LECTURE NOTE

ON

PROBABILITY AND SATISTICS 2

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LECTURE 2#

➢ Outline :-

✓ Discrete distributions

1- Binomial distribution

Definition

Expected value and Variance

Moment generating function

Characteristic function

Distribution function

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Binomial distribution

Consider an experiment having two possible outcomes: either success or failure. Suppose the experiment is repeated several times and the repetitions are independent of each other.

The total number of experiments where the outcome turns out to be a success is a random variable whose distribution is called binomial distribution.

The distribution has two parameters: the number n of repetitions of the experiment, and the probability p of success of an individual experiment.

Note A binomial distribution can be seen as a sum of mutually independent Bernoulli random variables

Binomial distribution

Definition:

A random variable *X* has the *binomial distribution with parameters n and p* if *X* has a discrete distribution for which the p.f. is as follows:

$$p(x|n, p) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x} & \text{for } x = 0, 1, 2, \ldots, n, \ 0 \le p \le 1. \\ 0 & \text{otherwise.} \end{cases}$$

In this distribution, *n* must be a positive integer, and *p* must lie in the interval

We will denote a binomial random variable with parameters *p* and *n* as *X ~ BIN*(*n, p*).

Binomial distribution

Proof :

Non-negativity is obvious. We need to prove that the sum of f(x) over its support equals 1. This is proved as follows:

$$\sum_{x=0}^{1} p(x) = \sum_{x=0}^{n} \binom{n}{x} p^{x} (1-p)^{n-x} = [p + (1-p)]^{n} = 1^{n} = 1$$

where we have used the formula for binomial expansions

$$(a+b)^{n} = \sum_{x=0}^{n} \binom{n}{x} a^{x} b^{n-x}$$

Binomial distribution

Example :

Find the probability of getting five heads and seven tails in 12 flips of a balanced coin.

Solution

Substituting $x = 5$, $n = 12$, and $p = \frac{1}{2}$ into the formula for the binomial distribution, we get

$$b\left(5; 12, \frac{1}{2}\right) = \binom{12}{5}\left(\frac{1}{2}\right)^5\left(1 - \frac{1}{2}\right)^{12-5}$$

and, looking up the value of $\binom{12}{5}$ in binomial table, we find that the result is $792\left(\frac{1}{2}\right)^{12}$, or approximately 0.19.

Probabilities for various binomial distributions can be obtained from the table given at the end of this book and from many statistical software programs.

Binomial Coefficients

n	$\binom{n}{0}$	$\binom{n}{1}$	$\binom{n}{2}$	$\binom{n}{3}$	$\binom{n}{4}$	$\binom{n}{5}$	$\binom{n}{6}$	$\binom{n}{7}$	$\binom{n}{8}$	$\binom{n}{9}$	$\binom{n}{10}$
0	1										
1	1	1									
2	1	2	1								
3	1	3	3	1							
4	1	4	6	4	1						
5	1	5	10	10	5	1					
6	1	6	15	20	15	6	1				
7	1	7	21	35	35	21	7	1			
8	1	8	28	56	70	56	28	8	1		
9	1	9	36	84	126	126	84	36	9	1	
10	1	10	45	120	210	252	210	120	45	10	1
11	1	11	55	165	330	462	462	330	165	55	11
12	1	12	66	220	495	792	924	792	495	220	66
13	1	13	78	286	715	1287	1716	1716	1287	715	286
14	1	14	91	364	1001	2002	3003	3432	3003	2002	1001
15	1	15	105	455	1365	3003	5005	6435	6435	5005	3003
16	1	16	120	560	1820	4368	8008	11440	12870	11440	8008
17	1	17	136	680	2380	6188	12376	19448	24310	24310	19448
18	1	18	153	816	3060	8568	18564	31824	43758	48620	43758
19	1	19	171	969	3876	11628	27132	50388	75582	92378	92378
20	1	20	190	1140	4845	15504	38760	77520	125970	167960	184756

Binomial distribution

H.W:

Find the probability that 7 of 10 persons will recover from a tropical disease if we can assume independence and the probability is 0.80 that any one of them will recover from the disease.

Note:

looking up the value of $\binom{10}{7}$ in binomial table

Binomial distribution

Theorem: The mean and the variance of the binomial distribution are

$$\mu = n\theta \quad \text{and} \quad \sigma^2 = n\theta(1-\theta)$$

Here p = θ

Proof

$$\mu = \sum_{x=0}^{n} x \cdot \binom{n}{x} \theta^x (1-\theta)^{n-x}$$

$$= \sum_{x=1}^{n} \frac{n!}{(x-1)!(n-x)!} \theta^x (1-\theta)^{n-x}$$

where we omitted the term corresponding to $x = 0$, which is 0, and canceled the x against the first factor of $x! = x(x-1)!$ in the denominator of $\binom{n}{x}$. Then, factoring out the factor n in $n! = n(n-1)!$ and one factor θ, we get

