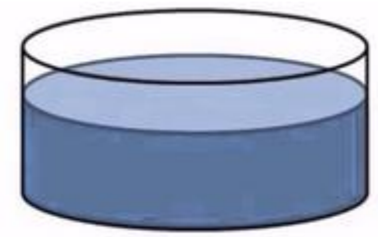
4 APPLICATIONS OF DIFFERENTIATION

Introduction

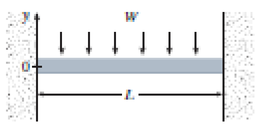
We use the derivative to determine the maximum and minimum values of particular functions (e.g. cost, strength, amount of material used in a building, profit, loss, etc.).

Change of velocity with time

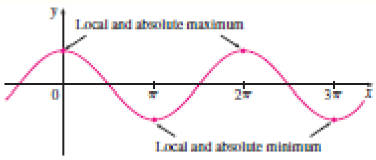
flow of tank



Displacement

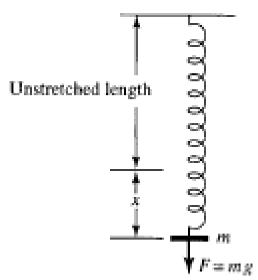


Maximum and Minimum Values



Simple circuit with light

Engineering mechanics



Summary

Mechanics

$v = \frac{dx}{dt}$, where v = velocity, x = distance, t = time.

$a = \frac{dv}{dt}$, where a = acceleration, v = velocity, t = time.

$F = \frac{dW}{dx}$, where F = force, W = work done (or energy used), x = distance moved in the direction of the force.

$F = \frac{dp}{dt}$, where F = force, p = momentum, t = time.

$P = \frac{dW}{dt}$, where P = power, W = work done (or energy used), t = time.

$\frac{dE}{dv} = p$, where E = kinetic energy, v = velocity, p = momentum.

Gases

$\frac{dW}{dV} = p$, where p = pressure, W = work done under isothermal expansion, V = volume.

Circuits

$I = \frac{dQ}{dt}$, where I = current, Q = charge, t = time.

$V = \left(L\frac{dI}{dt}\right)$, where V is the voltage drop across an inductor, L = inductance, I = current, t = time.

Electrostatics

$E = -\frac{dV}{dx}$, where V = potential, E = electric field, x = distance.

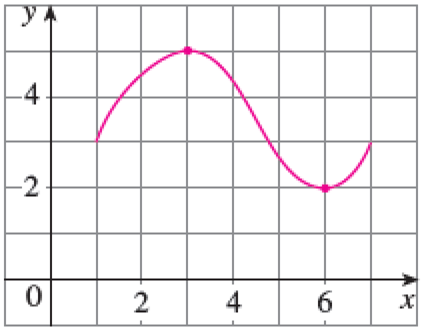
Maximum and Minimum Values

Some of the most important applications of differential calculus are optimization problems, in which we are required to find the optimal (best) way of doing something.

These problems can be reduced to finding the maximum or minimum values of a function.

Let's first explain exactly what we mean by maximum and minimum values.

We see that the highest point on the graph of the function f shown in Figure is the point (3,5). In other words, the largest value of f is f(3)= 5. Likewise, the smallest value is f(6)= 2. We say that f (3)= 5 is the ***absolute maximum*** of f and f (6)= 2 is the ***absolute minimum***.

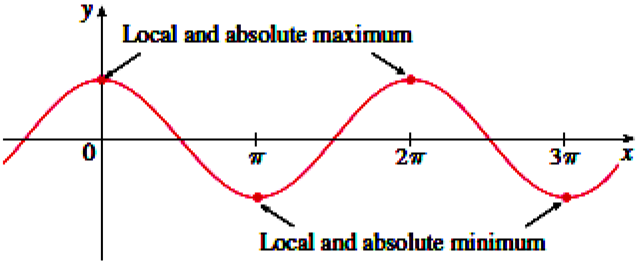


In general, we use the following definition

1 Definition Let c be a number in the domain D of a function f. Then $f(c)$ is the

- **absolute maximum** value of f on D if $f(c) \geqslant f(x)$ for all x in D.
- **absolute minimum** value of f on D if $f(c) \leqslant f(x)$ for all x in D.

Example 48 The function $f(x) = \cos x$ takes on its (local and absolute) maximum value of 1 infinitely many times, since $\cos 2n\pi = 1$ for any integer n and $-1 \leqslant \cos x \leqslant 1$ for all x. (See Figure) Likewise, $\cos(2n + 1)\pi = -1$ is its minimum value, where n is any integer.



