LECTURE NOTE

ON

PROBABILITY AND SATISTICS 2

BY

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LECTURE 3#

➢ Outline :-

✓ Discrete distributions

3- Poisson distribution

Definition

Expected value and Variance

Moment generating function

Characteristic function

Distribution function

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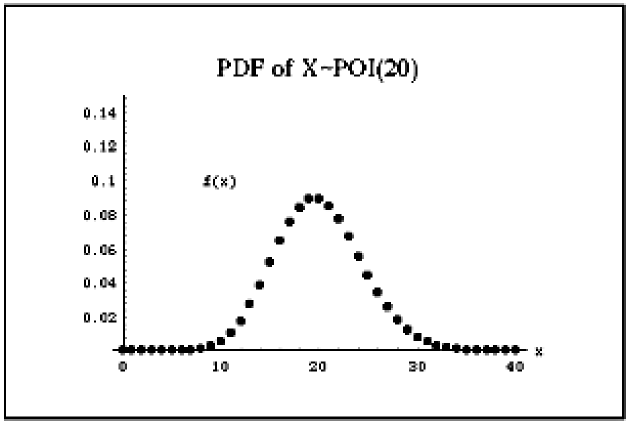
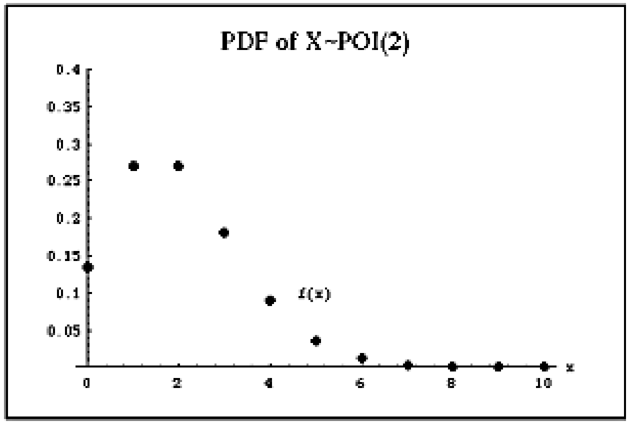
Poisson distribution

Definition : A random variable X is said to have a Poisson distribution if its probability mass function is given by

$$f(x) = \frac{e^{-\lambda}\,\lambda^{x}}{x!}, \qquad x = 0, 1, 2, \cdots, \infty,$$

where $0 < \lambda < \infty$ is a parameter. We denote such a random variable by $X \sim POI(\lambda)$.





The probability density function f is called the Poisson distribution after Simeon D. Poisson (1781-1840).

Poisson distribution

Proof :

It is easy to check $f(x) \geq 0$. We show that $\sum_{x=0}^{\infty} f(x)$ is equal to one

$$\sum_{x=0}^{\infty} f(x) = \sum_{x=0}^{\infty} \frac{e^{-\lambda} \lambda^{x}}{x!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{\lambda^{x}}{x!}$$

$$= e^{-\lambda} e^{\lambda} = 1.$$

Poisson distribution

Theorem: The mean , the variance the m.g.f. of Poisson distribution are:

$$E(X) = \lambda$$
$$Var(X) = \lambda$$
$$M(t) = e^{\lambda(e^t - 1)}.$$

Proof: First, we find the moment generating function of X.

$$\begin{aligned}
M(t) &= \sum_{x=0}^{\infty} e^{tx} f(x) \\
&= \sum_{x=0}^{\infty} e^{tx} \frac{e^{-\lambda}\lambda^x}{x!} \\
&= e^{-\lambda} \sum_{x=0}^{\infty} e^{tx} \frac{\lambda^x}{x!} \\
&= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(e^t\lambda)^x}{x!} \\
&= e^{-\lambda} e^{\lambda e^t} \\
&= e^{\lambda(e^t - 1)}.
\end{aligned}$$

Poisson distribution

Thus,

$$M'(t) = \lambda\, e^t\, e^{\lambda\,(e^t - 1)},$$

and

$$E(X) = M'(0) = \lambda.$$

Similarly,

$$M''(t) = \lambda\, e^t\, e^{\lambda\,(e^t - 1)} + \left(\lambda\, e^t\right)^2\, e^{\lambda\,(e^t - 1)}.$$

Hence

$$M''(0) = E(X^2) = \lambda^2 + \lambda.$$

Therefore

$$Var(X) = E(X^2) - (\,E(X)\,)^2 = \lambda^2 + \lambda - \lambda^2 = \lambda.$$

Poisson distribution

Example : A random variable *X* has Poisson distribution with a mean of 3. What is the probability that *X* is bounded by 1 and 3, that is,

$$P(1 \le X \le 3)?$$

Answer:

$$\mu_X = 3 = \lambda$$

$$f(x) = \frac{\lambda^x e^{-\lambda}}{x!}$$

Hence

$$f(x) = \frac{3^x e^{-3}}{x!}, \qquad x = 0, 1, 2, \ldots$$

Therefore

$$\begin{aligned} P(1 \le X \le 3) &= f(1) + f(2) + f(3) \\ &= 3\,e^{-3} + \frac{9}{2}\,e^{-3} + \frac{27}{6}\,e^{-3} \\ &= 12\,e^{-3}. \end{aligned}$$

Poisson distribution

Example : The number of tra!c accidents per week in a small city has a Poisson distribution with mean equal to 3. What is the probability of exactly 2 accidents occur in 2 weeks?

