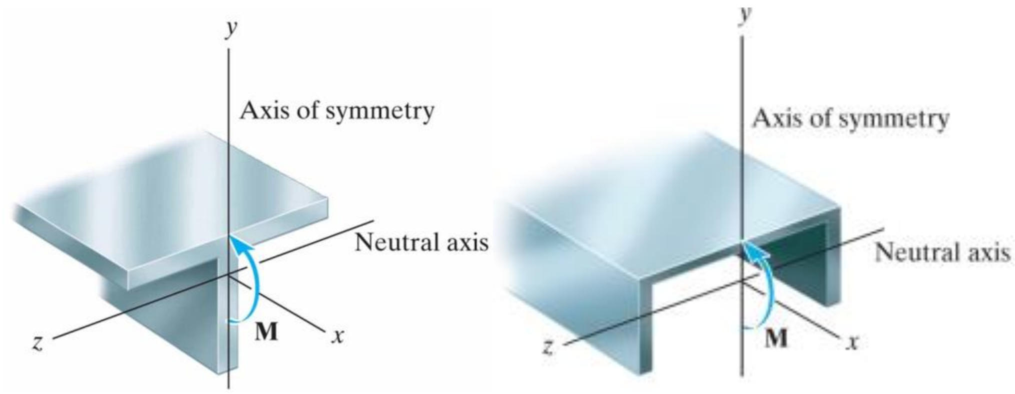
CHAPTER 6 BENDING

5. UNSYMMETRIC BENDING

The derivation of the flexure formula considered the case in which the cross–section area was symmetric about an axis perpendicular to the neutral axis; i.e. $I_{xy} = 0$, furthermore, the resultant internal moment M acts perpendicular to the axis of symmetry.



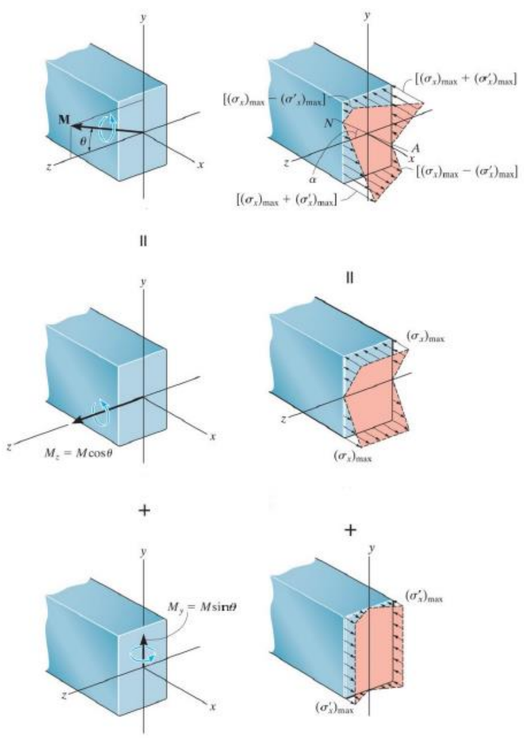
As it will be shown next, the relation $\sigma = -M\,y\,/\,I$ can be applied to any cross section with a bending moment as long as the moment is around a principal axis and I is a principal moment of inertia, i.e. $I_{xy} = 0$.

These conditions are actually unnecessary. The flexure formula can also be applied to a beam in the following two cases:

Case (I) The cross section is symmetric, so it is known where the principal axes are, but the moment vector M is not acting perpendicular to the axis of symmetry.

Case (II) The cross section area does not have an axis of symmetry with a moment applied.

• **Case (I) Moment Arbitrarily Applied**



Consider a rectangular cross section with a moment vector at an angle θ from the z axis in the plane of the cross section.

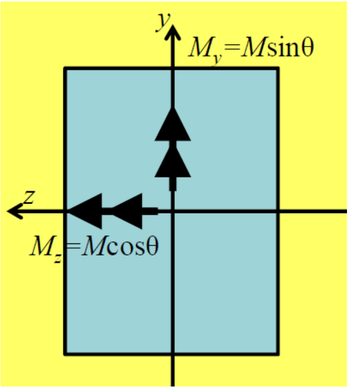
By expressing the moment vector in terms of its components about the y and z axes, the flexure equation can be applied to each moment and use superposition to add the stresses together.

