TORSIONAL AND FLEXURAL- TORSIONAL BUCKLING OF COLUMNS

INTROUDACTION

In the lectures presented heretofore, the assumption has been made that the material obeys Hooke's law. For this assumption to be valid, the stresses in the column must be below the proportional limit of the material. The linear elastic analysis is correct for slender columns. On the other hand, the axial stress in a shot column will exceed the proportional limit. Consequently, the elastic analysis is not valid for short columns, and the limiting load for short columns must be determined by taking inelastic behavior into account.Elastic torsional and torsional-flexural buckling of axially loaded columns mostly take place in thin-walled columns. Thin-walled columns are divided into two parts: Open section, Close section. Shear Centre is defined as the point in the cross-section through which the lateral (or transverse) loads must pass to produce bending without twisting. It is also the center of rotation, when only pure torque is applied. The shear center and the centroid of the cross section will coincide, when section has two axes of symmetry. The shear center will be on the axis of symmetry, when the cross section has one axis of symmetry.

FUNDAMENTAL RELATIONSHIPS OF TORSIONAL BEHAVIOR OF THIN-WALLED OPEN SECTION COLUMNS.

Noncircular sections of the torsional column are no longer plane during twisting. In another word, the section has a displacement along the axial direction. The sorts of torsion are: Uniform torsion & Nonuniform torsion If the member is allowed to warp freely, then the applied torque is resisted solely by St.Venant shearing stresses. If the member is restrained from warping freely, the applied torque is resisted by a combination of St.Venant shearing stresses and warping torsion .This behavior is called non uniform torsion. Hence (as stated above), the effect of torsion can be further split into two parts:

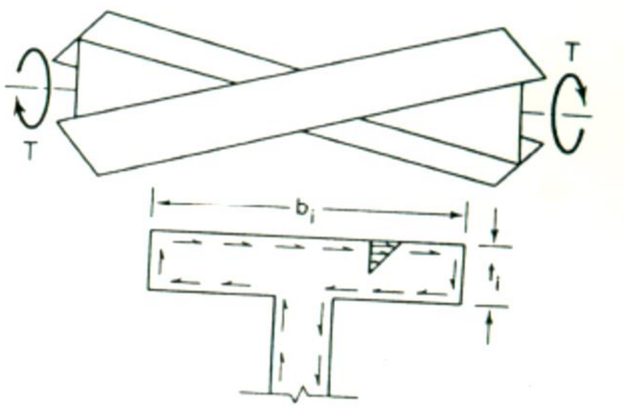
• Uniform or Pure Torsion (called St. Venant's torsion) - T_{sv}

• Non-Uniform Torsion, consisting of St.Venant's torsion (T_{sv}) and warping torsion (T_w).

UNIFORM TORSION OF THIN-WALLED BARS

If a couple of torques with opposite direction are applied to the both ends of the thin-walled open section (I section) column, uniform torsion will occur, as shown in Fig.1.Uniform torsion has two characteristics:

a. the same magnitude of twisting in every section. Thus, the longitudinal fibers do not have axial strain, and there is no normal stress but only shear stress caused by torsion in the section. The distribution of the shear stresses relates to the shape of the section and it is the same in each section.
b. the longitudinal fibers do not bend, the longitudinal fibers of the flanges and the web are still in line, there is just an angle (torsional angle) caused by torsion between the upper and lower flanges.



Torsional Constant (J) for members made up of rectangular plates may be computed approximately from

$$J = \frac{1}{3} \sum_i b_i (t_i)^3$$

in which b_i and t_i are length and thickness respectively of any element of the section.

In many cases, only uniform (or St. Venant's) torsion is applied to the section and the rate of change of angle of twist is constant along the member and the ends are free to warp as shown in Figure above.

