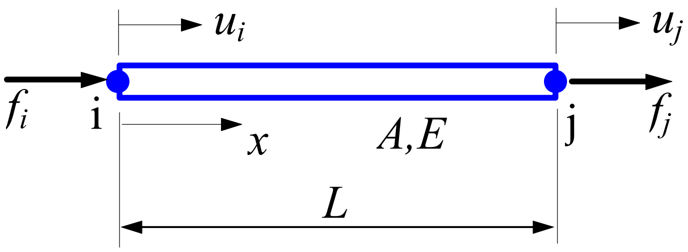
II. Bar Element

Consider a uniform prismatic bar:



L	length
A	cross-sectional area
E	elastic modulus
$u = u(x)$	displacement
$\varepsilon = \varepsilon(x)$	strain
$\sigma = \sigma(x)$	stress

Strain-displacement relation:

$$\varepsilon = \frac{du}{dx} \tag{1}$$

Stress-strain relation:

$$\sigma = E\varepsilon \tag{2}$$

Stiffness Matrix --- Direct Method

Assuming that the displacement u is *varying linearly* along the axis of the bar, i.e.,

$$u(x) = \left(1 - \frac{x}{L}\right)u_i + \frac{x}{L}u_j \tag{3}$$

we have

$$\varepsilon = \frac{u_j - u_i}{L} = \frac{\Delta}{L} \qquad (\Delta = \text{elongation}) \tag{4}$$

$$\sigma = E\varepsilon = \frac{E\Delta}{L} \tag{5}$$

We also have

$$\sigma = \frac{F}{A} \qquad (F = \text{force in bar}) \tag{6}$$

Thus, (5) and (6) lead to

$$F = \frac{EA}{L}\Delta = k\Delta \tag{7}$$

where $k = \dfrac{EA}{L}$ is the stiffness of the bar.

The bar is acting like a spring in this case and we conclude that element stiffness matrix is

$$\mathbf{k} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} = \begin{bmatrix} \frac{EA}{L} & -\frac{EA}{L} \\ -\frac{EA}{L} & \frac{EA}{L} \end{bmatrix}$$

or

$$\mathbf{k} = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{8}$$

This can be verified by considering the equilibrium of the forces at the two nodes.

Element equilibrium equation is

$$\frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_i \\ u_j \end{Bmatrix} = \begin{Bmatrix} f_i \\ f_j \end{Bmatrix} \tag{9}$$

Degree of Freedom (dof)

Number of components of the displacement vector at a node.

For 1-D bar element: one dof at each node.

Physical Meaning of the Coefficients in **k**

The jth column of **k** (here j = 1 or 2) represents the forces applied to the bar to maintain a deformed shape with unit displacement at node j and zero displacement at the other node.

Stiffness Matrix --- A Formal Approach

We derive the same stiffness matrix for the bar using a formal approach which can be applied to many other more complicated situations.

Define two *linear shape functions* as follows

$$N_i(\xi) = 1 - \xi, \qquad N_j(\xi) = \xi \tag{10}$$

where

$$\xi = \frac{x}{L}, \qquad 0 \le \xi \le 1 \tag{11}$$

From (3) we can write the displacement as

$$u(x) = u(\xi) = N_i(\xi)u_i + N_j(\xi)u_j$$

or

$$u = \begin{bmatrix} N_i & N_j \end{bmatrix} \begin{Bmatrix} u_i \\ u_j \end{Bmatrix} = \mathbf{Nu} \tag{12}$$

Strain is given by (1) and (12) as

$$\varepsilon = \frac{du}{dx} = \left[\frac{d}{dx}\mathbf{N} \right] \mathbf{u} = \mathbf{Bu} \tag{13}$$

where **B** is the element *strain-displacement matrix*, which is

$$\mathbf{B} = \frac{d}{dx}\begin{bmatrix} N_i(\xi) & N_j(\xi) \end{bmatrix} = \frac{d}{d\xi}\begin{bmatrix} N_i(\xi) & N_j(\xi) \end{bmatrix} \bullet \frac{d\xi}{dx}$$

i.e., $\qquad \mathbf{B} = \begin{bmatrix} -1/L & 1/L \end{bmatrix} \qquad (14)$

Stress can be written as

$$\sigma = E\varepsilon = E\mathbf{Bu} \tag{15}$$

Consider the *strain energy* stored in the bar

$$\begin{aligned} U &= \frac{1}{2}\int_V \sigma^{\mathrm{T}}\varepsilon dV = \frac{1}{2}\int_V \left(\mathbf{u}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}}E\mathbf{Bu}\right)dV \\ &= \frac{1}{2}\mathbf{u}^{\mathrm{T}}\left[\int_V \left(\mathbf{B}^{\mathrm{T}}E\mathbf{B}\right)dV\right]\mathbf{u} \end{aligned} \tag{16}$$

where (13) and (15) have been used.