Notice that in the equation $\sigma = M\,y\,/\,I$ the moment must be about an axis of symmetry, the moment of inertia must be with respect to the same axis as the moment, and y represents a distance measured perpendicular to the axis the moment is around.

Consider the stress due to M_z. The equation used will be $\sigma = M_z y / I_z$

There must be a sign placed on the stress based on whether the stress is tensile or compressive.

A general equation can be written by allowing y to be positive or negative based on the coordinate of the point where the stress needs to be calculated. At a point where y is positive, M_z causes compression based on the direction of the moment. At a point where z is negative, M_y causes compression, thus the resultant normal stress at any point on the cross section is:



$$\sigma = -\frac{M_z y}{I_z} + \frac{M_y z}{I_y}$$

This equation is good for any point in the cross section. The normal stress can be determined by substituting the appropriate coordinate values of the point of interest.

Since the stress at the neutral axis is zero, then this equation allows the determination of the neutral axis. By setting the general stress equation equal to zero:

$$y = \frac{M_y I_z}{M_z I_y} z$$

Given that $M_z = M\cos\theta$ and $M_y = M\sin\theta$:

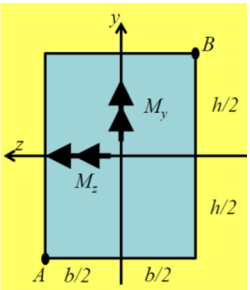
$$y = \left(\frac{I_z}{I_y}\tan\theta\right) z$$

This equation defines the neutral axis for the cross section. The slope of this line is $\tan\alpha = y/z$:

$$\tan\alpha = \frac{I_z}{I_y}\tan\theta$$

Through the general stress equation it is possible to determine where the maximum stress occurs by finding the point farthest away from the neutral axis.

Thus, σ_{max} occurs at points A and B where the stress at A is tensile and the stress at B is compressive.

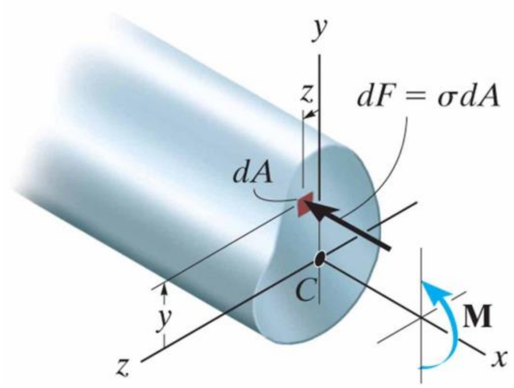


- **Case (II) Beam's Cross Section Area without Axis of Symmetry**

Consider the beam's cross section area without symmetry axis shown. The beam is subjected to a moment ***M*** as shown.

The objective is to determine the stress at an arbitrary point $P(y, z)$ of the cross–section area.

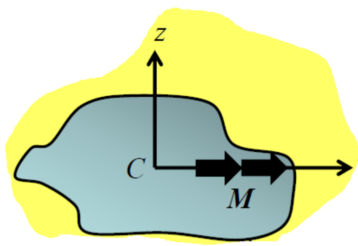
As it will be seen, the stress at P can be found through the use of the stress equation derived in the previous section. It only needs to be adapted to the unsymmetric area.



Consider, a cross section area as shown with a moment acting around the *y* axis.

The first step is to determine the centroid of the area and fix the origin of the coordinate system *y*–*z* to it.

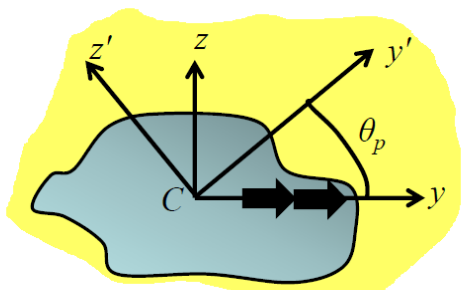
Then calculate the moments of inertia *Ix*, *Iy*, *Ixy*. In this case, since the area is not symmetric the product of inertia, *Ixy*, is not zero.



The principal axes and principal moments of inertia can be determined as

$$tan2\theta_p = \frac{-I_{yz}}{\frac{I_y - I_z}{2}}$$

$$I_{max,min} = \frac{I_y + I_z}{2} \pm \sqrt{\left(\frac{I_y + I_z}{2}\right)^2 + I_{yz}}$$



The stress at an arbitrary point can be found since the principal axes are now located in terms of the angle θ_P and $I_{y'}$ and $I_{z'}$ (I_{max}, I_{min}) are known, the moment ***M*** can be written in terms of its *y*' and *z*' components.