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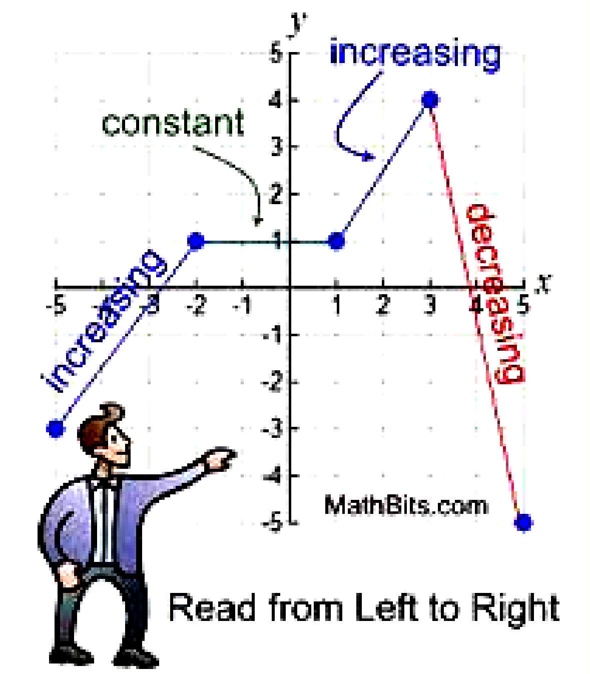
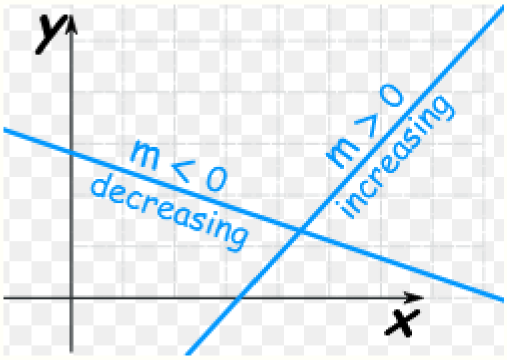
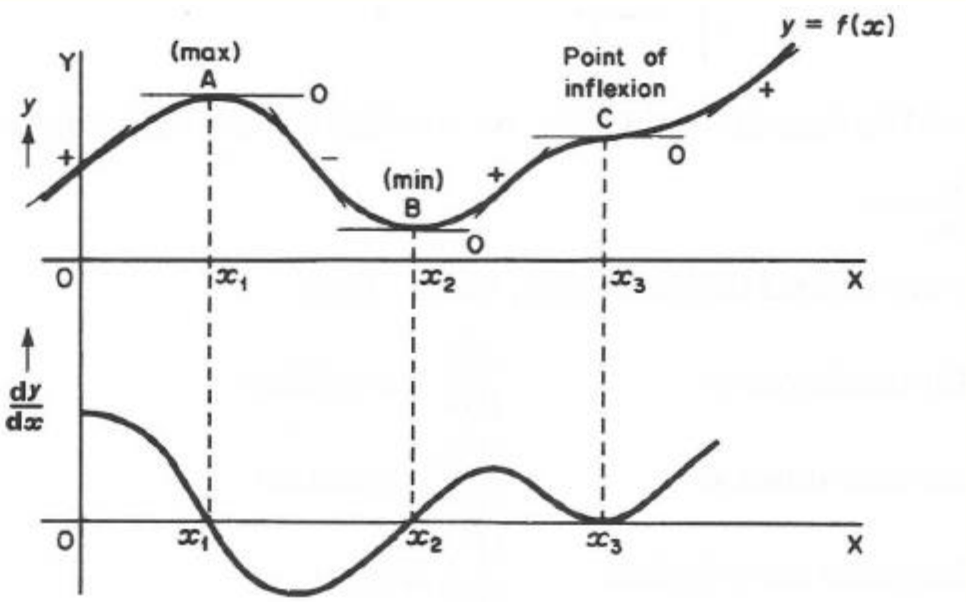




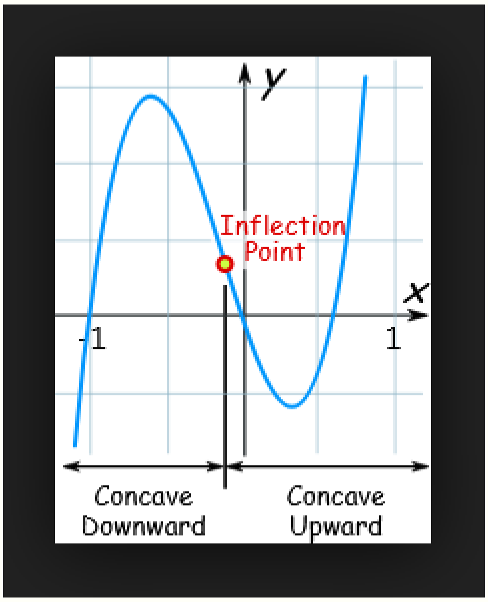
If f(x2) > f(x1) then the function is called **increasing** on its interval

If f(x2) < f(x1) then the function is called **decreasing** on its interval

If f(x2) = f(x1) then the function is called **constant** on its interval



Concavity



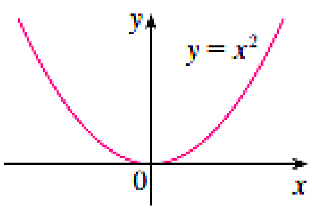
Remember:

The graph of y= f(x) is

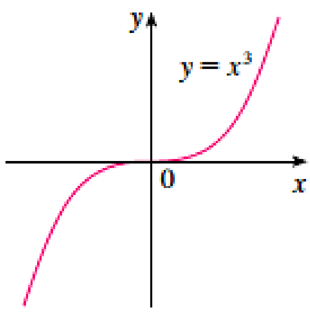
Concve up when y" > 0

Concave down when y" < 0

Example 49:



Mimimum value 0, no maximum

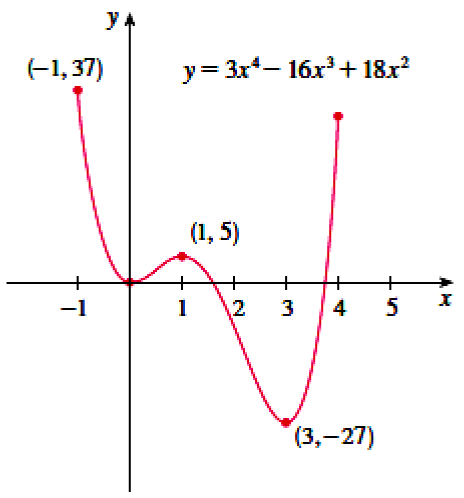


No mimimum, no maximum

Example 50: The graph of the function

$$f(x) = 3x^4 - 16x^3 + 18x^2 \qquad -1 \leqslant x \leqslant 4$$

is shown in Figure . You can see that $f(1) = 5$ is a local maximum, whereas the absolute maximum is $f(-1) = 37$. (This absolute maximum is not a local maximum because it occurs at an endpoint.) Also, $f(0) = 0$ is a local minimum and $f(3) = -27$ is both a local and an absolute minimum. Note that f has neither a local nor an absolute maximum at $x = 4$.



We have seen that some functions have extreme values, whereas others do not. The following theorem gives conditions under which a function is guaranteed to possess extreme values.

Extrema of a function (maxima and minima)

3 The Extreme Value Theorem If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$.

The Second Derivative Test Suppose f'' is continuous near c.

(a) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.

(b) If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.

