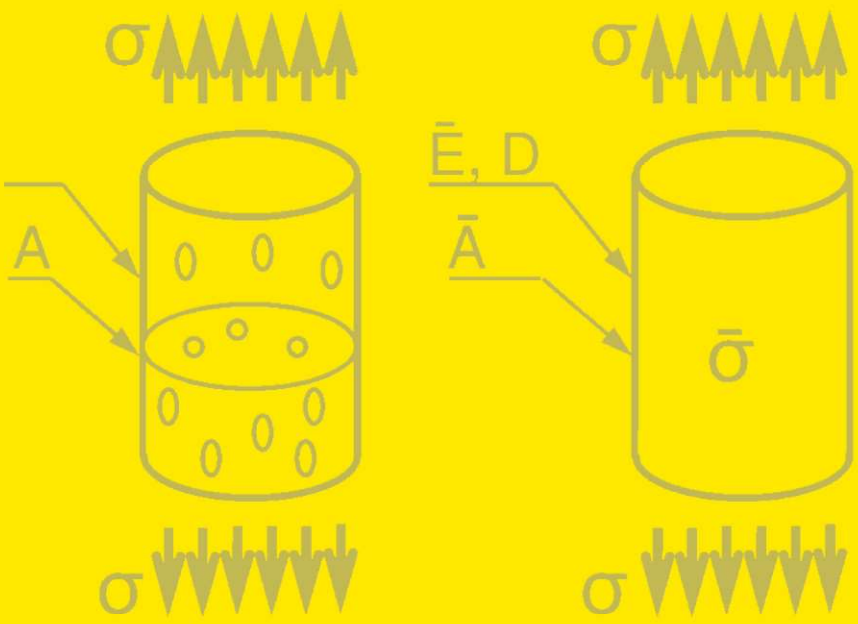
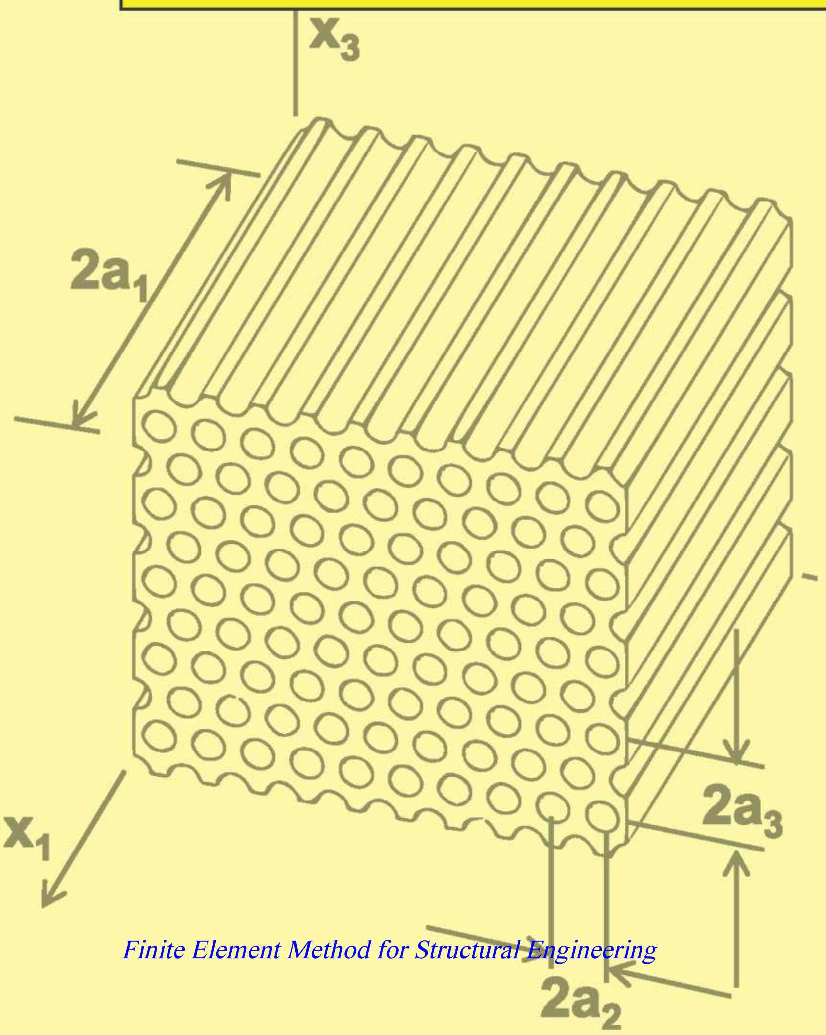


