LECTURE NOTE

ON

PROBABILITY AND SATISTICS 2

BY

ASSIST. PRF. DR. MUSTAFA I. NAIF

DEPARTMENT OF MATHEMATICS
COLLEGE OF EDUCATION FOR PURE SCIENCE
UNIVERISTY OF ANBAR

Contents

- Discrete distributions

(Bernoulli, Binomial , Poisson, Uniform, Hypergeometric, Negative Binomial, Some Special Discrete Bivariate Distributions)

- Continuous distributions

  (Exponential , Normal , Chi-square , Gamma , Student's t , F distribution, Multinomial, Multivariate normal, Multivariate Student's t, Wishart, Some Special Continuous Bivariate Distributions)

- Functions of random variables and their distribution

  Distribution Function Method, Transformation Method, Moment Method,

References

- Mathematical Statistics with Applications. D. D. Wackerly, William Mendenhall and Richard L. Scheaffer, seven edition, 2008
- Probability and Statistics. Morris H. DeGroot and Mark J. Schervish, Fourth Edition,2012
- A FIRST COURSE IN PROBABILITY. Sheldon Ross, Ninth Edition, 2014

LECTURE 1#

➢ Outline :-

✓ Discrete distributions

1- Bernoulli distribution

Definition

Expected value Variance

Moment generating function

Characteristic function

Distribution function

Relation to the binomial distribution

Solved exercises

Bernoulli distribution

Suppose you perform an experiment with two possible outcomes: either success or failure. Success happens with probability p, while failure happens with probability 1-p. A random variable that takes value 1 in case of success and 0 in case of failure is called a Bernoulli random variable (alternatively, it is said to have a Bernoulli distribution).

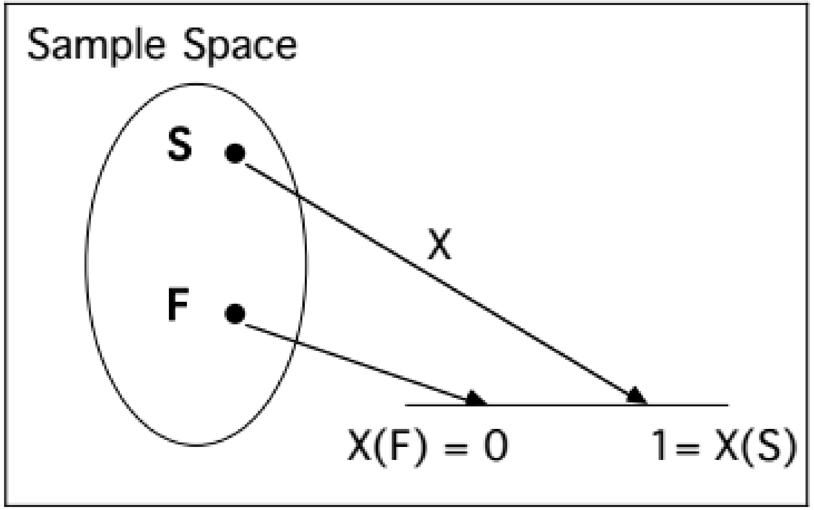
Bernoulli distribution

Definition:

The random variable *X* is called the Bernoulli random variable if its probability mass function is of the form

$$f(x) = p^x (1-p)^{1-x}, \qquad x = 0, 1$$

where p is the probability of success.



We denote the Bernoulli random variable by writing $X \sim BER(p)$.

Bernoulli distribution

Proof :

Non-negativity is obvious. We need to prove that the sum of f(x) over its support equals 1. This is proved as follows:

$$\sum_{x=0}^{1} f(x) = f(0) + f(1)$$

$$= 1 - p + p = 1$$

Bernoulli distribution

Example :

What is the probability of getting a score of not less than 5 in a throw of a six-sided die?

Answer: Although there are six possible scores $\{1, 2, 3, 4, 5, 6\}$, we are grouping them into two sets, namely $\{1, 2, 3, 4\}$ and $\{5, 6\}$. Any score in $\{1, 2, 3, 4\}$ is a failure and any score in $\{5, 6\}$ is a success. Thus, this is a Bernoulli trial with

$$P(X = 0) = P(\text{failure}) = \frac{4}{6} \qquad \text{and} \qquad P(X = 1) = P(\text{success}) = \frac{2}{6}.$$

Hence, the probability of getting a score of not less than 5 in a throw of a six-sided die is $\frac{2}{6}$.

Bernoulli distribution

Theorem :

If X is a Bernoulli random variable with parameter p, then the mean, variance and moment generating functions are respectively given by:

$$\mu_X = p$$

$$\sigma_X^2 = p(1-p)$$

$$M_X(t) = (1-p) + pe^t.$$

Bernoulli distribution

Proof:

The mean of the Bernoulli random variable is

$$\begin{aligned}\mu_X &= \sum_{x=0}^{1} x\, f(x) \\ &= \sum_{x=0}^{1} x\, p^x\, (1-p)^{1-x} \\ &= p.\end{aligned}$$

Similarly, the variance of X is given by

$$\begin{aligned}\sigma_X^2 &= \sum_{x=0}^{1} (x-\mu_X)^2\, f(x) \\ &= \sum_{x=0}^{1} (x-p)^2\, p^x\, (1-p)^{1-x} \\ &= p^2\,(1-p) + p\,(1-p)^2 \\ &= p\,(1-p)\,[p + (1-p)] \\ &= p\,(1-p).\end{aligned}$$