Example 51: Discuss the curve $y = x^4 - 4x^3$ with respect to concavity, points of inflection, and local maxima and minima. Use this information to sketch the curve

SOLUTION If $f(x) = x^4 - 4x^3$, then

$$f'(x) = 4x^3 - 12x^2 = 4x^2(x - 3)$$

$$f''(x) = 12x^2 - 24x = 12x(x - 2)$$

To find the critical numbers we set $f'(x) = 0$ and obtain $x = 0$ and $x = 3$. (Note that f' is a polynomial and hence defined everywhere.) To use the Second Derivative Test we evaluate f'' at these critical numbers:

$$f''(0) = 0 \qquad f''(3) = 36 > 0$$

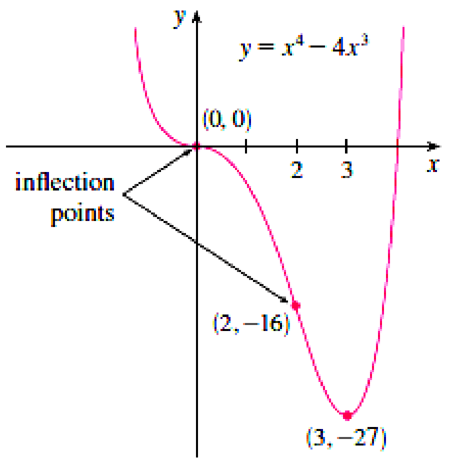
Since $f'(3) = 0$ and $f''(3) > 0$, $f(3) = -27$ is a local minimum. [In fact, the expression for $f'(x)$ shows that f decreases to the left of 3 and increases to the right of 3.] Since $f''(0) = 0$, the Second Derivative Test gives no information about the critical number 0. But since $f'(x) < 0$ for $x < 0$ and also for $0 < x < 3$, the First Derivative Test tells us that f does not have a local maximum or minimum at 0.

Since $f''(x) = 0$ when $x = 0$ or 2, we divide the real line into intervals with these numbers as endpoints and complete the following chart.

Interval	$f''(x) = 12x(x - 2)$	Concavity
$(-\infty, 0)$	+	upward
$(0, 2)$	−	downward
$(2, \infty)$	+	upward

The point $(0, 0)$ is an inflection point since the curve changes from concave up to concave downward there. Also $(2, -16)$ is an inflection point since the curve changes from concave downward to concave upward there.

Using the local minimum, the intervals of concavity, and the inflection points,



6 Definition **A critical number** of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

Example 52: Find the critical numbers of $f(x) = x^{3/5}(4 - x)$.

SOLUTION The Product Rule gives

n

$$f'(x) = x^{3/5}(-1) + (4 - x)\left(\tfrac{3}{5}x^{-2/5}\right) = -x^{3/5} + \frac{3(4 - x)}{5x^{2/5}}$$

$$= \frac{-5x + 3(4 - x)}{5x^{2/5}} = \frac{12 - 8x}{5x^{2/5}}$$

[The same result could be obtained by first writing $f(x) = 4x^{3/5} - x^{8/5}$.] Therefore $f'(x) = 0$ if $12 - 8x = 0$, that is, $x = \frac{3}{2}$, and $f'(x)$ does not exist when $x = 0$. Thus the critical numbers are $\frac{3}{2}$ and 0. ∎

Procedures for finding and distinguishing between stationary points:

1. Given y = f(x), determine dy/dx (i.e. f′(x)).
2. Let dy/dx = 0 and solve for the values of x.
3. Substitute the values of x into the original function y= f(x) to find the corresponding y ordinate values. This would establish the nature of stationary points.
4. Find d^2y/dx^2 and sub into the values found in 2 above. If the result is:

i. Positive then min. point

ii. Negative then max. point

iii. Zero then its point of inflexion (inflection)

5. Determine the sign of the gradient of the curve just before and just after the stationary points. If the sign changes for the gradient of the curve is:

a) Positive to negative then point is max.

b) Negative to positive then point is min

c) Positive to positive or negative to negative then it's a point of inflection.

Example 53: Find the local minimum and maximum values of the function *f*

$$f_{(x)} = x^3 - 3x^2 + 4$$

Solution

$f'_{(x)} = 3x^2 - 6x$, $f''_{(x)} = 6x - 6$

$f'_{(x)} = 0$, $0 = 3x^2 - 6x$

X = 0 or 3x – 6 = 0 then x = 2

Finding values of f″(x) at x = 0, 2

$f''_{(0)} = -6$ Relative maximum point

$f''_{(2)} = 6$ Relative minimum point

Summary of Curve Sketching

The following checklist is intended as a guide to sketching a curve y = f (x) by hand. Not every item is relevant to every function. (For instance, a given curve might not have an asymptote or possess symmetry.) But the guidelines provide all the information you need to make a sketch that displays the most important aspects of the function

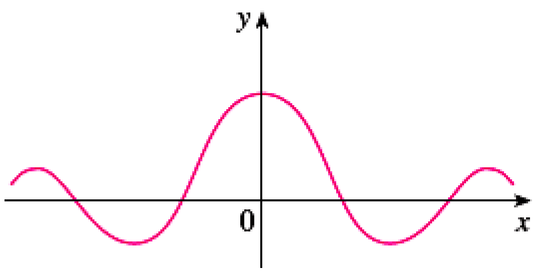
A. Domain It's often useful to start by determining the domain D of f, that is, the set of values of x for which $f(x)$ is defined.

B. Intercepts The y-intercept is $f(0)$ and this tells us where the curve intersects the y-axis. To find the x-intercepts, we set $y = 0$ and solve for x. (You can omit this step if the equation is difficult to solve.)

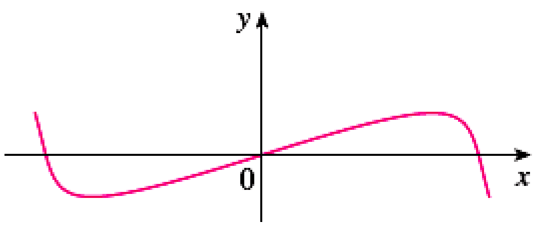
C. Symmetry

(i) If $f(-x) = f(x)$ for all x in D, that is, the equation of the curve is unchanged when x is replaced by $-x$, then f is an **even function** and the curve is symmetric about the y-axis. This means that our work is cut in half. If we know what the curve looks like for $x \geqslant 0$, then we need only reflect about the y-axis to obtain the complete curve [see Figure (a)]. Here are some examples: $y = x^2$, $y = x^4$, $y = |x|$, and $y = \cos x$.

