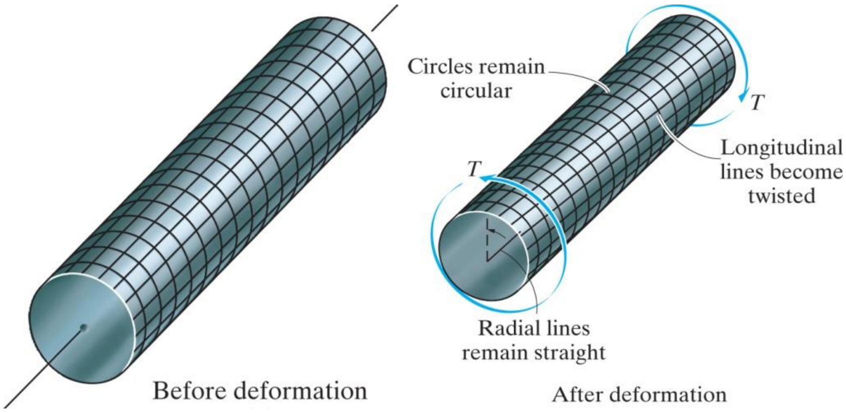
CHAPTER 5 TORSION

1. TORSIONAL DEFORMATION OF A CIRCULAR SHAFT



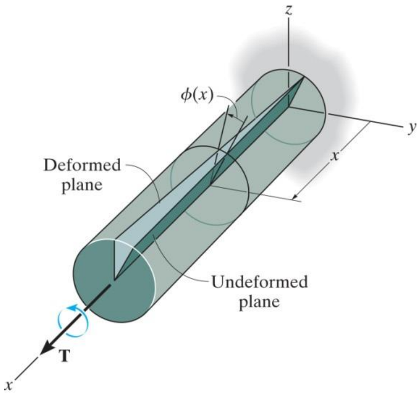
Torque is a moment that tends to twist a member about its longitudinal axis.

Consider a bar with a circular cross section.

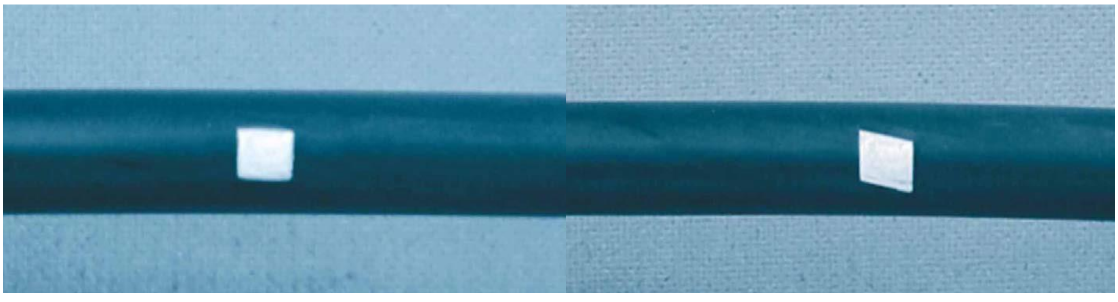
- After applying a torsional moment the original grid tends to distort as shown
- Twisting causes the circles to remain circles.
- Longitudinal lines deform into a helix that intersects the circles at equal angles.
- Cross section ends remain flat and radial lines remain straight.

Objective: Determine a relation between shear strain (γ) and the radial position (ρ) on a circular shaft undergoing torque.

- If the shaft is fixed at one end and torque is applied to the other, the shaft distorts into a skewed form.
- A radial line located on the cross section at a distance x of the fixed end of the shaft rotates at an angle $\phi(x)$.
- $\phi(x)$ is called ***angle of twist*** and is function of x.
- Isolate a small element located at a radial distance ρ from the axis of the shaft.



The angle of twist $\phi(x)$ increases as x increases.



From the definition of shear strain:

$$\gamma = \frac{\pi}{2} - \theta'$$

And from the geometry:

