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Theorem:

If $\{A_i\}_{i\in I}$ be an indexed family, then

① If $A_i \subseteq B \ \forall i \in I$, then $\bigcup_{i\in I} A_i \subseteq B$

② If $B \subseteq A_i \ \forall i \in I$, then $B \subseteq \bigcap_{i\in I} A_i$

Proof ①

Suppose that $A_i \subseteq B \ \forall i \in I$

Let $x \in \bigcup_{i\in I} A_i$

$\therefore \exists j \in I$ such that $x \in A_j$

$\because A_i \subseteq B \ \forall i \in I$

$\therefore A_j \subseteq B$

$\therefore x \in A_j \subseteq B$

$\Rightarrow x \in B$

$\Rightarrow \bigcup_{i\in I} A_i \subseteq B$

Proof ②

Suppose that $B \subseteq A_i \ \forall i \in I$.

Let $x \in B$

$\because B \subseteq A_i \ \forall i \in I$

$\therefore x \in B \subseteq A_i \ \forall i \in I$

$\therefore x \in A_i \ \forall i \in I$

$\therefore x \in \bigcap_{i\in I} A_i$

$\therefore B \subseteq A_{i \in I}$.

Theorem: DeMorgan's

If $\{A_i\}_{i\in I}$ be an indexed family, then

① $\left(\bigcup_{i\in I} A_i\right)^c = \bigcap_{i\in I} A_i^c$

Proof:

Let $x \in \left(\bigcup_{i\in I} A_i\right)^c$

$\Rightarrow x \notin \bigcup_{i\in I} A_i$

$\Rightarrow x \notin A_i \quad \forall i \in I$ deiyno *

$\Rightarrow x \in A_i^c \quad \forall i \in I$

$\Rightarrow x \in \bigcap_{i\in I} A_i^c$

$\Rightarrow \left(\bigcup_{i\in I} A_i\right)^c \subseteq \bigcap_{i\in I} A_i^c$ ①

Conversely.

Let $y \in \bigcap_{i\in I} A_i^c$

$\Rightarrow y \in A_i^c \quad \forall i \in I.$

$\Rightarrow y \notin A_i \quad \forall i \in I$

$\Rightarrow y \notin \bigcup_{i\in I} A_i$

$\Rightarrow y \in \left(\bigcup_{i\in I} A_i\right)^c$

$\Rightarrow \bigcap_{i\in I} A_i^c \subseteq \left(\bigcup_{i\in I} A_i\right)^c$ — ②

$\left(\bigcup_{i\in I} A_i\right)^c = \bigcap_{i\in I} A_i^c$

② $\left(\bigcap_{i\in I} A_i\right)^c = \bigcup_{i\in I} A_i^c$ (H.W)

Relations العلاقات

Cartesian Product الضرب الديكارتي

Definition: Ordered Pair الزوج المرتب

An ordered pair of elements a and b is denoted by (a,b) where a is called the first element and b is the second element.

Remark: Let a, b, c and d be four elements. Then:

1. $(a,b) \neq (c,d)$ in general
2. $(a,b) = (c,d)$ if and only if $a=c \wedge b=d$
3. $(a,b) = (b,a)$ if and only if $a=b$.

Cartesian Product

Consider two arbitrary sets A and B. The set of all ordered pairs (a,b) where $a \in A$ and $b \in B$ is called the product set of A and B or (cartesian product of A and B) and its denoted by $A \times B$.

Namely

$$A \times B = \{(a,b) : a \in A , b \in B\}$$

$$(a,b) \in A \times B \text{ iff } a \in A \wedge b \in B$$
$$(a,b) \notin A \times B \text{ iff } a \notin A \vee b \notin B.$$

one usually writes A^2 instead of $A \times A$.

عادة مايكتب A^2 بدلاً من $A \times A$

Examples:

1. Let $A=\{1,2\}$ and $B=\{a,b,c\}$. Then

$$A\times B=\{(1,a),(1,b),(1,c),(2,a),(2,b),(2,c)\}$$

$$B\times A=\{(a,1),(a,2),(b,1),(b,2),(c,1),(c,2)\}$$

$$A\times A=\{(1,1),(1,2),(2,1),(2,2)\}$$

Remark

1. $A\times B\neq B\times A$ In general
2. $n(A\times B)=n(A)\cdot n(B)$.

where A and B are finite sets and $n(A)=$ number of elements in A.

The above example we notice that

$$n(A\times B)=6=2\cdot 3=n(A)\cdot n(B).$$
$$n(B\times A)=6=3\cdot 2=n(B)\cdot n(A)$$
$$n(A\times A)=4=2\cdot 2=n(A)\cdot n(A).$$

Product of Three or More Sets

The idea of a Product of sets can be extended to any finite number of sets. Specifically, for any sets $A_1, A_2, \dots, A_m$, the set of all m-element lists $(a_1, a_2, \dots, a_m)$, where each $a_i \in A_i$ is called the (cartesian) Product of the sets $A_1, A_2, \dots, A_m$; it is denoted by

$$A_1 \times A_2 \times \dots \times A_m \quad \text{or equivalently} \quad \prod_{i=1}^{m} A_i$$

Remark:

1. If either A or B infinite set, then $A \times B$ is infinite.

2. If either A or B empty set then $A \times B$ is empty.

Theorem:
If A and B are two non-empty sets, then
$A \times B = B \times A$ iff $A = B$.

Proof: Suppose that $A \times B = B \times A$.
Let $a \in A$
$\therefore (a,b) \in A \times B \quad \forall b \in B$
$\therefore (a,b) \in A \times B = B \times A$
$\therefore (a,b) \in B \times A$
$\therefore a \in B \wedge b \in A$
$\because a \in A \longrightarrow a \in B$
$\therefore A \subseteq B$.
In similar way we prove $B \subseteq A$, So $A = B$.
(اللكاه الكي)
Suppose that $A = B$.
$\because \quad A \times A = A \times A$
$\therefore \quad A \times B = B \times A$