(ii) If $f(-x) = -f(x)$ for all x in D, then f is an **odd function** and the curve is symmetric about the origin. Again we can obtain the complete curve if we know what it looks like for $x \geqslant 0$. [Rotate 180° about the origin; see Figure (b).] Some simple examples of odd functions are $y = x$, $y = x^3$, $y = x^5$, and $y = \sin x$.

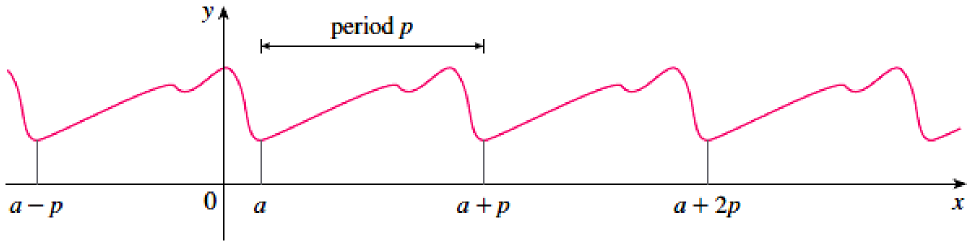


(a) Even function: reflectional symmetry



(b) Odd function: rotational symmetry

(iii) If $f(x+p)=f(x)$ for all x in D, where p is a positive constant, then f is called a **periodic function** and the smallest such number p is called the **period**. For instance, $y=\sin x$ has period 2π and $y=\tan x$ has period π. If we know what the graph looks like in an interval of length p, then we can use translation to sketch the entire graph (see Figure).



D. Asymptotes

(i) *Horizontal Asymptotes.* Recall from chapter 2 that if either $\lim_{x\to\infty} f(x)=L$ or $\lim_{x\to-\infty} f(x)=L$, then the line $y=L$ is a horizontal asymptote of the curve $y=f(x)$. If it turns out that $\lim_{x\to\infty} f(x)=\infty$ (or $-\infty$), then we do not have an asymptote to the right, but this fact is still useful information for sketching the curve.

(ii) *Vertical Asymptotes.* Recall from chapter 2 that the line $x=a$ is a vertical asymptote if at least one of the following statements is true:

1

$$\lim_{x\to a^+} f(x)=\infty \qquad \lim_{x\to a^-} f(x)=\infty$$

$$\lim_{x\to a^+} f(x)=-\infty \qquad \lim_{x\to a^-} f(x)=-\infty$$

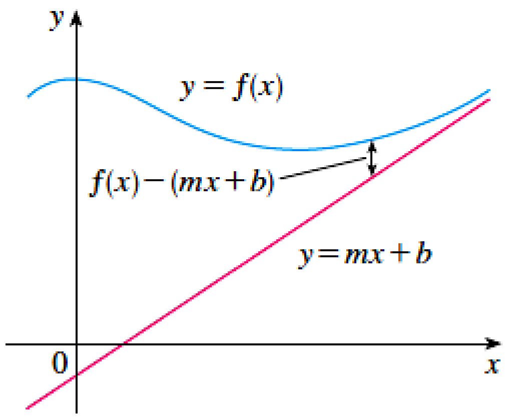
(For rational functions you can locate the vertical asymptotes by equating the denominator to 0 after canceling any common factors. But for other functions this method does not apply.) Furthermore, in sketching the curve it is very useful to know exactly which of the statements in (1) is true. If $f(a)$ is not defined but a is an endpoint of the domain of f, then you should compute $\lim_{x\to a^-} f(x)$ or $\lim_{x\to a^+} f(x)$, whether or not this limit is infinite.

(iii) *Slant Asymptotes.*

Some curves have asymptotes that are *oblique*, that is, neither horizontal nor vertical. If

$$\lim_{x\to\infty} [f(x)-(mx+b)]=0$$

where $m\neq 0$, then the line $y=mx+b$ is called a **slant asymptote** because the vertical distance between the curve $y=f(x)$ and the line $y=mx+b$ approaches 0, as in Figure . (A similar situation exists if we let $x\to-\infty$.) For rational functions, slant asymptotes occur when the degree of the numerator is one more than the degree of the denominator. In such a case the equation of the slant asymptote can be found by long division as in the following example.



E. Intervals of Increase or Decrease Use the I/D Test. Compute $f'(x)$ and find the intervals on which $f'(x)$ is positive (f is increasing) and the intervals on which $f'(x)$ is negative (f is decreasing).

F. Local Maximum and Minimum Values Find the critical numbers of f [the numbers c where $f'(c) = 0$ or $f'(c)$ does not exist]. Then use the First Derivative Test. If f' changes from positive to negative at a critical number c, then $f(c)$ is a local maximum. If f' changes from negative to positive at c, then $f(c)$ is a local minimum. Although it is usually preferable to use the First Derivative Test, you can use the Second Derivative Test if $f'(c) = 0$ and $f''(c) \neq 0$. Then $f''(c) > 0$ implies that $f(c)$ is a local minimum, whereas $f''(c) < 0$ implies that $f(c)$ is a local maximum.

G. Concavity and Points of Inflection Compute $f''(x)$ and use the Concavity Test. The curve is concave upward where $f''(x) > 0$ and concave downward where $f''(x) < 0$. Inflection points occur where the direction of concavity changes.

H. Sketch the Curve Using the information in items A–G, draw the graph. Sketch the asymptotes as dashed lines. Plot the intercepts, maximum and minimum points, and inflection points. Then make the curve pass through these points, rising and falling according to E, with concavity according to G, and approaching the asymptotes.

Example 54:

Use the guidelines to sketch the curve $y = \dfrac{2x^2}{x^2 - 1}$.

A. The domain is

$$\{x \mid x^2 - 1 \neq 0\} = \{x \mid x \neq \pm 1\} = (-\infty, -1) \cup (-1, 1) \cup (1, \infty)$$

B. The x- and y-intercepts are both 0.

C. Since $f(-x) = f(x)$, the function f is even. The curve is symmetric about the y-axis.

D.
$$\lim_{x\to\pm\infty} \frac{2x^2}{x^2 - 1} = \lim_{x\to\pm\infty} \frac{2}{1 - 1/x^2} = 2$$

Therefore the line $y = 2$ is a horizontal asymptote.