In this case the applied torque is resisted entirely by shear stresses and no warping stresses result. The total angle of twist φ is given by

$$\phi = \frac{T\ z}{GJ}$$

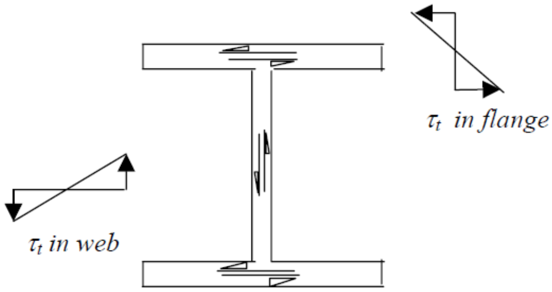
where T = Applied Torsion = T_{sv}

(Note: in this case only St.Venant's Torsion is applied)

The maximum shear stress in the element of thickness t is given by

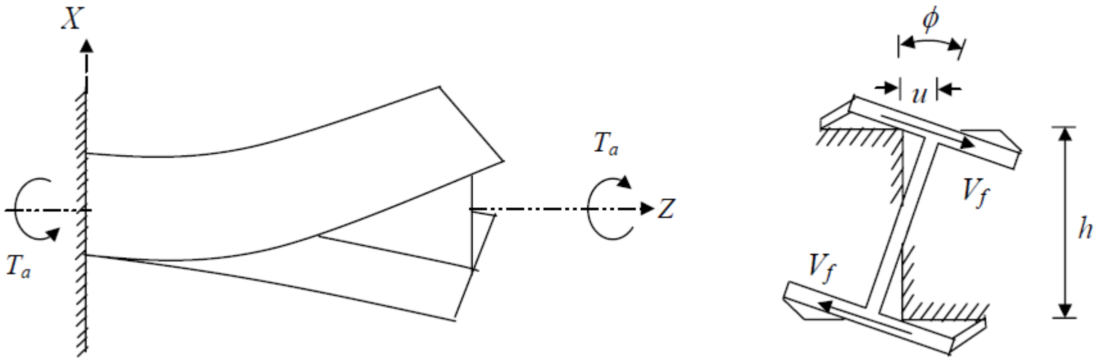
$$\tau_t = G t \phi'$$

Figure below gives the corresponding stress pattern for an I section.



NON-UNIFORM TORSION OF THIN-WALLED BARS

In previous section Pure (uniform) Torsion was assumed that when a torque T_{sv} twists the member, all cross sections were completely free to warp. When warping deformation is constrained, the member undergoes non-uniform torsion. Non-uniform torsion is illustrated in Figure below. Where an I-section fixed at one end is subjected to torsion at the other end. Here the member is restrained from warping freely as one end is fixed. The warping restraint causes bending deformation of the flanges in their plane in addition to twisting. The bending deformation is accompanied by a shear force in each flange.



The total non-uniform torsion (T_n) is given by

$$T_n = T_{sv} + T_w$$

where T_w is the warping torsion.

Shear force V_f in each flange is given by

$$V_f = -\frac{dM_f}{dz}$$

where M_f is the bending moment in each flange. Since, the flanges bend in opposite directions, the shear forces in the two flanges are oppositely directed and form a couple. This couple, which acts to resist the applied torque, is called ***warping torsion.*** For the *I*-section warping torsion is given by

$$T_w = V_f . h$$

The bending moment in the upper flange is given by $M_f = EI_f \frac{d^2u}{dz^2}$

in which I_f is the moment of inertia of flange about its strong axis (i.e. the vertical axis) and *u*, the lateral displacement of the flange centerline which is given by

$$u = \phi h / 2 \qquad M_f = \frac{EI_f h}{2}\frac{d^2\phi}{dz^2} = \frac{E I_f \ h}{2} \ \phi''$$

The term $I_f h^2 /2$ is called the warping constant (Γ) for the cross-section.

$$T_w = -E\Gamma \frac{d^3\phi}{dz^3} = -E\Gamma \ \phi'''$$

$$\Gamma = \frac{I_f \cdot h^2}{2} \qquad \text{(for an I-section)}$$

$E\Gamma$ is termed as the warping rigidity of the section, analogous to GJ, the St. Venant's torsional stiffness. The torque will be resisted by a combination of St.Venant's shearing stresses and warping torsion. Non-uniform torsional resistance (T_n) at any cross-section is therefore given by the sum of St.Venant's torsion (T_{sv}) and warping torsion (T_w). Thus, the differential equation for non-uniform torsional resistance $T_n(z)$ can be written as the algebraic sum of the two effects, due to St.Venant's Torsion and Warping Torsion.