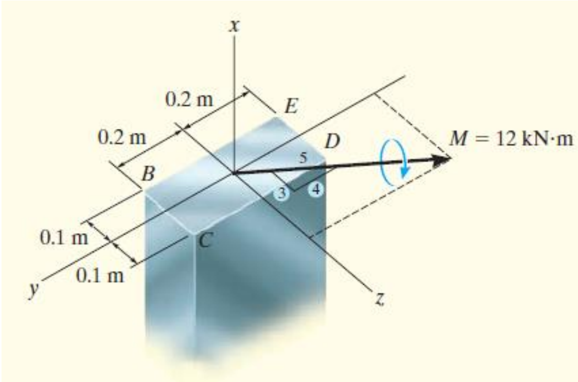
$$\sigma = -\frac{M_{z'}y'}{I_{z'}} + \frac{M_{y'}z'}{I_{y'}}$$

Where: $y' = ycos\theta + zsin\theta \qquad z' = -ysin\theta + zcos\theta$

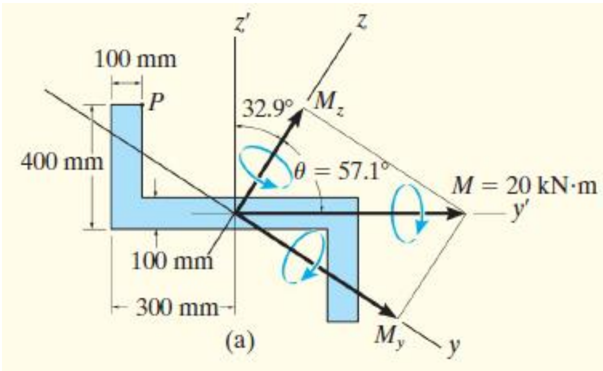
The point where the maximum stress occurs may not be obvious.

The easiest way to find the maximum stress for a general cross section is to locate the neutral axis as before, then determine which point in the cross section is furthest away from that axis.

Example 19: The rectangular cross section shown in Figure is subjected to a bending moment of M = 12 kN . m. Determine the normal stress developed at each corner of the section, and specify the orientation of the neutral axis.



Example 20: The Z-section shown in Figure is subjected to the bending moment of M = 20 kN . m. The principal axes y and z are oriented as shown, such that they represent the minimum and maximum principal moments of inertia, $I_y = 0.960(10^{-3})$ m^4 and $I_z = 7.54(10^{-3})$ m^4, respectively. Determine the normal stress at point P and the orientation of the neutral axis.



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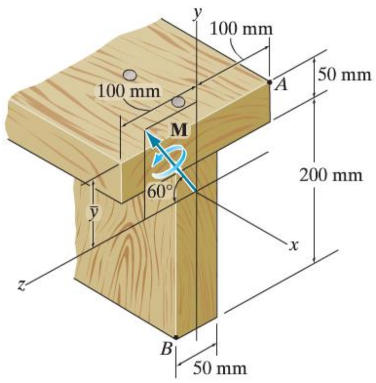
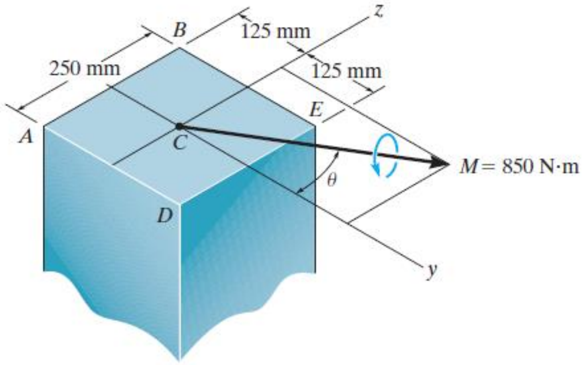
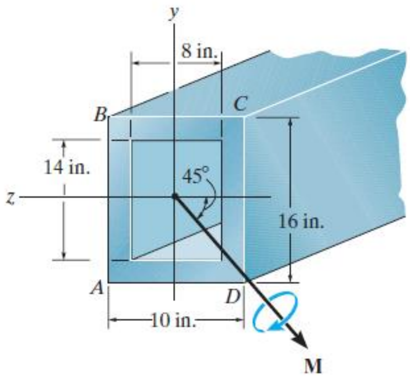


Q 1: If the wood used for the T-beam has an allowable tensile and compressive stress of $(\sigma_{allow})_t$ = 4 MPa and $(\sigma_{allow})_c$ = 6 MPa, respectively, determine the maximum allowable internal moment M that can be applied to the beam.