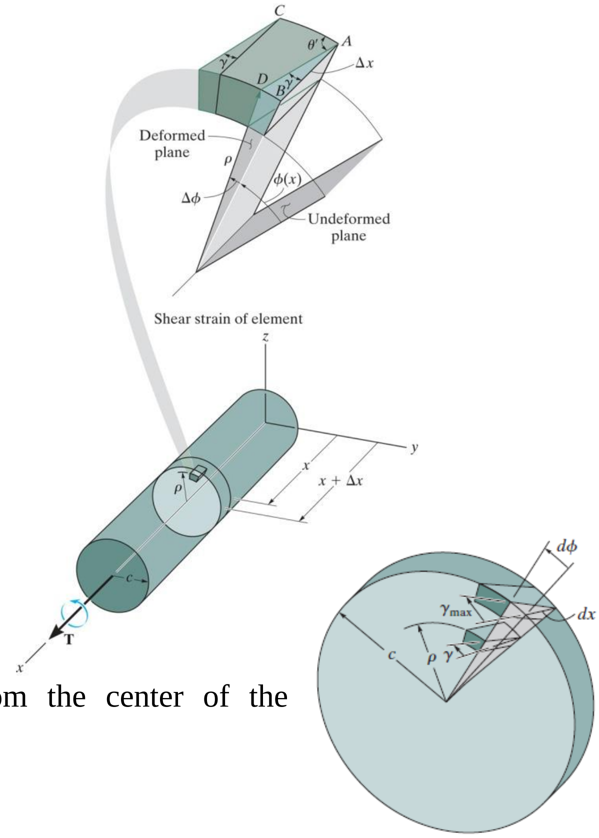
$$BD = \rho\Delta\phi = \Delta x\,\gamma$$

$$\gamma = \rho\frac{d\phi}{dx}$$

Since, $d\phi/dx = \gamma/\rho = \gamma_{max}/c$ (c is the radius),

$$\gamma = \left(\frac{\rho}{c}\right)\gamma_{\max}$$

This means that ***shear strain (γ) varies linearly with the distance ρ*** measured from the center of the cylindrical member.



The shear strain at points on the cross section increases linearly with ρ, i.e., $\gamma = (\rho/c)\gamma_{max}$.

2. THE TORSION FORMULA

When an external torque is applied to a shaft, an internal torque is created within the shaft.

Objective: To develop an expression to relate the internal torque in a shaft to the shear stress distribution on the cross section area of a circular shaft or pipe.

• Assuming that the material remains within the linear–elastic region, i.e. Hooke's law is valid.

$$\tau = G\gamma$$

Therefore, from the relation developed in the previous section:

$$\tau = \left(\frac{\rho}{c}\right)\tau_{\max}$$

It is possible to express:

$$\tau = G\gamma = G\gamma_{max}\left(\frac{\rho}{c}\right) = \tau_{max}\left(\frac{\rho}{c}\right)$$

This equation relates the shear stress distribution over the cross section area in terms of the radial position ρ, that is, the ***shear stress (τ) varies linearly from the center of the cross section***.

• Since the torque produced by the stress distribution over the entire cross section of the shaft needs to be equal to the resultant internal torque T at that section, then

$$dF = \tau\, dA \qquad dT = \rho dF = \rho \tau dA$$

• By integrating over the entire cross section area:

$$T = \int_A \rho(\tau\, dA) = \int_A \rho\left(\frac{\rho}{c}\right)\tau_{\max}\, dA$$

• And since τ_{max}/c is constant:

$$T = \frac{\tau_{\max}}{c}\int_A \rho^2\, dA$$

The integral depends only on the geometry of the shaft. It represents ***the polar moment of inertia*** of the shaft's cross-sectional area about the shaft's longitudinal axis. It is a geometric property so is always positive.

$$J = \int_A \rho^2 dA$$

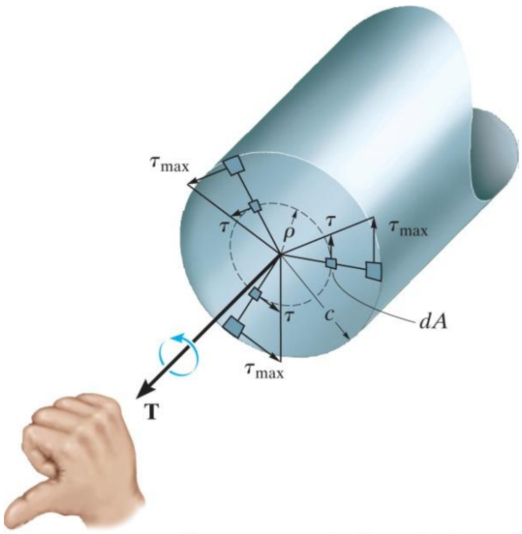
Where: J = Ix + Iy

• Thus:

$$\tau_{\max} = \frac{Tc}{J}$$

Then, the shear stress at a location ρ of the shaft cross section area is

$$\tau = \frac{T\rho}{J}$$



Shear stress varies linearly along each radial line of the cross section.

These two last formulas are known as ***the torsion formula***.