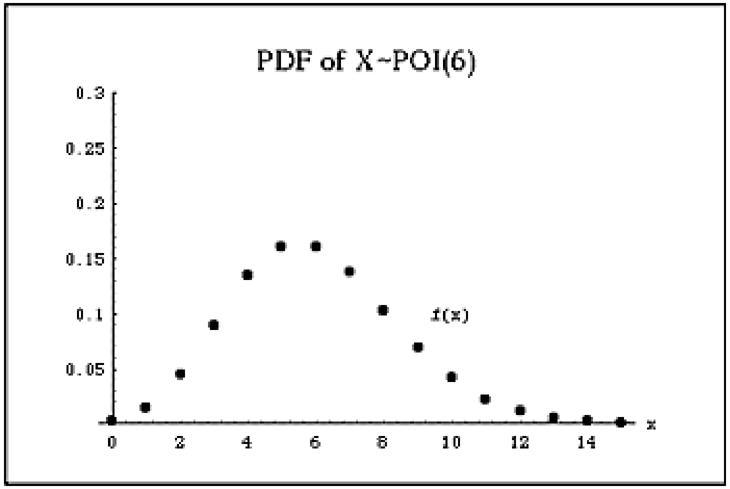
Answer: The mean tra!c accident is 3. Thus, the mean accidents in two weeks are $\lambda = (3)\,(2) = 6.$

Since

$$f(x) = \frac{\lambda^x\, e^{-\lambda}}{x!}$$

we get

$$f(2) = \frac{6^2\, e^{-6}}{2!} = 18\, e^{-6}.$$



Poisson distribution

Characteristic function:

The characteristic function of Poisson random variable X is

$$\varphi_X(t) = \exp(\lambda[\exp(it) - 1])$$

Proof:

$$\begin{aligned}
\varphi_X(t) &= \mathrm{E}[\exp(itX)] \\
&= \sum_{x \in R_X} \exp(itx)\, p_X(x) \\
&= \sum_{x \in R_X} [\exp(it)]^x \exp(-\lambda) \frac{1}{x!} \lambda^x \\
&= \exp(-\lambda) \sum_{x=0}^{\infty} \frac{(\lambda \exp(it))^x}{x!} \\
&= \exp(-\lambda) \exp(\lambda \exp(it)) \\
&= \exp(\lambda[\exp(it) - 1])
\end{aligned}$$

where:

$$\exp(\lambda \exp(it)) = \sum_{x=0}^{\infty} \frac{(\lambda \exp(it))^x}{x!}$$

is the usual Taylor series expansion of the exponential function

Poisson distribution

Distribution function: The distribution function of a Poisson random variable X is

$$F_X(x) = \begin{cases} \exp(-\lambda) \sum_{s=0}^{\lfloor x \rfloor} \frac{1}{s!}\lambda^s & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Where $\lfloor x \rfloor$ is the largest integer not greater than x.

Proof:

$$\begin{aligned} F_X(x) &= \mathrm{P}(X \leq x) \\ &= \sum_{s=0}^{\lfloor x \rfloor} \mathrm{P}(X = s) \\ &= \sum_{s=0}^{\lfloor x \rfloor} p_X(s) \\ &= \sum_{s=0}^{\lfloor x \rfloor} \exp(-\lambda) \frac{1}{s!}\lambda^s \\ &= \exp(-\lambda) \sum_{s=0}^{\lfloor x \rfloor} \frac{1}{s!}\lambda^s \end{aligned}$$

1

Solved exercises

Let *X* have a Poisson distribution with parameter λ = 1. What is the probability that $X \geq 2$ given that $X \leq 4$?

Solution

$$P(X \geq 2 \,/\, X \leq 4) = \frac{P(2 \leq X \leq 4)}{P(X \leq 4)}.$$

$$\begin{aligned} P(2 \leq X \leq 4) &= \sum_{x=2}^{4} \frac{\lambda^x e^{-\lambda}}{x!} \\ &= \frac{1}{e} \sum_{x=2}^{4} \frac{1}{x!} \\ &= \frac{17}{24\,e}. \end{aligned}$$

Similarly

$$\begin{aligned} P(X \leq 4) &= \frac{1}{e} \sum_{x=0}^{4} \frac{1}{x!} \\ &= \frac{65}{24\,e}. \end{aligned}$$

Therefore, we have

$$P(X \geq 2 \,/\, X \leq 4) = \frac{17}{65}.$$

2

Solved exercises

If the moment generating function of a random variable *X* is $M(t) = e^{4.6\,(e^t-1)}$, then what are the mean and variance of *X*? What is the probability that *X* is between 3 and 6, that is $P(3 < X < 6)$?

Solution: Since the moment generating function of *X* is given by

$M(t) = e^{4.6\,(e^t-1)}$

we conclude that $X \sim POI(\lambda)$ with $\lambda = 4.6$. Thus, by

$$E(X) = 4.6 = Var(X).$$

$$\begin{aligned} P(3 < X < 6) &= f(4) + f(5) \\ &= F(5) - F(3) \\ &= 0.686 - 0.326 \\ &= 0.36. \end{aligned}$$