Since the denominator is 0 when $x = \pm 1$, we compute the following limits:

$$\lim_{x\to 1^+} \frac{2x^2}{x^2 - 1} = \infty \qquad \lim_{x\to 1^-} \frac{2x^2}{x^2 - 1} = -\infty$$

$$\lim_{x\to -1^+} \frac{2x^2}{x^2 - 1} = -\infty \qquad \lim_{x\to -1^-} \frac{2x^2}{x^2 - 1} = \infty$$

Therefore the lines $x = 1$ and $x = -1$ are vertical asymptotes. This information about limits and asymptotes enables us to draw the preliminary sketch in Figure 5, showing the parts of the curve near the asymptotes.

E.
$$f'(x) = \frac{(x^2 - 1)(4x) - 2x^2 \cdot 2x}{(x^2 - 1)^2} = \frac{-4x}{(x^2 - 1)^2}$$

Since $f'(x) > 0$ when $x < 0$ $(x \neq -1)$ and $f'(x) < 0$ when $x > 0$ $(x \neq 1)$, f is increasing on $(-\infty, -1)$ and $(-1, 0)$ and decreasing on $(0, 1)$ and $(1, \infty)$.

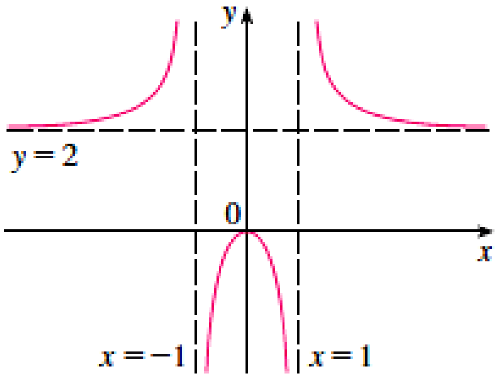
F. The only critical number is $x = 0$. Since f' changes from positive to negative at 0, $f(0) = 0$ is a local maximum by the First Derivative Test.

G.
$$f''(x) = \frac{(x^2 - 1)^2(-4) + 4x \cdot 2(x^2 - 1)2x}{(x^2 - 1)^4} = \frac{12x^2 + 4}{(x^2 - 1)^3}$$

Since $12x^2 + 4 > 0$ for all x, we have

$$f''(x) > 0 \iff x^2 - 1 > 0 \iff |x| > 1$$

and $f''(x) < 0 \iff |x| < 1$. Thus the curve is concave upward on the intervals $(-\infty, -1)$ and $(1, \infty)$ and concave downward on $(-1, 1)$. It has no point of inflection since 1 and -1 are not in the domain of f.



Example 55:

Sketch the graph of $f(x) = \dfrac{x^2}{\sqrt{x+1}}$.

A. Domain $= \{x \mid x + 1 > 0\} = \{x \mid x > -1\} = (-1, \infty)$

B. The x- and y-intercepts are both 0.

C. Symmetry: None

D. Since

$$\lim_{x\to\infty} \frac{x^2}{\sqrt{x+1}} = \infty$$

there is no horizontal asymptote. Since $\sqrt{x+1} \to 0$ as $x \to -1^+$ and $f(x)$ is always positive, we have

$$\lim_{x\to -1^+} \frac{x^2}{\sqrt{x+1}} = \infty$$

and so the line $x = -1$ is a vertical asymptote.

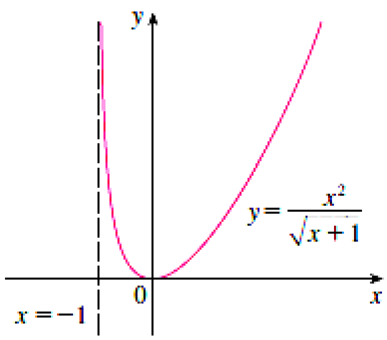
E. $$f'(x) = \frac{\sqrt{x+1}\,(2x) - x^2 \cdot 1/\left(2\sqrt{x+1}\right)}{x+1} = \frac{3x^2 + 4x}{2(x+1)^{3/2}} = \frac{x(3x+4)}{2(x+1)^{3/2}}$$

We see that $f'(x) = 0$ when $x = 0$ (notice that $-\frac{4}{3}$ is not in the domain of f), so the only critical number is 0. Since $f'(x) < 0$ when $-1 < x < 0$ and $f'(x) > 0$ when $x > 0$, f is decreasing on $(-1, 0)$ and increasing on $(0, \infty)$.

F. Since $f'(0) = 0$ and f' changes from negative to positive at 0, $f(0) = 0$ is a local (and absolute) minimum by the First Derivative Test.

G. $$f''(x) = \frac{2(x+1)^{3/2}(6x+4) - (3x^2+4x)3(x+1)^{1/2}}{4(x+1)^3} = \frac{3x^2 + 8x + 8}{4(x+1)^{5/2}}$$

Note that the denominator is always positive. The numerator is the quadratic $3x^2 + 8x + 8$, which is always positive because its discriminant is $b^2 - 4ac = -32$, which is negative, and the coefficient of x^2 is positive. Thus $f''(x) > 0$ for all x in the domain of f, which means that f is concave upward on $(-1, \infty)$ and there is no point of inflection.



Example 56:

Sketch the graph of $f(x) = \dfrac{\cos x}{2 + \sin x}$.

A. The domain is $\mathbb{R}$.

B. The y-intercept is $f(0) = \frac{1}{2}$. The x-intercepts occur when $\cos x = 0$, that is, $x = (\pi/2) + n\pi$, where n is an integer.

C. f is neither even nor odd, but $f(x + 2\pi) = f(x)$ for all x and so f is periodic and has period 2π. Thus, in what follows, we need to consider only $0 \leqslant x \leqslant 2\pi$ and then extend the curve by translation in part H.

D. Asymptotes: None

E.
$$f'(x) = \frac{(2 + \sin x)(-\sin x) - \cos x\,(\cos x)}{(2 + \sin x)^2} = -\frac{2\sin x + 1}{(2 + \sin x)^2}$$

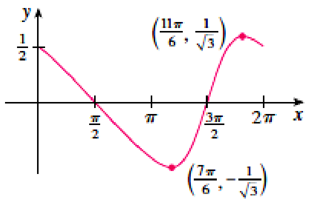
The denominator is always positive, so $f'(x) > 0$ when $2\sin x + 1 < 0 \iff \sin x < -\frac{1}{2} \iff 7\pi/6 < x < 11\pi/6$. So f is increasing on $(7\pi/6, 11\pi/6)$ and decreasing on $(0, 7\pi/6)$ and $(11\pi/6, 2\pi)$.