Q 2: The member has a square cross section and is subjected to a resultant internal bending moment of M = 850 N.m as shown. Determine the stress at each corner and sketch the stress distribution produced by **M**. Set θ = 45°.



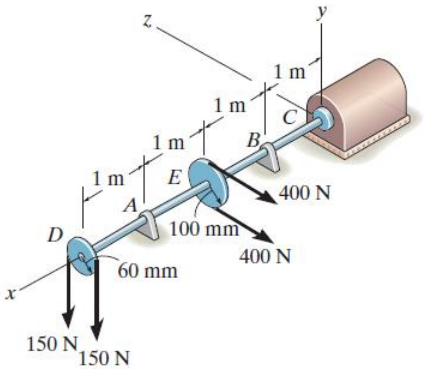


Q 3: The beam is subjected to a bending moment of M = 20 kip . ft directed as shown.

(1) Determine the maximum bending stress in the beam and the orientation of the neutral axis.

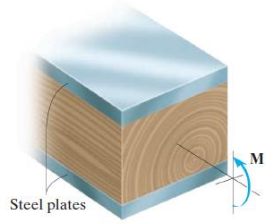
(2) Determine the maximum magnitude of the bending moment M that can be applied to the beam so that the bending stress in the member does not exceed 12 ksi.

Q 4: The 30-mm-diameter shaft is subjected to the vertical and horizontal loadings of two pulleys as shown. It is supported on two journal bearings at A and B which offer no resistance to axial loading. Furthermore, the coupling to the motor at C can be assumed not to offer any support to the shaft. Determine the maximum bending stress developed in the shaft.



6. COMPOSITE BEAMS

Beams constructed of two or more different materials are referred to as ***composite beams.*** An example is a beam made of wood with straps of steel at its top and bottom, Figure.



Since the flexure formula was developed only for beams made of homogeneous material, this formula cannot be applied to directly determine the normal stress in a composite beam. In this section, however, we will develop a method for modifying or "transforming" a composite beam's cross section into one made of a single material. Once this has been done, the flexure formula can then be used to determine the bending stress in the beam.

To explain how to do this we will consider a composite beam made of two materials, 1 and 2, bonded together as shown in Fig. *a*. If a bending moment is applied to this beam, then, like one that is homogeneous, the total cross-sectional area will *remain plane* after bending, and hence the normal strains will vary linearly from zero at the neutral axis to a maximum farthest from this axis, Fig. *b*. Provided the material is linear elastic, then at any point the normal stress in material 1 is determined from $\sigma = E_1\varepsilon$, and for material 2 the stress is found from $\sigma = E_2\varepsilon$. Assuming material 1 is stiffer than material 2, then $E_1 > E_2$ and so the stress distribution will look like that shown in Fig. *c* or *d*. In particular, notice the jump in stress that occurs at the juncture of the two materials. Here the *strain* is the *same,* but since the modulus of elasticity for the materials suddenly changes, so does the stress.

Rather than using this complicated stress distribution, it is simpler to transform the beam into one made of a single material. For example, if the beam is thought to consist entirely of the less stiff material 2, then the cross section will look like that shown in Fig. *e*. Here the height *h* of the beam remains the *same,* since the strain distribution in Fig. *b* must be the same. However, the upper portion of the beam must be widened in order to carry a load *equivalent* to that carried by the stiffer material 1 in Fig. *d*. This necessary width can be determined by considering the force *d***F** acting on an area $dA = dz\, dy$ of the beam in Fig. *a*. It is $dF = \sigma\, dA = E_1\varepsilon\,(dz\, dy)$. Assuming the width of a *corresponding element* of height *dy* in Fig. *e* is *n dz,* then $dF' = \sigma\_dA' = E_2\varepsilon\,(n\, dz\, dy)$. Equating these forces, so that they produce the same moment about the *z* (neutral) axis, we have

