Divisible Module

Definition// Let R be an integral domain and let M be an R-module. An element $m \in M$ is called divisible if for every nonzero element $r \in R$, there exists an element $x \in M$ such that

$$m = rx$$

خواص هذا الموديل

① The set of all divisible element of M denoted by $\delta(M)$.

② $\delta(M)$ is a submodule of M ??

Def// A module M is called divisible, if $\delta(M) = M$ that is divisible iff $rM = M$ $\forall\, 0 \neq r \in R$.

Remark// If R is an integral domain and M is an R-module, then M is divisible.

Examples// ① $\mathbb{Z}$ as a $\mathbb{Z}$-Module is not divisible, because $3 \in \mathbb{Z}$ is not divisible element, If $0 \neq r = 2 \in \mathbb{Z} \Longrightarrow \nexists\, y \in \mathbb{Z} \ni 2 \cdot y = 3$

② The only divisible element in $\mathbb{Z}$ is 0.

Or $2\mathbb{Z} = E \neq \mathbb{Z} \Longrightarrow \mathbb{Z}$ is not divisible.

$n\mathbb{Z}$ لأنه ماذا عن هنا عفاة

$10 = 6 \cdot \square$?

③ $\mathbb{Z}_n$ is not divisible $\mathbb{Z}$-Module

$n\mathbb{Z}_n = \{\bar{0}\} \neq \mathbb{Z}_n$

اي صفة موجودة بالاين وغير موجودة بالأب

④ $\mathbb{Q}$ as a $\mathbb{Z}$-Module ($\mathbb{Q}_{\mathbb{Z}}$) is divisible

because: Let $x \in \mathbb{Q} \wedge 0 \neq r \in \mathbb{Z}$

$x = \frac{a}{b}$, $a, b \in \mathbb{Z} \wedge b \neq 0$

take $y = \frac{a}{rb} \Rightarrow y \in \mathbb{Z}\mathbb{Q} \Longrightarrow ry = r \cdot \frac{a}{rb} = \frac{a}{b} = x$

$\therefore x$ is

⑤ Z_6 as a Z-module is not divisible in Z_6.

because $(\bar{0})$ is the only divisible in Z_6.

$\nexists\, x \in Z_6 \ni 6x = m$, for example

$2 \neq 6 \cdot \boxed{}$

Z_6 ما عنصر يستحيل ان $Z_6 = \{0, 1, 2, 3, 4, 5\}$

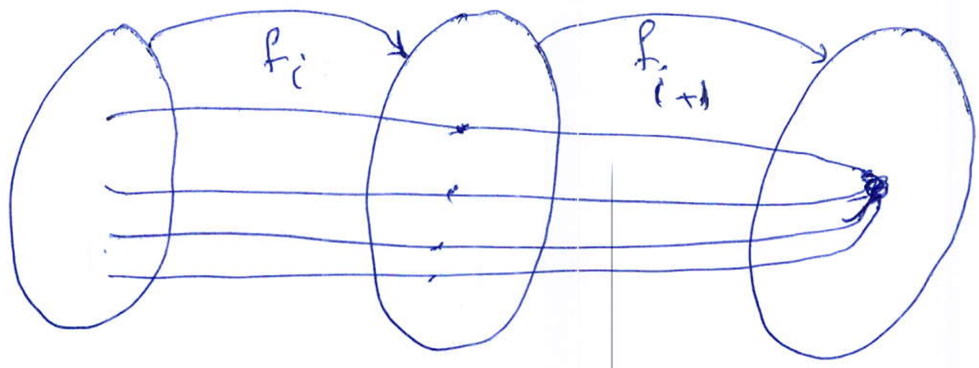
Exact Sequence

المتتابعة التامة

Definition// Let $\{M_i\}$, $i \in I$ be a family of R-modules with corresponding family $\{f_i\}_{i \in I}$ of R-homomorphisms with $f_i : M_i \longrightarrow M_{i+1}$, the sequence

$$\cdots \longrightarrow M_{i-2} \xrightarrow{f_{i-2}} M_{i-1} \xrightarrow{f_{i-1}} M_i \xrightarrow{f_i} M_{i+1} \xrightarrow{f_{i+1}} M_{i+2} \xrightarrow{f_{i+2}} \cdots$$

is called an exact sequence iff $\operatorname{Im} f_i = \ker f_{i+1}$ $\forall\, i$



If for $k > i$, $M_i = 0$ the seq is of the form

$$\cdots \longrightarrow M_{i-2} \xrightarrow{f_{i-2}} M_{i-1} \xrightarrow{f_{i-1}} M_i \xrightarrow{f_i = \alpha} 0$$

Similarly, if for $k < i$, $M_i = 0$, the seq is of the form

$$0 \xrightarrow{\alpha} M_i \xrightarrow{f_i} M_{i+1} \xrightarrow{f_{i+1}} M_{i+2} \longrightarrow \cdots$$

An exact seq of the form

$$0 \longrightarrow M_{i-1} \xrightarrow{f_{i-1}} M_i \xrightarrow{f_i} M_{i+1} \xrightarrow{f_{i+1}} 0$$

is called short exact sequence اقل شي 3 حدود

Example// $0 \longrightarrow 2\mathbb{Z} \xrightarrow{i} \mathbb{Z} \xrightarrow{\pi} \frac{\mathbb{Z}}{2\mathbb{Z}} \simeq \mathbb{Z}_2 \longrightarrow 0$
is a short exact sequence, because i and π are homomorphisms and $Im(i) = ker(\pi)$;
$Im(i) = 2\mathbb{Z} = ker(\pi)$. i.e $f : \mathbb{Z} \longrightarrow \frac{\mathbb{Z}}{2\mathbb{Z}} \ni f(x) = x + 2\mathbb{Z} \ \forall x \in \mathbb{Z}$.