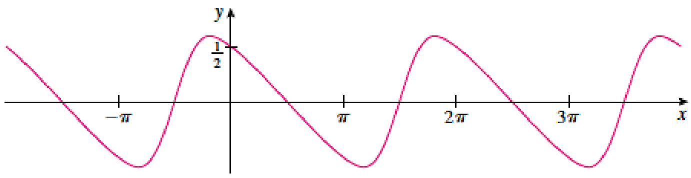
F. From part E and the First Derivative Test, we see that the local minimum value is $f(7\pi/6) = -1/\sqrt{3}$ and the local maximum value is $f(11\pi/6) = 1/\sqrt{3}$.

G. If we use the Quotient Rule again and simplify, we get

$$f''(x) = -\frac{2\cos x\,(1 - \sin x)}{(2 + \sin x)^3}$$

Because $(2 + \sin x)^3 > 0$ and $1 - \sin x \geqslant 0$ for all x, we know that $f''(x) > 0$ when $\cos x < 0$, that is, $\pi/2 < x < 3\pi/2$. So f is concave upward on $(\pi/2, 3\pi/2)$ and concave downward on $(0, \pi/2)$ and $(3\pi/2, 2\pi)$. The inflection points are $(\pi/2, 0)$ and $(3\pi/2, 0)$.





Example 57:

Sketch the graph of $y = \ln(4 - x^2)$.

A. The domain is

$$\{x \mid 4 - x^2 > 0\} = \{x \mid x^2 < 4\} = \{x \mid |x| < 2\} = (-2, 2)$$

B. The y-intercept is $f(0) = \ln 4$. To find the x-intercept we set

$$y = \ln(4 - x^2) = 0$$

We know that $\ln 1 = 0$, so we have $4 - x^2 = 1 \Rightarrow x^2 = 3$ and therefore the x-intercepts are $\pm\sqrt{3}$.

C. Since $f(-x) = f(x)$, f is even and the curve is symmetric about the y-axis.

D. We look for vertical asymptotes at the endpoints of the domain. Since $4 - x^2 \to 0^+$ as $x \to 2^-$ and also as $x \to -2^+$, we have

$$\lim_{x\to 2^-} \ln(4 - x^2) = -\infty \qquad \lim_{x\to -2^+} \ln(4 - x^2) = -\infty$$

Thus the lines $x = 2$ and $x = -2$ are vertical asymptotes.

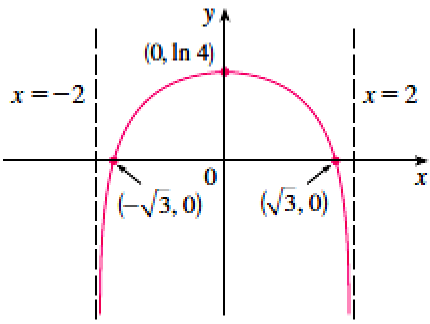
E. $$f'(x) = \frac{-2x}{4 - x^2}$$

Since $f'(x) > 0$ when $-2 < x < 0$ and $f'(x) < 0$ when $0 < x < 2$, f is increasing on $(-2, 0)$ and decreasing on $(0, 2)$.

F. The only critical number is $x = 0$. Since f' changes from positive to negative at 0, $f(0) = \ln 4$ is a local maximum by the First Derivative Test.

G. $$f''(x) = \frac{(4 - x^2)(-2) + 2x(-2x)}{(4 - x^2)^2} = \frac{-8 - 2x^2}{(4 - x^2)^2}$$

Since $f''(x) < 0$ for all x, the curve is concave downward on $(-2, 2)$ and has no inflection point.



The Mean Value Theorem

The Mean Value Theorem Let f be a function that satisfies the following hypotheses:

1. f is continuous on the closed interval $[a, b]$.

2. f is differentiable on the open interval (a, b).

Then there is a number c in (a, b) such that

1
$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

or, equivalently,

2
$$f(b) - f(a) = f'(c)(b - a)$$

Before proving this theorem, we can see that it is reasonable by interpreting it geometrically. Figures 3 and 4 show the points $A(a, f(a))$ and $B(b, f(b))$ on the graphs of two differentiable functions. The slope of the secant line AB is

3
$$m_{AB} = \frac{f(b) - f(a)}{b - a}$$

Before proving this theorem, we can see that it is reasonable by interpreting it geometrically. Figures 3 and 4 show the points $A(a, f(a))$ and $B(b, f(b))$ on the graphs of two differentiable functions. The slope of the secant line AB is

3 $$m_{AB} = \frac{f(b) - f(a)}{b - a}$$

which is the same expression as on the right side of Equation 1. Since $f'(c)$ is the slope of the tangent line at the point $(c, f(c))$, the Mean Value Theorem, in the form given by Equation 1, says that there is at least one point $P(c, f(c))$ on the graph where the slope of the tangent line is the same as the slope of the secant line AB. In other words, there is a point P where the tangent line is parallel to the secant line AB. (Imagine a line far away that stays parallel to AB while moving toward AB until it touches the graph for the first time.)

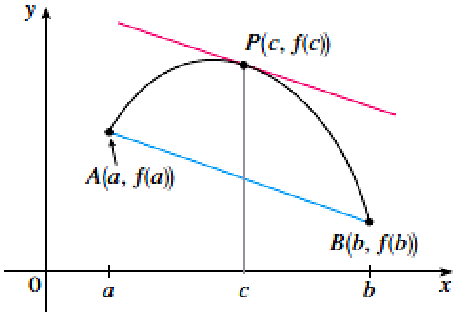


FIGURE 3

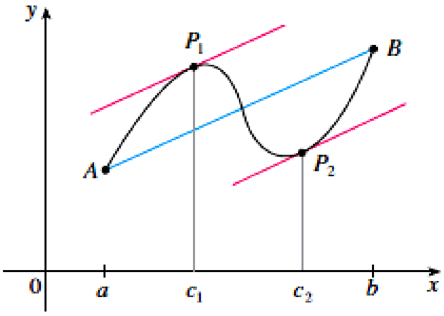


FIGURE 4

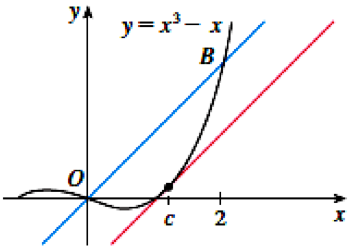
To illustrate the Mean Value Theorem with a specific function, let's consider $f(x) = x^3 - x$, $a = 0$, $b = 2$. Since f is a polynomial, it is continuous and differentiable for all x, so it is certainly continuous on $[0, 2]$ and differentiable on $(0, 2)$. Therefore, by the Mean Value Theorem, there is a number c in $(0, 2)$ such that