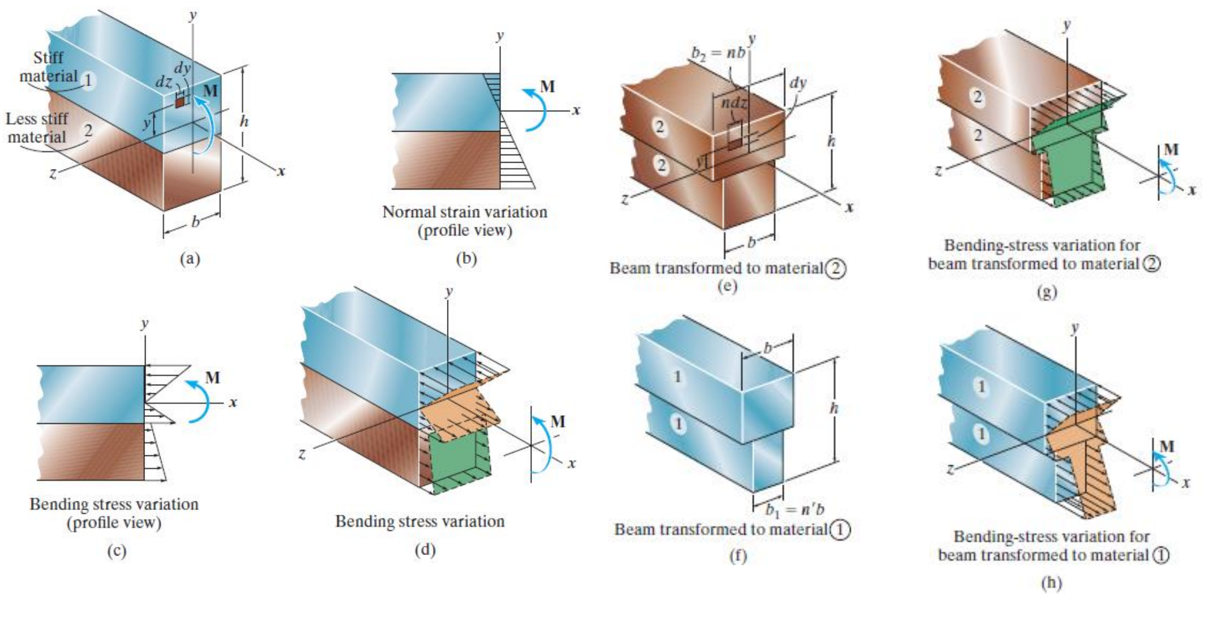
$$E_1\epsilon\,(dz\,dy) = E_2\epsilon\,(n\,dz\,dy) \quad \text{Or} \quad n = \frac{E_1}{E_2}$$

This dimensionless number n is called the ***transformation factor.*** It indicates that the cross section, having a width b on the original beam, Fig. *a*, must be increased in width to $b_2 = nb$ in the region where material 1 is being transformed into material 2, Fig. *e*.

In a similar manner, if the less stiff material 2 is transformed into the stiffer material 1, the cross section will look like that shown in Fig. *f*. Here the width of material 2 has been changed to $b_1 = n'b$, where $n' = E_2 / E_1$. In this case the transformation factor n' will be *less than one* since $E_1 > E_2$. In other words, we need less of the stiffer material to support the moment.

Once the beam has been transformed into one having a single material, the normal-stress distribution over the transformed cross section will be linear as shown in Fig. *g* or Fig. *h*. Consequently, the flexure formula can now be applied in the usual manner to determine the stress at each point on the transformed beam. Of course, the stress in the transformed beam will be equivalent to the stress in the same material of the actual beam; however, the stress in the transformed material has to be multiplied by the transformation factor n (or n') to obtain the stress in any other actual material that was transformed. This is because the area of the transformed material, $dA' = n\,dz\,dy$, is n times the area of actual material $dA = dz\,dy$. That is,

$$dF = \sigma\,dA = \sigma' dA'$$
$$\sigma\,dz\,dy = \sigma' n\,dz\,dy$$
$$\sigma = n\sigma'$$



(a)

Normal strain variation (profile view)
(b)

Beam transformed to material ②
(e)

Bending-stress variation for beam transformed to material ②
(g)