Finite Element Method

Lecture Notes

[redacted]

Type text here



[redacted]

[redacted]

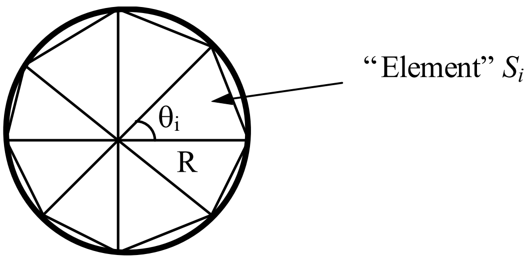
Chapter 1. Introduction

I. Basic Concepts

The *finite element method* (FEM), or *finite element analysis* (FEA), is based on the idea of building a complicated object with simple blocks, or, dividing a complicated object into small and manageable pieces. Application of this simple idea can be found everywhere in everyday life as well as in engineering.

Examples:

- Lego (kids' play)
- Buildings
- Approximation of the area of a circle:



Area of one triangle: $S_i = \frac{1}{2} R^2 \sin\theta_i$

Area of the circle: $S_N = \sum_{i=1}^{N} S_i = \frac{1}{2} R^2 N \sin\left(\frac{2\pi}{N}\right) \rightarrow \pi R^2 \;\; as\; N \rightarrow \infty$

where N = total number of triangles (elements).

Why Finite Element Method?

- *Design analysis*: hand calculations, experiments, and computer simulations
- FEM/FEA is the most widely applied computer simulation method in engineering
- Closely integrated with CAD/CAM applications
- ...

Applications of FEM in Engineering

- Mechanical/Aerospace/Civil/Automobile Engineering
- Structure analysis (static/dynamic, linear/nonlinear)
- Thermal/fluid flows
- Electromagnetics
- Geomechanics
- Biomechanics
- ...

Examples:

...

A Brief History of the FEM

- 1943 ----- Courant (Variational methods)
- 1956 ----- Turner, Clough, Martin and Topp (Stiffness)
- 1960 ----- Clough ("Finite Element", plane problems)

- 1970s ----- Applications on mainframe computers
- 1980s ----- Microcomputers, pre- and postprocessors
- 1990s ----- Analysis of large structural systems

FEM in Structural Analysis

Procedures:

- Divide structure into pieces (elements with nodes)
- Describe the behavior of the physical quantities on each element
- Connect (assemble) the elements at the nodes to form an approximate system of equations for the whole structure
- Solve the system of equations involving unknown quantities at the nodes (e.g., displacements)
- Calculate desired quantities (e.g., strains and stresses) at selected elements

Example:

Computer Implementations

- Preprocessing (build FE model, loads and constraints)
- FEA solver (assemble and solve the system of equations)
- Postprocessing (sort and display the results)

Available Commercial FEM Software Packages

- *ANSYS* (General purpose, PC and workstations)
- *SDRC/I-DEAS* (Complete CAD/CAM/CAE package)
- *NASTRAN* (General purpose FEA on mainframes)
- *ABAQUS* (Nonlinear and dynamic analyses)
- *COSMOS* (General purpose FEA)
- *ALGOR* (PC and workstations)
- *PATRAN* (Pre/Post Processor)
- *HyperMesh* (Pre/Post Processor)
- *Dyna-3D* (Crash/impact analysis)
- ...

Objectives of This FEM Course

- Understand the fundamental ideas of the FEM
- Know the behavior and usage of each type of elements covered in this course
- Be able to prepare a suitable FE model for given problems
- Can interpret and evaluate the quality of the results (know the physics of the problems)
- Be aware of the limitations of the FEM (don't misuse the FEM - a numerical tool)

II. Review of Matrix Algebra

Linear System of Algebraic Equations

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\
\ldots\ldots\ldots & \\
a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n &= b_n
\end{aligned}
\qquad (1)
$$

where $x_1, x_2, \ldots, x_n$ are the unknowns.