$$T_n(z) = GJ\frac{d\phi}{dz} - E\Gamma\frac{d^3\phi}{dz^3} = GJ\phi' - E\Gamma \ \phi'''$$

$$or, T_n(z) = GJ\phi' - EI_f . \frac{h^2}{2} . \phi''' \qquad \text{(for an } I\text{-section)}$$

In the above, the first term on the right hand side (depending on *GJ*) represents the resistance of the section to twist and the second term represents the resistance to warping and is dependent on *E*Γ.
In the example considered (Figure above), the applied torque T_a is constant along the length, λ, of the beam . For equilibrium, the applied torque, T_a, should be equal to torsional resistance T_n. The boundary conditions are: (i) the slope of the beam is zero when *z = 0* and (ii) the BM is zero when z = λ i.e. at the free end.

$$\frac{d\phi}{dz} = 0 \qquad when \qquad z = 0$$

$$\frac{d^2\phi}{dz^2} = 0 \qquad when \qquad z = \lambda$$

The solution of T_n equation is

$$\frac{d\phi}{dz} = \frac{T_n}{GJ}\left(1 - \frac{Cos\,h\dfrac{\lambda - z}{a}}{Cos\,h\dfrac{\lambda}{a}}\right)$$

$$\text{in which} \qquad a^2 = \frac{E\Gamma}{GJ}$$

Since the flexural rigidity EI_f and torsional rigidity *GJ* are both measured in the same units (N.mm^2), *a* has the dimensions of length and *depends on the proportions of the beam*. Because of the presence of the second term in the equation the angle of twist per unit length varies along the length of the beam even though the *dz.* Applied torsion, T_a ,remains constant. When is known, the St. Venant's torsion *d*φ (T_{sv}) and the warping torsion (T_w) may be calculated or any cross section. At the *dz* built-in section (*z = 0*) and T_{sv} *= 0*. At this *d*φ point, the entire torque is balanced by the moment of the shearing forces in each of the flanges.

$$V_f = -T_n/h$$

At the end z = λ,

$$\frac{d\phi}{dz} = \frac{T_n}{GJ}\left(1 - \frac{1}{Cos\,h\left(\lambda/a\right)}\right)$$

If the length of the beam is large in comparison with the cross sectional dimensions $\left(1 - \dfrac{1}{Cosh\left(\frac{\lambda}{a}\right)}\right)$ tends to approach 1, as the second term is negligible. Hence $\dfrac{d\phi}{dz}$ approaches T_n / GJ .

The bending moment in the flange is found from

$$V_f = \frac{dM_f}{dz} = EI_f \cdot \frac{d^3\phi}{dz^3} \cdot \frac{h}{2}$$

where M_f is the bending moment in each flange.

$$M_f = EI_f \cdot \frac{h}{2} \cdot \frac{d^2\phi}{dz^2}$$

Substituting for $\dfrac{d\phi}{dz}$ from eq. (14) we obtain

$$M_f = \frac{a}{h} . T_n . \frac{Sin\,h\left(\frac{\lambda - z}{a}\right)}{Cos\,h\,\lambda / a}$$

The maximum bending moment at the fixed end is given by

$$M_{f\,max} = \frac{a}{h} \cdot T_n \cdot tanh\left(\frac{\lambda}{a}\right)$$

When λ is several times larger than a, $tan\,h\,(\lambda / a)$ approaches 1, so that

$$M_{f\,max} \cong \frac{a.T_n}{h}$$

In other words, the maximum bending moment in each of the flanges will be the same as

that of cantilever of length a, and loaded at the free end by a force of $\left(\dfrac{T_n}{h}\right)$. For a short beam λ is small in comparison with a, so $tanh\left(\dfrac{\lambda}{a}\right) \cong \left(\dfrac{\lambda}{a}\right)$

$$\text{Hence} \quad M_{f\,max} = \frac{T_n . \lambda}{h}$$

To calculate the angle of twist, φ , we integrate the right hand side of equation $d\varphi/dz$ equation:

$$\phi = \frac{T_n}{GJ}\left[z + \frac{a \sin h\left(\frac{\lambda - z}{a}\right)}{Cos\, h\left(\frac{\lambda}{a}\right)} - a \tan h\left(\frac{\lambda}{a}\right)\right]$$