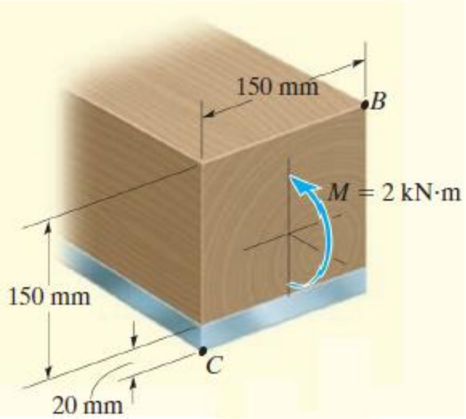
Bending stress variation (profile view)
(c)

Bending stress variation
(d)

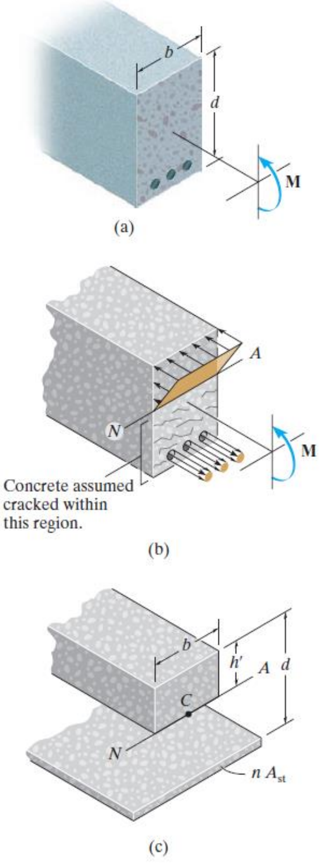
Beam transformed to material ①
(f)

Bending-stress variation for beam transformed to material ①
(h)

Example 21: The composite beam in Figure is made of wood and reinforced with a steel strap located on its bottom side. If the beam is subjected to a bending moment of M = 2 kN.m, determine the normal stress at points B and C. Take E_w = 12 GPa and E_{st} = 200 GPa.



7. REINFORCED CONCRETE BEAMS



All beams subjected to pure bending must resist both tensile and compressive stresses. Concrete, however, is very susceptible to cracking when it is in tension, and therefore by itself it will not be suitable for resisting a bending moment. In order to circumvent this shortcoming, engineers place steel reinforcing rods within a concrete beam at a location where the concrete is in tension, Fig. *a*. To be most effective, these rods are located farthest from the beam's neutral axis, so that the moment created by the forces developed in them is greatest about the neutral axis. Furthermore, the rods are required to have some concrete coverage to protect them from corrosion or loss of strength in the event of a fire. Codes used for actual reinforced concrete design assume the concrete will not be able to support any tensile loading, since the possible cracking of concrete is unpredictable. As a result, the normal-stress distribution acting on the cross-sectional area of a reinforced concrete beam is assumed to look like that shown in Fig. *b*.

The stress analysis requires locating the neutral axis and determining the maximum stress in the steel and concrete. To do this, the area of steel A_{st} is first transformed into an equivalent area of concrete using the transformation factor $n = E_{st} / E_{conc}$. This ratio, which gives $n > 1$, requires a "greater" amount of concrete to replace the steel. The transformed area is nA_{st} and

$$\Sigma \tilde{y} A \text{, must be zero, since } \bar{y} = \Sigma \tilde{y} A / \Sigma A = 0. \text{ Thus,}$$

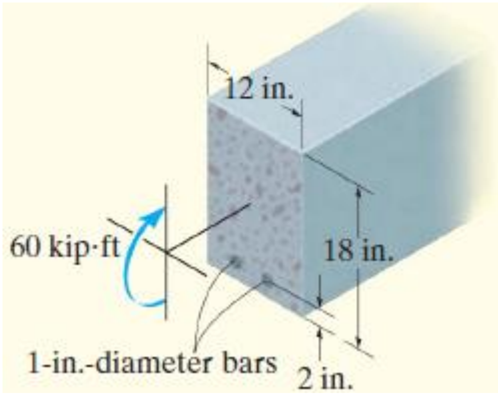
the transformed section looks like that shown in Fig. *c*. Here *d* represents the distance from the top of the beam to the thin strip of (transformed) steel, *b* is the beam's width, and *h*' is the yet unknown distance from the top of the beam to the neutral axis. To obtain *h*', we require the neutral axis to pass through the centroid *C* of the cross-sectional area of the transformed section, Fig. *c*. With reference to the neutral axis, therefore, the moment of the two areas together,

$$bh'\left(\frac{h'}{2}\right) - nA_{st}(d - h') = 0$$

$$\frac{b}{2}h'^2 + nA_{st}h' - nA_{st}d = 0$$

Once h' is obtained from this quadratic equation, the solution proceeds in the usual manner for obtaining the stress in the beam.