In *matrix form*:

$$\mathbf{A}\mathbf{x} = \mathbf{b} \qquad (2)$$

where

$$
\mathbf{A} = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix}
$$

$$
\mathbf{x} = \{x_i\} = \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix} \qquad \mathbf{b} = \{b_i\} = \begin{Bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{Bmatrix} \qquad (3)
$$

A is called a $n \times n$ (square) matrix, and **x** and **b** are (column) vectors of dimension n.

Row and Column Vectors

$$\mathbf{v} = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \qquad \mathbf{w} = \begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix}$$

Matrix Addition and Subtraction

For two matrices **A** and **B**, both of the *same size* ($m \times n$), the addition and subtraction are defined by

$$\mathbf{C} = \mathbf{A} + \mathbf{B} \qquad \text{with} \quad c_{ij} = a_{ij} + b_{ij}$$

$$\mathbf{D} = \mathbf{A} - \mathbf{B} \qquad \text{with} \quad d_{ij} = a_{ij} - b_{ij}$$

Scalar Multiplication

$$\lambda \mathbf{A} = \begin{bmatrix} \lambda a_{ij} \end{bmatrix}$$

Matrix Multiplication

For two matrices **A** (of size $l \times m$) and **B** (of size $m \times n$), the product of **AB** is defined by

$$\mathbf{C} = \mathbf{AB} \qquad \text{with } c_{ij} = \sum_{k=1}^{m} a_{ik} b_{kj}$$

where $i = 1, 2, ..., l$; $j = 1, 2, ..., n$.

Note that, in general, $\mathbf{AB} \neq \mathbf{BA}$, but $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$ (associative).

Transpose of a Matrix

If **A** = [a_{ij}], then the transpose of **A** is

$$\mathbf{A}^T = \left[a_{ji}\right]$$

Notice that $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$.

Symmetric Matrix

A *square* ($n \times n$) matrix **A** is called symmetric, if

$$\mathbf{A} = \mathbf{A}^T \qquad \text{or} \qquad a_{ij} = a_{ji}$$

Unit (Identity) Matrix

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

Note that **AI** = **A**, **Ix** = **x**.

Determinant of a Matrix

The determinant of *square* matrix **A** is a scalar number denoted by det **A** or |**A**|. For 2×2 and 3×3 matrices, their determinants are given by

$$\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

and

$$\det\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{21}a_{32}a_{13}$$

$$- a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{23}a_{32}a_{11}$$

Singular Matrix

A *square* matrix **A** is *singular* if det **A** = 0, which indicates problems in the systems (nonunique solutions, degeneracy, etc.)

Matrix Inversion

For a *square* and *nonsingular* matrix **A** ($\det \mathbf{A} \neq 0$), its *inverse* $\mathbf{A}^{-1}$ is constructed in such a way that

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$

The *cofactor matrix* **C** of matrix **A** is defined by

$$C_{ij} = (-1)^{i+j} M_{ij}$$

where M_{ij} is the determinant of the smaller matrix obtained by eliminating the *i*th row and *j*th column of **A**.

Thus, the inverse of **A** can be determined by

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \mathbf{C}^T$$

We can show that $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.

Examples:

$$(1)\quad \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{(ad-bc)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Checking,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{(ad-bc)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$(2)\quad \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}^{-1} = \frac{1}{(4-2-1)}\begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}^{T} = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Checking,

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

If det $\mathbf{A} = 0$ (i.e., $\mathbf{A}$ is singular), then $\mathbf{A}^{-1}$ does not exist!

The solution of the linear system of equations (Eq.(1)) can be expressed as (assuming the coefficient matrix **A** is nonsingular)

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$$

Thus, the main task in solving a linear system of equations is to found the inverse of the coefficient matrix.

Solution Techniques for Linear Systems of Equations

- Gauss elimination methods
- Iterative methods

Positive Definite Matrix

A *square* ($n \times n$) matrix **A** is said to be *positive definite*, if for any nonzero vector **x** of dimension n,

$$\mathbf{x}^T \mathbf{A}\mathbf{x} > 0$$

Note that positive definite matrices are nonsingular.

Differentiation and Integration of a Matrix

Let

$$\mathbf{A}(t) = \left[a_{ij}(t)\right]$$

then the differentiation is defined by

$$\frac{d}{dt}\mathbf{A}(t) = \left[\frac{da_{ij}(t)}{dt}\right]$$

and the integration by

$$\int \mathbf{A}(t)dt = \left[\int a_{ij}(t)dt\right]$$