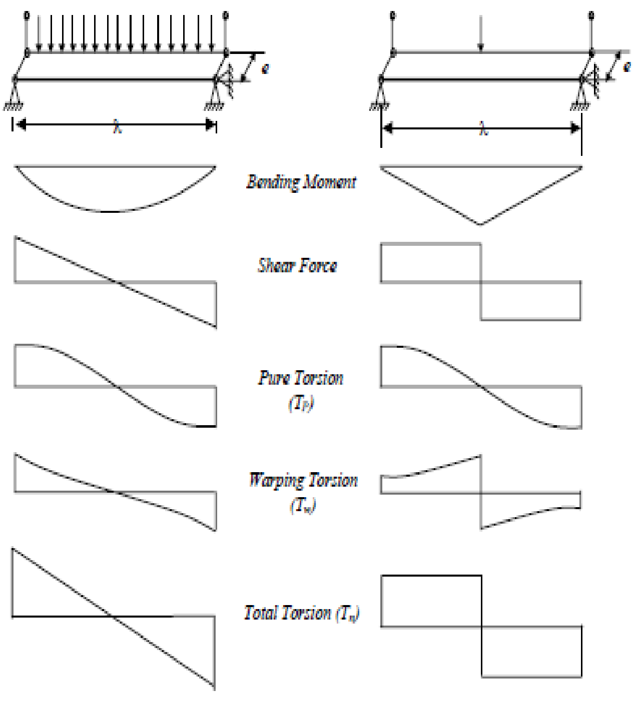
We can obtain the value of φ at the end (i.e.) when z = λ

$$(\phi)_{z=\lambda} = \frac{T_n}{GJ}\left(\lambda - a \tan h \frac{\lambda}{a}\right)$$

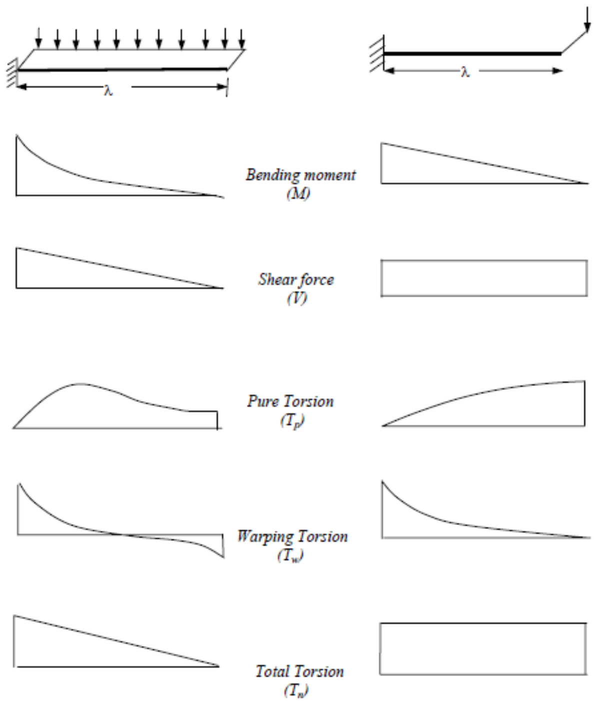
For long beams $\tan h\left(\frac{\lambda}{a}\right) \cong 1$

$$(\phi)_{z=\lambda} = \frac{T_n}{GJ}(\lambda - a)$$

The effect of the warping restraint on the angle of twist is equivalent to diminishing the length λ of the beam to *(λ - a)*. Certain simple cases of the effect of Torsion in simply supported beams and cantilever are illustrated in Figures below .



Torsion in simply supported beam with free end warping



Torsion in Cantilevers