Binomial distribution

$$\mu = n\theta \cdot \sum_{x=1}^{n} \binom{n-1}{x-1} \theta^{x-1}(1-\theta)^{n-x}$$

and, letting $y = x - 1$ and $m = n - 1$, this becomes

$$\mu = n\theta \cdot \sum_{y=0}^{m} \binom{m}{y} \theta^{y}(1-\theta)^{m-y} = n\theta$$

$$E[X(X-1)] = \sum_{x=0}^{n} x(x-1) \binom{n}{x} \theta^{x}(1-\theta)^{n-x}$$

$$= \sum_{x=2}^{n} \frac{n!}{(x-2)!(n-x)!} \theta^{x}(1-\theta)^{n-x}$$

$$= n(n-1)\theta^2 \cdot \sum_{x=2}^{n} \binom{n-2}{x-2} \theta^{x-2}(1-\theta)^{n-x}$$

since the last summation is the sum of all the values of a binomial distribution with the parameters m and θ, and hence equal to 1.

To find expressions for μ_2' and σ^2, let us make use of the fact that $E(X^2) = E[X(X-1)] + E(X)$ and first evaluate $E[X(X-1)]$. Duplicating for all practical purposes the steps used before, we thus get

Binomial distribution

and, letting $y = x - 2$ and $m = n - 2$, this becomes

$$E[X(X-1)] = n(n-1)\theta^2 \cdot \sum_{y=0}^{m} \binom{m}{y} \theta^y (1-\theta)^{m-y}$$
$$= n(n-1)\theta^2$$

Therefore,

$$\mu_2' = E[X(X-1)] + E(X) = n(n-1)\theta^2 + n\theta$$

and, finally,

$$\sigma^2 = \mu_2' - \mu^2$$
$$= n(n-1)\theta^2 + n\theta - n^2\theta^2$$
$$= n\theta(1-\theta)$$

Relation to the Bernoulli distribution

Proposition 1: A random variable has a binomial distribution with parameters n and p, with n = 1, if and only if it has a Bernoulli distribution with parameter p.

Proof: We demonstrate that the two distributions are equivalent by showing that they have the same probability mass function.

The probability mass function of a binomial distribution with parameters n and p, with n = 1, is:

$$p(x) = \begin{cases} \binom{1}{x} p^x (1-p)^{1-x} & \text{if } x \in \{0,1\} \\ 0 & \text{if } x \notin \{0,1\} \end{cases} \quad , \quad \text{but,}$$

$$p(0) = \binom{1}{0} p^0 (1-p)^{1-0} = \frac{1!}{0!1!}(1-p) = 1-p \text{ , and,}$$

$$p(1) = \binom{1}{1} p^1 (1-p)^{1-1} = \frac{1!}{1!0!} p = p$$

Relation to the Bernoulli distribution

Proof:

Therefore, the probability mass function can be written as

$$f(x) = \begin{cases} p & \text{if } x = 1 \\ 1-p & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases} \longrightarrow$$

which is the probability mass function of a Bernoulli random variable.

Proposition 2 : A random variable has a binomial distribution with parameters n and p if and only if it can be written as a sum of n jointly independent Bernoulli random variables with parameter p.

Proof: We will prove that later:

Binomial distribution

Theorem :

The moment generating function of a binomial random variable X is defined for any $t \in R$ as : $M_X(t) = (1 - p + p\exp(t))^n$

Proof:

$$
\begin{aligned}
M_X(t) &= \mathrm{E}[\exp(tX)] \\
&= \mathrm{E}[\exp(t(Y_1 + \ldots + Y_n))] \\
&= \mathrm{E}[\exp(tY_1) \cdot \ldots \cdot \exp(tY_n)] \\
&= \mathrm{E}[\exp(tY_1)] \cdot \ldots \cdot \mathrm{E}[\exp(tY_n)] \\
&= M_{Y_1}(t) \cdot \ldots \cdot M_{Y_n}(t) \\
&= (1 - p + p\exp(t)) \cdot \ldots \cdot (1 - p + p\exp(t)) \\
&= (1 - p + p\exp(t))^n
\end{aligned}
$$

The definition of m. g. f.

X can be represented as a sum of n independent Bernoulli r.v.

Y1 ; …; Yn are jointly independent

The definition of m. g. f. Y1,…Yn

The formula for the moment generating function of a Ber. r.v.

Since the m.g.f. Ber. .v. exists,so is the m.g.f. of a binomial random variable exists .

Binomial distribution

Characteristic function:

The characteristic function of a binomial random variable X is

$$\varphi_X(t) = (1 - p + p\exp(it))^n$$

Proof: Similar to the previous proof

$$\begin{aligned}
\varphi_X(t) &= \mathrm{E}[\exp(itX)] \\
&= \mathrm{E}[\exp(it(Y_1 + \ldots + Y_n))] \\
&= \mathrm{E}[\exp(itY_1) \cdot \ldots \cdot \exp(itY_n)] \\
&= \mathrm{E}[\exp(itY_1)] \cdot \ldots \cdot \mathrm{E}[\exp(itY_n)] \\
&= \varphi_{Y_1}(t) \cdot \ldots \cdot \varphi_{Y_n}(t) \\
&= (1 - p + p\exp(it)) \cdot \ldots \cdot (1 - p + p\exp(it)) \\
&= (1 - p + p\exp(it))^n
\end{aligned}$$

