is the squaring function, so the Chain Rule gives

$$\frac{dy}{dx} = \frac{d}{dx}\underbrace{\sin}_{\substack{\text{outer}\\\text{function}}}\underbrace{(x^2)}_{\substack{\text{evaluated}\\\text{at inner}\\\text{function}}} = \underbrace{\cos}_{\substack{\text{derivative}\\\text{of outer}\\\text{function}}}\underbrace{(x^2)}_{\substack{\text{evaluated}\\\text{at inner}\\\text{function}}}\cdot\underbrace{2x}_{\substack{\text{derivative}\\\text{of inner}\\\text{function}}}$$

$$= 2x\cos(x^2)$$

(b) Note that $\sin^2 x = (\sin x)^2$. Here the outer function is the squaring function and the inner function is the sine function. So

$$\frac{dy}{dx} = \frac{d}{dx}\underbrace{(\sin x)^2}_{\substack{\text{inner}\\\text{function}}} = \underbrace{2}_{\substack{\text{derivative}\\\text{of outer}\\\text{function}}}\cdot\underbrace{(\sin x)}_{\substack{\text{evaluated}\\\text{at inner}\\\text{function}}}\cdot\underbrace{\cos x}_{\substack{\text{derivative}\\\text{of inner}\\\text{function}}}$$

Example 46: Write the composite function in the form $f(g(x))$. [Identify the inner function $u = g(x)$ and the outer function $y = f(u)$.] Then find the derivative dy/dx.

1. $y = \sqrt[3]{1 + 4x}$
2. $y = (2x^3 + 5)^4$
3. $y = \tan \pi x$
4. $y = \sin(\cot x)$
5. $y = e^{\sqrt{x}}$
6. $y = \sqrt{2 - e^x}$

Solution

1. Let $u = g(x) = 1 + 4x$ and $y = f(u) = \sqrt[3]{u}$. Then $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = (\frac{1}{3}u^{-2/3})(4) = \frac{4}{3\sqrt[3]{(1+4x)^2}}$.

2. Let $u = g(x) = 2x^3 + 5$ and $y = f(u) = u^4$. Then $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = (4u^3)(6x^2) = 24x^2(2x^3+5)^3$.

3. Let $u = g(x) = \pi x$ and $y = f(u) = \tan u$. Then $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = (\sec^2 u)(\pi) = \pi \sec^2 \pi x$.

4. Let $u = g(x) = \cot x$ and $y = f(u) = \sin u$. Then $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = (\cos u)(-\csc^2 x) = -\cos(\cot x)\csc^2 x$.

5. Let $u = g(x) = \sqrt{x}$ and $y = f(u) = e^u$. Then $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = (e^u)\left(\frac{1}{2}x^{-1/2}\right) = e^{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}} = \frac{e^{\sqrt{x}}}{2\sqrt{x}}$.

6. Let $u = g(x) = 2 - e^x$ and $y = f(u) = \sqrt{u}$. Then $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = (\frac{1}{2}u^{-1/2})(-e^x) = -\frac{e^x}{2\sqrt{2-e^x}}$.

Implicit Differentiation

To find dy/dx for any equation involving x and y differentiation each of term in the equation with respect to x instead of finding y in terms of x.

Example 47 Find y'' if $x^4 + y^4 = 16$.

Solution Differentiating the equation implicitly with respect to x, we get

$$4x^3 + 4y^3y' = 0$$

Solving for y' gives

$$y' = -\frac{x^3}{y^3} \qquad \boxed{3}$$

To find y'' we differentiate this expression for y' using the Quotient Rule and remembering that y is a function of x:

$$y'' = \frac{d}{dx}\left(-\frac{x^3}{y^3}\right) = -\frac{y^3\,(d/dx)(x^3) - x^3\,(d/dx)(y^3)}{(y^3)^2}$$

$$= -\frac{y^3 \cdot 3x^2 - x^3(3y^2y')}{y^6}$$

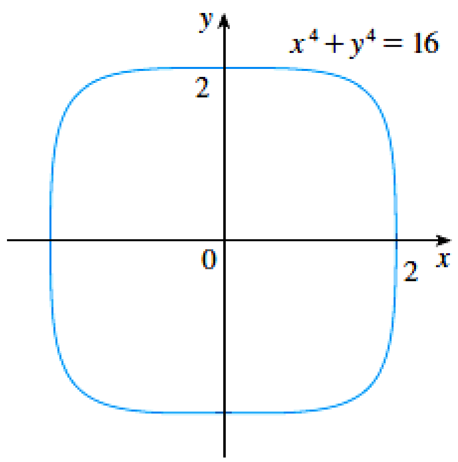
If we now substitute Equation 3 into this expression, we get