The *work* done by the two nodal forces is

$$W = \frac{1}{2}f_i u_i + \frac{1}{2}f_j u_j = \frac{1}{2}\mathbf{u}^{\mathrm{T}}\mathbf{f} \tag{17}$$

For conservative system, we state that

$$U = W \tag{18}$$

which gives

$$\frac{1}{2}\mathbf{u}^{\mathrm{T}}\left[\int_V \left(\mathbf{B}^{\mathrm{T}}E\mathbf{B}\right)dV\right]\mathbf{u} = \frac{1}{2}\mathbf{u}^{\mathrm{T}}\mathbf{f}$$

We can conclude that

$$\left[\int_V \left(\mathbf{B}^{\mathrm{T}}E\mathbf{B}\right)dV\right]\mathbf{u} = \mathbf{f}$$

or

$$\mathbf{k}\mathbf{u} = \mathbf{f} \tag{19}$$

where

$$\mathbf{k} = \int_V \left(\mathbf{B}^{\mathrm{T}} E \mathbf{B}\right) dV \tag{20}$$

is the *element stiffness matrix*.

Expression (20) is a general result which can be used for the construction of other types of elements. This expression can also be derived using other more rigorous approaches, such as the *Principle of Minimum Potential Energy*, or the *Galerkin's Method*.

Now, we evaluate (20) for the bar element by using (14)

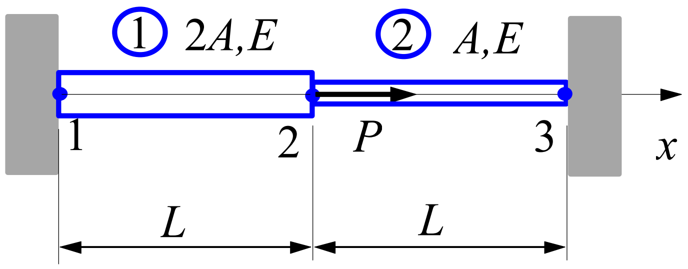
$$\mathbf{k} = \int_0^L \begin{Bmatrix} -1/L \\ 1/L \end{Bmatrix} E \begin{bmatrix} -1/L & 1/L \end{bmatrix} A dx = \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

which is the same as we derived using the direct method.

Note that from (16) and (20), the strain energy in the element can be written as

$$U = \frac{1}{2} \mathbf{u}^{\mathrm{T}} \mathbf{k} \mathbf{u} \tag{21}$$

Example 2.1



Problem: Find the stresses in the two bar assembly which is loaded with force *P*, and constrained at the two ends, as shown in the figure.

Solution: Use two 1-D bar elements.

Element 1,

$$\begin{array}{cc} & \begin{array}{cc} u_1 & u_2 \end{array} \\ \mathbf{k}_1 = \dfrac{2EA}{L} & \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array}$$

Element 2,

$$\begin{array}{cc} & \begin{array}{cc} u_2 & u_3 \end{array} \\ \mathbf{k}_2 = \dfrac{EA}{L} & \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array}$$

Imagine a frictionless pin at node 2, which connects the two elements. We can assemble the global FE equation as follows,

$$\frac{EA}{L}\begin{bmatrix} 2 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$$

Load and boundary conditions (BC) are,

$$u_1 = u_3 = 0, \qquad F_2 = P$$

FE equation becomes,

$$\frac{EA}{L}\begin{bmatrix} 2 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ u_2 \\ 0 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ P \\ F_3 \end{Bmatrix}$$

Deleting the 1st row and column, and the 3rd row and column, we obtain,

$$\frac{EA}{L}[3]\{u_2\} = \{P\}$$

Thus,

$$u_2 = \frac{PL}{3EA}$$

and

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \frac{PL}{3EA}\begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix}$$

Stress in element 1 is

$$\sigma_1 = E\varepsilon_1 = E\mathbf{B}_1\mathbf{u}_1 = E\begin{bmatrix} -1/L & 1/L \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$= E\frac{u_2 - u_1}{L} = \frac{E}{L}\left(\frac{PL}{3EA} - 0\right) = \frac{P}{3A}$$

Similarly, stress in element 2 is

$$\sigma_2 = E\varepsilon_2 = E\mathbf{B}_2\mathbf{u}_2 = E\begin{bmatrix}-1/L & 1/L\end{bmatrix}\begin{Bmatrix}u_2\\ u_3\end{Bmatrix}$$

$$= E\frac{u_3 - u_2}{L} = \frac{E}{L}\left(0 - \frac{PL}{3EA}\right) = -\frac{P}{3A}$$

which indicates that bar 2 is in compression.

Check the results!

Notes:

- In this case, the calculated stresses in elements 1 and 2 are exact within the linear theory for 1-D bar structures. It will not help if we further divide element 1 or 2 into smaller finite elements.
- For tapered bars, averaged values of the cross-sectional areas should be used for the elements.
- We need to find the displacements first in order to find the stresses, since we are using the *displacement based FEM*.