Example 22: The reinforced concrete beam has the cross-sectional area shown in Figure. If it is subjected to a bending moment of M = 60 kip . ft, determine the normal stress in each of the steel reinforcing rods and the maximum normal stress in the concrete. Take $E_{st} = 29(10^3)$ ksi and $E_{conc} = 3.6(10^3)$ ksi.



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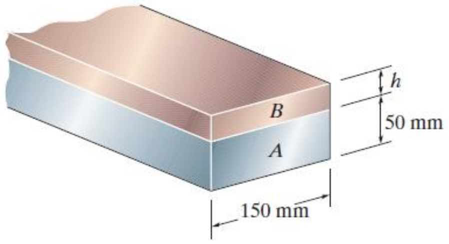
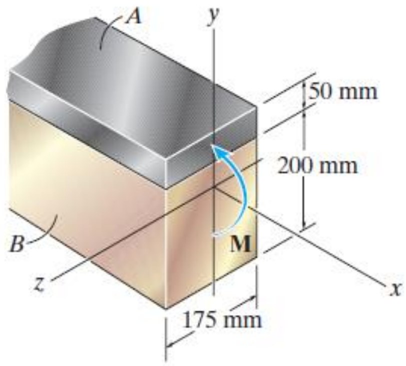
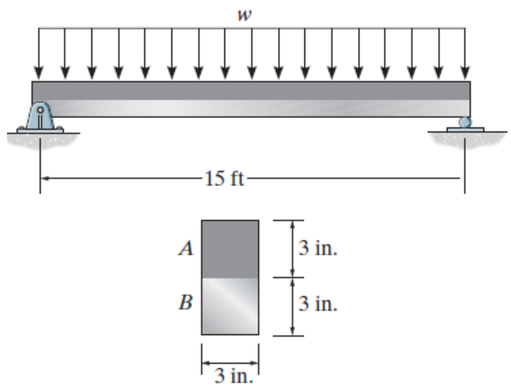
Q 1: The composite beam is made of 6061-T6 aluminum (A) and C83400 red brass (B). Determine the dimension h of the brass strip so that the neutral axis of the beam is located at the seam of the two metals. What maximum moment will this beam support if the allowable bending stress for the aluminum is $(\sigma_{allow})_{al}$ = 128 MPa and for the brass $(\sigma_{allow})_{br}$ = 35 MPa?

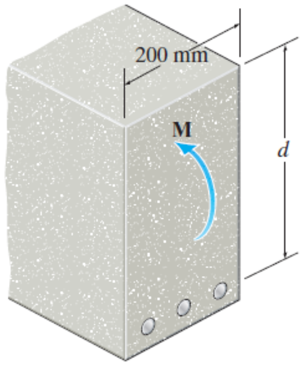




Q 2: The composite beam is made of steel (A) bonded to brass (B) and has the cross section shown. If it is subjected to a moment of M = 6.5 kN m, determine the maximum bending stress in the brass and steel. Also, what is the stress in each material at the seam where they are bonded together? E_{br} = 100 GPa. E_{st} = 200 GPa.

Q 3: Segment A of the composite beam is made from 2014-T6 aluminum alloy and segment B is A-36 steel. The allowable bending stress for the aluminum and steel are $(\sigma_{allow})_{al}$ = 15 ksi and $(\sigma_{allow})_{st}$ = 22 ksi. Determine the maximum allowable intensity w of the uniform distributed load.





Q 4: The concrete beam is reinforced with three 20-mm diameter steel rods. Assume that the concrete cannot support tensile stress. If the allowable compressive stress for concrete is $(\sigma_{allow})_{con}$ = 12.5 MPa and the allowable tensile stress for steel is $(\sigma_{allow})_{st}$ = 220 MPa, determine the required dimension d so that both the concrete and steel achieve their allowable stress simultaneously. This condition is said to be 'balanced'. Also, compute the corresponding maximum allowable internal moment **M** that can be applied to the beam. The moduli of elasticity for concrete and steel are E_{con} = 25 GPa and E_{st} = 200 GPa, respectively.