$$y'' = -\frac{3x^2y^3 - 3x^3y^2\left(-\dfrac{x^3}{y^3}\right)}{y^6}$$

$$= -\frac{3(x^2y^4 + x^6)}{y^7} = -\frac{3x^2(y^4 + x^4)}{y^7}$$

But the values of x and y must satisfy the original equation $x^4 + y^4 = 16$. So the answer simplifies to

$$y'' = -\frac{3x^2(16)}{y^7} = -48\,\frac{x^2}{y^7}$$



Related Rates

In a related rates problem the idea is to compute the rate of change of one quantity in terms of the rate of change of another quantity (which may be more easily measured). The procedure is to find an equation that relates the two quantities and then use the Chain Rule to differentiate both sides with respect to time.

Problem Solving Strategy

1. Read the problem carefully.
2. Draw a diagram if possible.
3. Introduce notation. Assign symbols to all quantities that are functions of time.
4. Express the given information and the required rate in terms of derivatives.
5. Write an equation that relates the various quantities of the problem. If necessary, use the geometry of the situation to eliminate one of the variables by substitution.
6. Use the Chain Rule to differentiate both sides of the equation with respect to *t*.
7. Substitute the given information into the resulting equation and solve for the unknown rate.

Example 72:

Air is being pumped into a spherical balloon so that its volume increases at a rate of 100 cm^3/s. How fast is the radius of the balloon increasing when the diameter is 50 cm?

SOLUTION We start by identifying two things:

the *given information:*

the rate of increase of the volume of air is 100 cm^3/s

and the *unknown:*

the rate of increase of the radius when the diameter is 50 cm

In order to express these quantities mathematically, we introduce some suggestive *notation:*

Let V be the volume of the balloon and let r be its radius.

The key thing to remember is that rates of change are derivatives. In this problem, the volume and the radius are both functions of the time t. The rate of increase of the volume with respect to time is the derivative dV/dt, and the rate of increase of the radius is dr/dt. We can therefore restate the given and the unknown as follows:

Given: $\quad \dfrac{dV}{dt} = 100 \text{ cm}^3/\text{s}$

Unknown: $\quad \dfrac{dr}{dt}$ when $r = 25$ cm

In order to connect dV/dt and dr/dt, we first relate V and r by the formula for the volume of a sphere:

$$V = \tfrac{4}{3}\pi r^3$$

In order to use the given information, we differentiate each side of this equation with respect to t. To differentiate the right side, we need to use the Chain Rule:

$$\frac{dV}{dt} = \frac{dV}{dr}\frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}$$

Now we solve for the unknown quantity:

$$\frac{dr}{dt} = \frac{1}{4\pi r^2}\frac{dV}{dt}$$

If we put $r = 25$ and $dV/dt = 100$ in this equation, we obtain

$$\frac{dr}{dt} = \frac{1}{4\pi(25)^2}100 = \frac{1}{25\pi}$$

The radius of the balloon is increasing at the rate of $1/(25\pi) \approx 0.0127$ cm/s.

Example 73:

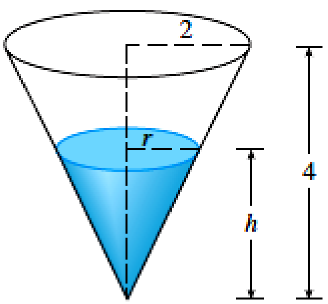
A water tank has the shape of an inverted circular cone with base radius 2 m and height 4 m. If water is being pumped into the tank at a rate of 2 m^3/min, find the rate at which the water level is rising when the water is 3 m deep.

SOLUTION We first sketch the cone and label it as in Figure 3. Let V, r, and h be the volume of the water, the radius of the surface, and the height of the water at time t, where t is measured in minutes.

We are given that $dV/dt = 2$ m^3/min and we are asked to find dh/dt when h is 3 m. The quantities V and h are related by the equation

$$V = \tfrac{1}{3}\pi r^2 h$$

but it is very useful to express V as a function of h alone. In order to eliminate r, we use



the similar triangles in Figure 3 to write

$$\frac{r}{h} = \frac{2}{4} \qquad r = \frac{h}{2}$$

and the expression for V becomes

$$V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{\pi}{12}h^3$$

Now we can differentiate each side with respect to t:

$$\frac{dV}{dt} = \frac{\pi}{4}h^2\frac{dh}{dt}$$

so

$$\frac{dh}{dt} = \frac{4}{\pi h^2}\frac{dV}{dt}$$

Substituting $h = 3$ m and $dV/dt = 2\ \text{m}^3/\text{min}$, we have

$$\frac{dh}{dt} = \frac{4}{\pi(3)^2}\cdot 2 = \frac{8}{9\pi}$$

The water level is rising at a rate of $8/(9\pi) \approx 0.28$ m/min.