Binomial distribution

Distribution function: The distribution function of a binomial random variable X is

$$F_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ \sum_{s=0}^{x} \binom{n}{s} p^s (1-p)^{n-s} & \text{if } 0 \leq x \leq n \\ 1 & \text{if } x > n \end{cases}$$

Proof. For $x < 0$, $F_X(x) = 0$, because X cannot be smaller than 0. For $x > n$, $F_X(x) = 1$, because X is always smaller than or equal to n. For $0 \leq x \leq n$:

$$\begin{aligned} F_X(x) &= \mathrm{P}(X \leq x) \\ &= \sum_{s=0}^{x} \mathrm{P}(X = s) \\ &= \sum_{s=0}^{x} p_X(s) = \sum_{s=0}^{x} \binom{n}{s} p^s (1-p)^{n-s} \end{aligned}$$

1

Solved exercises

Suppose you independently flip a coin 4 times and the outcome of each toss can be either head (with probability 1/2) or tails (also with probability 1=2). What is the probability of obtaining exactly 2 tails?

Solution

Denote by X the number of times the outcome is tails (out of the 4 tosses). X has a binomial distribution with parameters n = 4 and p = 1/2. The probability of obtaining exactly 2 tails can be computed from the probability mass function of X as follows:

$$
\begin{aligned}
p_X(2) &= \binom{n}{2} p^2 (1-p)^{n-2} = \binom{4}{2} \left(\frac{1}{2}\right)^2 \left(1-\frac{1}{2}\right)^{4-2} \\
&= \frac{4!}{2!2!} \frac{1}{4} \frac{1}{4} = \frac{4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1 \cdot 2 \cdot 1} \frac{1}{16} = \frac{6}{16} = \frac{3}{8}
\end{aligned}
$$

2

Solved exercises

Suppose you independently throw a dart 10 times. Each time you throw a dart, the probability of hitting the target is 3/4. What is the probability of hitting the target less than 5 times (out of the 10 total times you throw a dart)?

Solution

Denote by X the number of times you hit the target. X has a binomial distribution with parameters n = 10 and p = 3/4. The probability of hitting the target less than 5 times can be computed from the distribution function of X as follows:

$$\begin{aligned} \mathrm{P}(X<5) &= \mathrm{P}(X\leq 4) = F_X(4) \\ &= \sum_{s=0}^{4}\binom{n}{s}p^s(1-p)^{n-s} \\ &= \sum_{s=0}^{4}\binom{10}{s}\left(\frac{3}{4}\right)^s\left(\frac{1}{4}\right)^{10-s} \simeq 0.0197 \end{aligned}$$

3

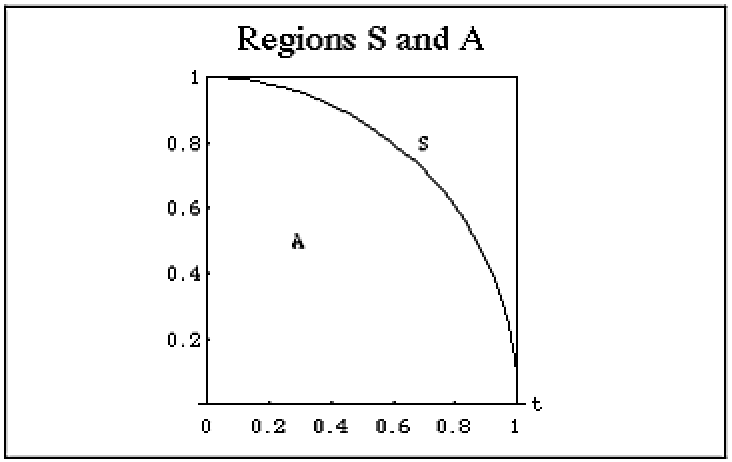
Exercises

1) On a five-question multiple-choice test there are five possible answers, of which one is correct. If a student guesses randomly and independently, what is the probability that she is correct only on two questions?
2) What is the probability of rolling two sixes and three nonsixes in 5 independent casts of a fair die?
3) What is the probability of rolling at most two sixes in 5 independent casts of a fair die?
4) Suppose that 2000 points are selected independently and at random from the unit squares $S = \{(x, y) \,|\, 0 \le x, y \le 1\}$. Let *X* equal the number of points that fall in $A = \{(x, y) \,|\, x^2 + y^2 < 1\}$. How is *X* distributed? What are the mean, variance and standard deviation of *X*?

Exercises

4) Hinte : If a point falls in *A*, then it is a success. If a point falls in the complement of *A*, then it is a failure. The probability of success is

$$p = \frac{\text{area of A}}{\text{area of S}} = \frac{1}{4}\pi.$$



5) Let the probability that the birth weight (in grams) of babies in America is less than 2547 grams be 0.1. If *X* equals the number of babies that weigh less than 2547 grams at birth among 20 of these babies selected at random, then what is $P(X \leq 3)$?