Next, we find the moment generating function of the Bernoulli random variable

$$\begin{aligned}M(t) &= E\left(e^{tX}\right) \\ &= \sum_{x=0}^{1} e^{tx}\, p^x\, (1-p)^{1-x} \\ &= (1-p) + e^t\, p.\end{aligned}$$

Bernoulli distribution

Characteristic function

Definition Let X be a random variable. The characteristic function $\phi(t)$ of X is defined as

$$\begin{aligned}\phi(t) &= E\left(e^{i\,t\,X}\right)\\ &= E\left(cos(tX) + i\,sin(tX)\right)\\ &= E\left(cos(tX)\right) + i\,E\left(sin(tX)\right).\end{aligned}$$

The probability density function can be recovered from the characteristic function by using the following formula

$$f(x) = \frac{1}{2\,\pi} \int_{-\infty}^{\infty} e^{-i\,t\,x}\,\phi(t)\,dt.$$

Bernoulli distribution

Characteristic function

The characteristic function of a Bernoulli random variable X is

$$\varphi_X(t) = 1 - p + p\exp(it)$$

Proof. Using the definition of characteristic function:

$$\begin{aligned}
\varphi_X(t) &= \mathrm{E}[\exp(itX)] \\
&= \sum_{x \in R_X} \exp(itx)\, p_X(x) \\
&= \exp(it \cdot 1) \cdot p_X(1) + \exp(it \cdot 0) \cdot p_X(0) \\
&= \exp(it) \cdot p + 1 \cdot (1 - p) \\
&= 1 - p + p\exp(it)
\end{aligned}$$

–

Bernoulli distribution

Distribution function

The distribution function of a Bernoulli random variable X is

$$F_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1-p & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x \geq 1 \end{cases}$$

Proof. Remember the definition of distribution function:

$$F_X(x) = \mathrm{P}(X \leq x)$$

and the fact that X can take either value 0 or value 1. If $x < 0$, then $\mathrm{P}(X \leq x) = 0$, because X can not take values strictly smaller than 0. If $0 \leq x < 1$, then $\mathrm{P}(X \leq x) = 1 - p$, because 0 is the only value strictly smaller than 1 that X can take. Finally, if $x \geq 1$, then $\mathrm{P}(X \leq x) = 1$, because all values X can take are smaller than or equal to 1. ■

1

Solved exercises

Let X be a Bernoulli random variable with parameter $p = 1/2$. Find its tenth moment.

Solution

The moment generating function of X is

$$M_X(t) = \frac{1}{2} + \frac{1}{2}\exp(t)$$

The tenth moment of X is equal to the tenth derivative of its moment generating function, evaluated at $t = 0$:

$$\mu_X(10) = \mathrm{E}\left[X^{10}\right] = \left.\frac{d^{10}M_X(t)}{dt^{10}}\right|_{t=0}$$

But

$$\begin{aligned}
\frac{dM_X(t)}{dt} &= \frac{1}{2}\exp(t) \\
\frac{d^2M_X(t)}{dt^2} &= \frac{1}{2}\exp(t) \\
&\vdots \\
\frac{d^{10}M_X(t)}{dt^{10}} &= \frac{1}{2}\exp(t)
\end{aligned}$$

so that:

$$\begin{aligned}
\mu_X(10) &= \left.\frac{d^{10}M_X(t)}{dt^{10}}\right|_{t=0} \\
&= \frac{1}{2}\exp(0) = \frac{1}{2}
\end{aligned}$$

2

Solved exercises

Let X and Y be two independent Bernoulli random variables with parameter p.
Derive the probability mass function of their sum: Z = X + Y?

Solution

The probability mass function of X is

$$p_X(x) = \begin{cases} p & \text{if } x = 1 \\ 1 - p & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}$$

The probability mass function of Y is

$$p_Y(y) = \begin{cases} p & \text{if } y = 1 \\ 1 - p & \text{if } y = 0 \\ 0 & \text{otherwise} \end{cases}$$

The support of Z (the set of values Z can take) is

$$R_Y = \{0, 1, 2\}$$

3

The formula for the probability mass function of a sum of two independent variables

$$p_Z(z) = \sum_{y \in R_Y} p_X(z-y)\, p_Y(y)$$

where R_Y is the support of Y. When $z = 0$, the formula gives:

$$\begin{aligned} p_Z(0) &= \sum_{y \in R_Y} p_X(-y)\, p_Y(y) \\ &= p_X(-0)\, p_Y(0) + p_X(-1)\, p_Y(1) \\ &= (1-p)(1-p) + 0 \cdot p = (1-p)^2 \end{aligned}$$

When $z = 1$, the formula gives:

$$\begin{aligned} p_Z(1) &= \sum_{y \in R_Y} p_X(1-y)\, p_Y(y) \\ &= p_X(1-0)\, p_Y(0) + p_X(1-1)\, p_Y(1) \\ &= p \cdot (1-p) + (1-p) \cdot p = 2p(1-p) \end{aligned}$$

When $z = 2$, the formula gives:

$$\begin{aligned} p_Z(2) &= \sum_{y \in R_Y} p_X(2-y)\, p_Y(y) \\ &= p_X(2-0)\, p_Y(0) + p_X(2-1)\, p_Y(1) \\ &= 0 \cdot (1-p) + p \cdot p = p^2 \end{aligned}$$

4

Therefore, the probability mass function of Z is

$$p_Z(z) = \begin{cases} (1-p)^2 & \text{if } z = 0 \\ 2p(1-p) & \text{if } z = 1 \\ p^2 & \text{if } z = 2 \\ 0 & \text{otherwise} \end{cases}$$