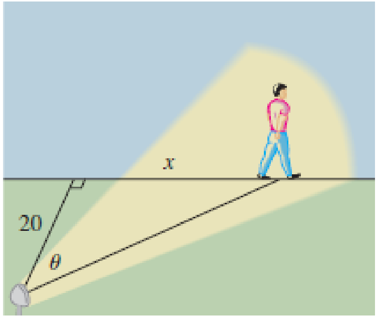
Example 74:

A man walks along a straight path at a speed of 4 ft/s. A searchlight is located on the ground 20 ft from the path and is kept focused on the man. At what rate is the searchlight rotating when the man is 15 ft from the point on the path closest to the searchlight?

SOLUTION We draw Figure 5 and let x be the distance from the man to the point on the path closest to the searchlight. We let θ be the angle between the beam of the searchlight and the perpendicular to the path.

We are given that $dx/dt = 4$ ft/s and are asked to find $d\theta/dt$ when $x = 15$. The equation that relates x and θ can be written from Figure 5:

$$\frac{x}{20} = \tan\theta \qquad x = 20\tan\theta$$



Differentiating each side with respect to t, we get

$$\frac{dx}{dt} = 20\sec^2\theta\,\frac{d\theta}{dt}$$

so

$$\frac{d\theta}{dt} = \frac{1}{20}\cos^2\theta\,\frac{dx}{dt}$$

$$= \frac{1}{20}\cos^2\theta\,(4) = \frac{1}{5}\cos^2\theta$$

When $x = 15$, the length of the beam is 25, so $\cos\theta = \frac{4}{5}$ and

$$\frac{d\theta}{dt} = \frac{1}{5}\left(\frac{4}{5}\right)^2 = \frac{16}{125} = 0.128$$

The searchlight is rotating at a rate of 0.128 rad/s. ■

Indeterminate Forms and L'Hospital's Rule

John Bernoulli discovered a rule using derivatives to calculate limits of fractions whose numerators and denominators both approach zero or +∞. The rule is known today as l'Hôpital's Rule, after Guillaume de l'Hôpital.

Indeterminate Form 0/0

If we want to know how the function

$$F(x) = \frac{x - \sin x}{x^3}$$

behaves *near* $x = 0$ (where it is undefined), we can examine the limit of $F(x)$ as $x \to 0$. We cannot apply the Quotient Rule for limits (Theorem 1 of Chapter 2) because the limit of the denominator is 0. Moreover, in this case, *both* the numerator and denominator approach 0, and $0/0$ is undefined. Such limits may or may not exist in general, but the limit does exist for the function $F(x)$ under discussion by applying l'Hôpital's Rule, as we will see in Example 1d.

If the continuous functions $f(x)$ and $g(x)$ are both zero at $x = a$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

cannot be found by substituting $x = a$. The substitution produces $0/0$, a meaningless expression, which we cannot evaluate. We use $0/0$ as a notation for an expression known as an **indeterminate form**. Other meaningless expressions often occur, such as ∞/∞, $\infty \cdot 0$, $\infty - \infty$, 0^0, and 1^∞, which cannot be evaluated in a consistent way; these are called indeterminate forms as well. Sometimes, but not always, limits that lead to indeterminate forms may be found by cancelation, rearrangement of terms, or other algebraic

THEOREM 6—L'Hôpital's Rule Suppose that $f(a) = g(a) = 0$, that f and g are differentiable on an open interval I containing a, and that $g'(x) \neq 0$ on I if $x \neq a$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

assuming that the limit on the right side of this equation exists.

Example 75:

The following limits involve $0/0$ indeterminate forms, so we apply l'Hôpital's Rule. In some cases, it must be applied repeatedly.

(a) $\lim_{x\to 0}\frac{3x-\sin x}{x}=\lim_{x\to 0}\frac{3-\cos x}{1}=\frac{3-\cos x}{1}\bigg|_{x=0}=2$

(b) $\lim_{x\to 0}\frac{\sqrt{1+x}-1}{x}=\lim_{x\to 0}\frac{\frac{1}{2\sqrt{1+x}}}{1}=\frac{1}{2}$

(c) $\lim_{x\to 0}\frac{\sqrt{1+x}-1-x/2}{x^2}$ $\quad\frac{0}{0}$; apply l'Hôpital's Rule.