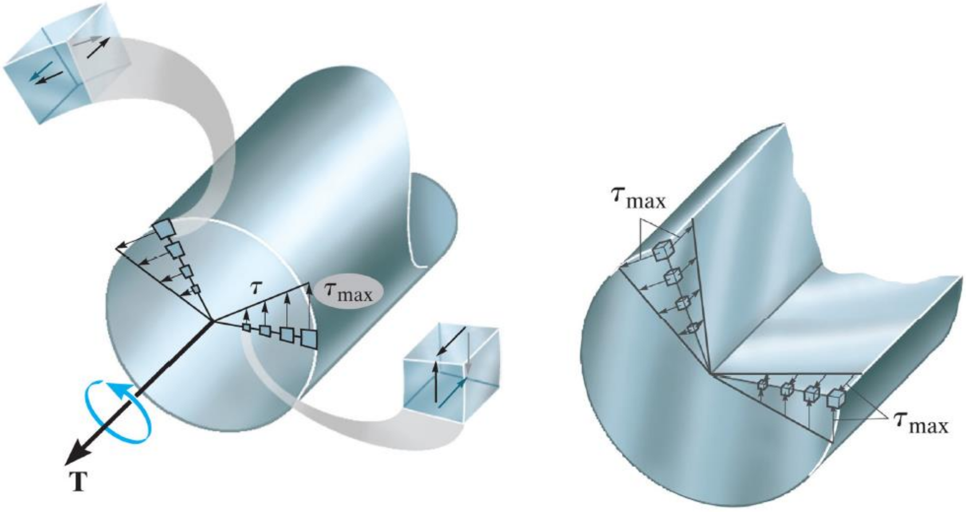
These formulas are valid only if the shaft is circular, the material is homogeneous and elastic.

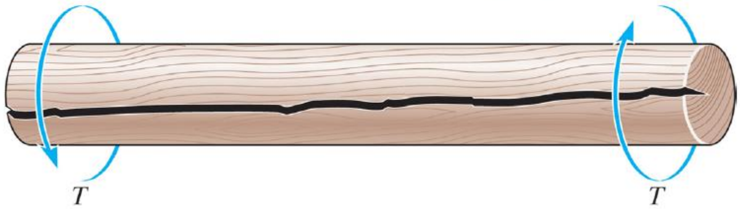
For a solid shaft: $\boldsymbol{J = \pi c^4/2}$

For a hollow (tubular) shaft: $\boldsymbol{J = \pi (c_o^4 - c_i^4)/2}$,

Where c_o and c_i are outer and inner radius.

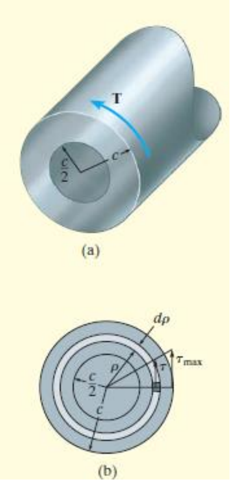
Now if we isolate an element of material on the cross section of the shaft, due to the complementary property of shear, equal shear stresses must also act on four of its adjacent faces as shown in the figure. Hence, ***not only does the internal torque T develop a linear distribution of shear stress along each radial line in the plane of the cross-sectional area, but also an associated shear-stress distribution is developed along an axial plane***.

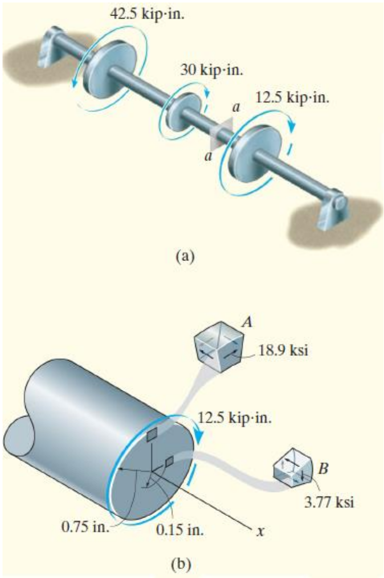




Failure of a wooden shaft due to torsion.

Example 1: The solid shaft of radius c is subjected to a torque T, Figure. Determine the fraction of T that is resisted by the material contained within the outer region of the shaft, which has an inner radius c/2 of and outer radius c.





Example 2: The 1.5-in.-diameter shaft shown in Figure (a) is supported by two bearings and is subjected to three torques. Determine the shear stress developed at points A and B, located at section a–a of the shaft, Figure (b).

Example 3: The pipe shown in Figure has an inner radius of 40 mm and an outer radius of 50 mm. If its end is tightened against the support at A using the torque wrench, determine the shear stress developed in the material at the inner and outer walls along the central portion of the pipe.