$$f(2) - f(0) = f'(c)(2 - 0)$$

Now $f(2) = 6$, $f(0) = 0$, and $f'(x) = 3x^2 - 1$, so this equation becomes

$$6 = (3c^2 - 1)2 = 6c^2 - 2$$

which gives $c^2 = \frac{4}{3}$, that is, $c = \pm 2/\sqrt{3}$. But c must lie in $(0, 2)$, so $c = 2/\sqrt{3}$.



Optimization Problems

In solving such practical problems the greatest challenge is often to convert the word problem into a mathematical optimization problem by setting up the function that is to be maximized or minimized

Solving Applied Optimization Problems

1. *Read the problem.* Read the problem until you understand it. What is given? What is the unknown quantity to be optimized?
2. *Draw a picture.* Label any part that may be important to the problem.
3. *Introduce variables.* List every relation in the picture and in the problem as an equation or algebraic expression, and identify the unknown variable.
4. *Write an equation for the unknown quantity.* If you can, express the unknown as a function of a single variable or in two equations in two unknowns. This may require considerable manipulation.
5. *Test the critical points and endpoints in the domain of the unknown.* Use what you know about the shape of the function's graph. Use the first and second derivatives to identify and classify the function's critical points.

Example 58:
A cylindrical can is to be made to hold 1 L of oil. Find the dimensions that will minimize the cost of the metal to manufacture the can?.

SOLUTION Draw the diagram as in Figure 3, where r is the radius and h the height (both in centimeters). In order to minimize the cost of the metal, we minimize the total surface area of the cylinder (top, bottom, and sides). From Figure 4 we see that the sides are made from a rectangular sheet with dimensions $2\pi r$ and h. So the surface area is

$$A = 2\pi r^2 + 2\pi rh$$

We would like to express A in terms of one variable, r. To eliminate h we use the fact that the volume is given as 1 L, which is equivalent to 1000 cm^3. Thus

$$\pi r^2 h = 1000$$

which gives $h = 1000/(\pi r^2)$. Substitution of this into the expression for A gives

$$A = 2\pi r^2 + 2\pi r\left(\frac{1000}{\pi r^2}\right) = 2\pi r^2 + \frac{2000}{r}$$

We know r must be positive, and there are no limitations on how large r can be. Therefore the function that we want to minimize is

$$A(r) = 2\pi r^2 + \frac{2000}{r} \qquad r > 0$$

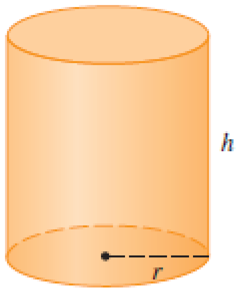


FIGURE 3

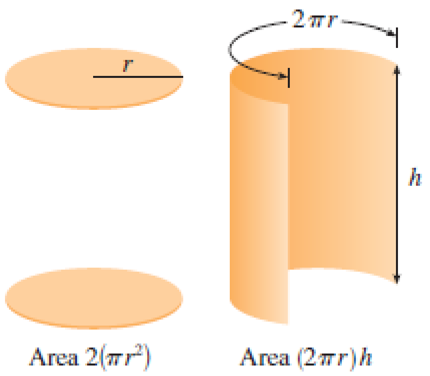


FIGURE 4

To find the critical numbers, we differentiate:

$$A'(r) = 4\pi r - \frac{2000}{r^2} = \frac{4(\pi r^3 - 500)}{r^2}$$

Then $A'(r) = 0$ when $\pi r^3 = 500$, so the only critical number is $r = \sqrt[3]{500/\pi}$.

The value of h corresponding to $r = \sqrt[3]{500/\pi}$ is

$$h = \frac{1000}{\pi r^2} = \frac{1000}{\pi(500/\pi)^{2/3}} = 2\sqrt[3]{\frac{500}{\pi}} = 2r$$

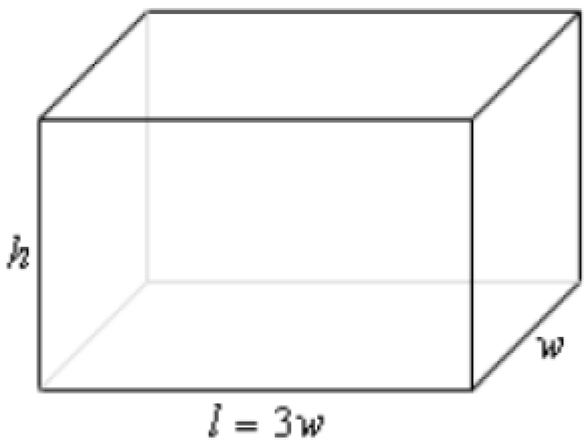
Thus, to minimize the cost of the can, the radius should be $\sqrt[3]{500/\pi}$ cm and the height should be equal to twice the radius, namely, the diameter. ■

Example 59

We want to construct a box whose base length is 3 times the base width. The material used to build the top and bottom cost $10/ft^2 and the material used to build the sides cost $6/ft^2. If the box must have a volume of 50ft^3 determine the dimensions that will minimize the cost to build the box.

Solution:

First, we sketch a figure as below:



We want to minimize the cost of the materials subject to the constraint that the volume must be 50ft^3. Note as well that the cost for each side is just the area of that side times the appropriate cost.