$=\lim_{x\to 0}\frac{(1/2)(1+x)^{-1/2}-1/2}{2x}$ $\quad$ Still $\frac{0}{0}$; apply l'Hôpital's Rule again.

$=\lim_{x\to 0}\frac{-(1/4)(1+x)^{-3/2}}{2}=-\frac{1}{8}$ $\quad$ Not $\frac{0}{0}$; limit is found.

(d) $\lim_{x\to 0}\frac{x-\sin x}{x^3}$ $\quad\frac{0}{0}$; apply l'Hôpital's Rule.

$=\lim_{x\to 0}\frac{1-\cos x}{3x^2}$ $\quad$ Still $\frac{0}{0}$; apply l'Hôpital's Rule again.

$=\lim_{x\to 0}\frac{\sin x}{6x}$ $\quad$ Still $\frac{0}{0}$; apply l'Hôpital's Rule again.

$=\lim_{x\to 0}\frac{\cos x}{6}=\frac{1}{6}$ $\quad$ Not $\frac{0}{0}$; limit is found.

Example 76: Calculate $\lim_{x\to\infty}\frac{\ln x}{\sqrt{x}}$.

SOLUTION Since $\ln x\to\infty$ and $\sqrt{x}\to\infty$ as $x\to\infty$, l'Hospital's Rule applies:

$$\lim_{x\to\infty}\frac{\ln x}{\sqrt{x}}=\lim_{x\to\infty}\frac{1/x}{\frac{1}{2}x^{-1/2}}=\lim_{x\to\infty}\frac{1/x}{1/(2\sqrt{x})}$$

Notice that the limit on the right side is now indeterminate of type $\frac{0}{0}$. But instead of applying l'Hospital's Rule a second time as we did in Example 2, we simplify the expression and see that a second application is unnecessary:

$$\lim_{x\to\infty}\frac{\ln x}{\sqrt{x}}=\lim_{x\to\infty}\frac{1/x}{1/(2\sqrt{x})}=\lim_{x\to\infty}\frac{2}{\sqrt{x}}=0$$

Example 77: Find the limits of these ∞/∞ forms:

(a) $\lim_{x\to\pi/2} \frac{\sec x}{1+\tan x}$ **(b)** $\lim_{x\to\infty} \frac{\ln x}{2\sqrt{x}}$ **(c)** $\lim_{x\to\infty} \frac{e^x}{x^2}$.

Solution

(a) The numerator and denominator are discontinuous at $x = \pi/2$, so we investigate the one-sided limits there. To apply l'Hôpital's Rule, we can choose I to be any open interval with $x = \pi/2$ as an endpoint.

$$\lim_{x\to(\pi/2)^-} \frac{\sec x}{1+\tan x}$$ $\frac{\infty}{\infty}$ from the left so we apply l'Hôpital's Rule.

$$= \lim_{x\to(\pi/2)^-} \frac{\sec x \tan x}{\sec^2 x} = \lim_{x\to(\pi/2)^-} \sin x = 1$$

The right-hand limit is 1 also, with $(-\infty)/(-\infty)$ as the indeterminate form. Therefore, the two-sided limit is equal to 1.

(b) $$\lim_{x\to\infty} \frac{\ln x}{2\sqrt{x}} = \lim_{x\to\infty} \frac{1/x}{1/\sqrt{x}} = \lim_{x\to\infty} \frac{1}{\sqrt{x}} = 0$$ $\frac{1/x}{1/\sqrt{x}} = \frac{\sqrt{x}}{x} = \frac{1}{\sqrt{x}}$

(c) $$\lim_{x\to\infty} \frac{e^x}{x^2} = \lim_{x\to\infty} \frac{e^x}{2x} = \lim_{x\to\infty} \frac{e^x}{2} = \infty$$ ■

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Example 78: `

Evaluate $\lim_{x\to 0^+} x \ln x$.

SOLUTION The given limit is indeterminate because, as $x \to 0^+$, the first factor (x) approaches 0 while the second factor $(\ln x)$ approaches $-\infty$. Writing $x = 1/(1/x)$, we have $1/x \to \infty$ as $x \to 0^+$, so l'Hospital's Rule gives

$$\lim_{x\to 0^+} x \ln x = \lim_{x\to 0^+} \frac{\ln x}{1/x} = \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+} (-x) = 0$$ ■