The two functions we'll be working with here this time are,

$$\text{Minimize}: C = 10(2lw) + 6(2wh + 2lh) = 60w^2 + 48wh$$
$$\text{Constraint}: 50 = lwh = 3w^2h$$

As with the first example, we will solve the constraint for one of the variables and plug this into the cost. It will definitely be easier to solve the constraint for h so let's do that.

$$h = \frac{50}{3w^2}$$

Plugging this into the cost gives,

$$C(w) = 60w^2 + 48w\left(\frac{50}{3w^2}\right) = 60w^2 + \frac{800}{w}$$

Now, let's get the first and second (we'll be needing this later…) derivatives,

$$C'(w) = 120w - 800w^{-2} = \frac{120w^3 - 800}{w^2} \qquad C''(w) = 120 + 1600w^{-3}$$

The next critical point will come from determining where the numerator is zero.

$$120w^3 - 800 = 0 \qquad \Rightarrow \qquad w = \sqrt[3]{\frac{800}{120}} = \sqrt[3]{\frac{20}{3}} = 1.8821$$

First, we know that whatever the value of w that we get it will have to be positive and we can see second derivative above that

provided $w > 0$ we will have $C''(w) > 0$ and so in the interval of possible optimal values the cost function will always be concave up and so $w = 1.8821$ must give the absolute minimum cost.

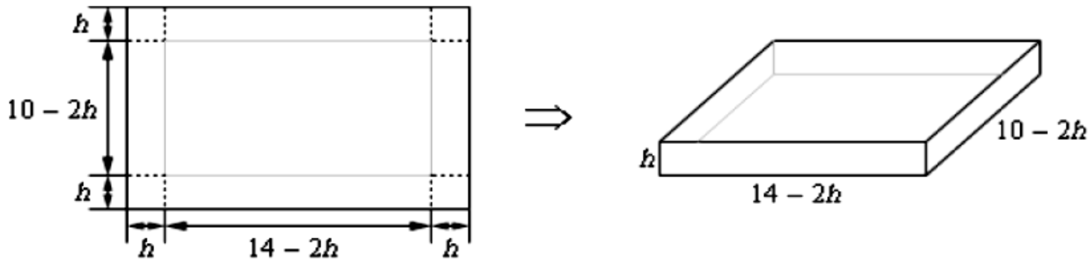
All we need to do now is to find the remaining dimensions.

$$w = 1.8821$$
$$l = 3w = 3(1.8821) = 5.6463$$
$$h = \frac{50}{3w^2} = \frac{50}{3(1.8821)^2} = 4.7050$$

Also, even though it was not asked for, the minimum cost is : $C(1.8821) = \$637.60$.

Example 60:

We have a piece of cardboard that is 14 inches by 10 inches and we're going to cut out the corners as shown below and fold up the sides to form a box, also shown below. Determine the height of the box that will give a maximum volume.



In this case we want to maximize the volume. Here is the volume, in terms of h and its first derivative.

$$V(h) = h(14-2h)(10-2h) = 140h - 48h^2 + 4h^3 \qquad V'(h) = 140 - 96h + 12h^2$$

Setting the first derivative equal to zero and solving gives the following two critical points,

$$h = \frac{12 \pm \sqrt{39}}{3} = 1.9183,\quad 6.0817$$

So, knowing that whatever h is it must be in the range $0 \le h \le 5$ we can see that the second critical point is outside this range and so the only critical point that we need to worry about is 1.9183.

Finally, since the volume is defined and continuous on $0 \le h \le 5$ all we need to do is plug in the critical points and endpoints into the volume to determine which gives the largest volume. Here are those function evaluations.

$$V(0) = 0 \qquad V(1.9183) = 120.1644 \qquad V(5) = 0$$

So, if we take $h = 1.9183$ we get a maximum volume.

Example 61: a rectangle is to be inscribed in a semicircle of radius 2. What is the largest area the rectangle can have, and what are its dimensions?

Solution Let $(x, \sqrt{4-x^2})$ be the coordinates of the corner of the rectangle obtained by placing the circle and rectangle in the coordinate plane (Figure 4.40). The length, height, and area of the rectangle can then be expressed in terms of the position x of the lower right-hand corner:

$$\text{Length: } 2x, \qquad \text{Height: } \sqrt{4-x^2}, \qquad \text{Area: } 2x\sqrt{4-x^2}.$$

Notice that the values of x are to be found in the interval $0 \le x \le 2$, where the selected corner of the rectangle lies.

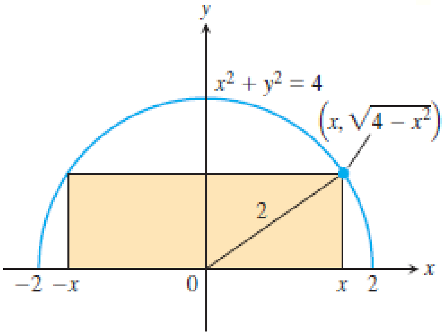
Our goal is to find the absolute maximum value of the function

$$A(x) = 2x\sqrt{4-x^2}$$

on the domain $[0, 2]$.

The derivative

$$\frac{dA}{dx} = \frac{-2x^2}{\sqrt{4-x^2}} + 2\sqrt{4-x^2}$$



is not defined when $x = 2$ and is equal to zero when

$$\frac{-2x^2}{\sqrt{4-x^2}} + 2\sqrt{4-x^2} = 0$$
$$-2x^2 + 2(4-x^2) = 0$$
$$8 - 4x^2 = 0$$
$$x^2 = 2$$
$$x = \pm\sqrt{2}.$$

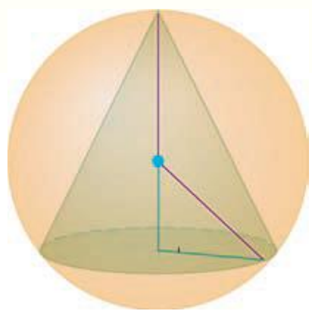
Of the two zeros, $x = \sqrt{2}$ and $x = -\sqrt{2}$, only $x = \sqrt{2}$ lies in the interior of A's domain and makes the critical-point list. The values of A at the endpoints and at this one critical point are

Critical point value: $A(\sqrt{2}) = 2\sqrt{2}\sqrt{4-2} = 4$

Endpoint values: $A(0) = 0, \qquad A(2) = 0.$

The area has a maximum value of 4 when the rectangle is $\sqrt{4-x^2} = \sqrt{2}$ units high and $2x = 2\sqrt{2}$ units long.

Example 62**(Homework)** Find the volume of the largest right circular cone that can be inscribed in a sphere of radius 3?



V = 32π / 3