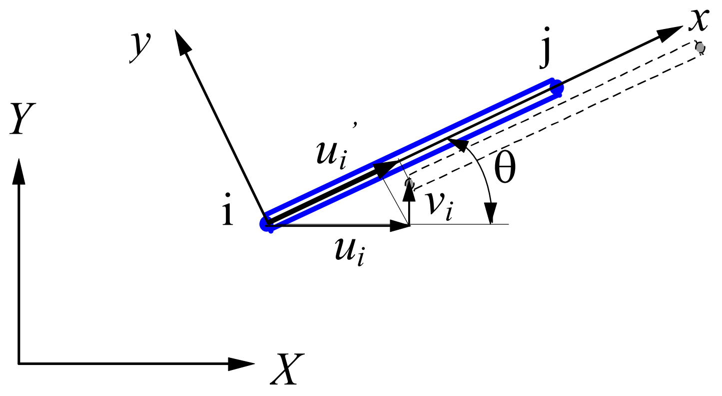
Bar Elements in 2-D and 3-D Space

2-D Case



Local	*Global*
x, y	*X, Y*
$u_i^{'}, v_i^{'}$	u_i, v_i
1 dof at node	2 dof's at node

Note: Lateral displacement $v_i^{'}$ does not contribute to the stretch of the bar, within the linear theory.

Transformation

$$u_i^{'} = u_i \cos\theta + v_i \sin\theta = \begin{bmatrix} l & m \end{bmatrix} \begin{Bmatrix} u_i \\ v_i \end{Bmatrix}$$

$$v_i^{'} = -u_i \sin\theta + v_i \cos\theta = \begin{bmatrix} -m & l \end{bmatrix} \begin{Bmatrix} u_i \\ v_i \end{Bmatrix}$$

where $l = \cos\theta,\ m = \sin\theta$.

In matrix form,

$$\begin{Bmatrix} u_i' \\ v_i' \end{Bmatrix} = \begin{bmatrix} l & m \\ -m & l \end{bmatrix} \begin{Bmatrix} u_i \\ v_i \end{Bmatrix} \tag{26}$$

or,

$$\mathbf{u}_i' = \tilde{\mathbf{T}} \mathbf{u}_i$$

where the *transformation matrix*

$$\tilde{\mathbf{T}} = \begin{bmatrix} l & m \\ -m & l \end{bmatrix} \tag{27}$$

is *orthogonal*, that is, $\tilde{\mathbf{T}}^{-1} = \tilde{\mathbf{T}}^T$.

For the two nodes of the bar element, we have

$$\begin{Bmatrix} u_i' \\ v_i' \\ u_j' \\ v_j' \end{Bmatrix} = \begin{bmatrix} l & m & 0 & 0 \\ -m & l & 0 & 0 \\ 0 & 0 & l & m \\ 0 & 0 & -m & l \end{bmatrix} \begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \end{Bmatrix} \tag{28}$$

or,

$$\mathbf{u}' = \mathbf{T}\mathbf{u} \qquad \text{with } \mathbf{T} = \begin{bmatrix} \tilde{\mathbf{T}} & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{T}} \end{bmatrix} \tag{29}$$

The nodal forces are transformed in the same way,

$$\mathbf{f}' = \mathbf{T}\mathbf{f} \tag{30}$$

Stiffness Matrix in the 2-D Space

In the local coordinate system, we have

$$\frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_i^{'} \\ u_j^{'} \end{Bmatrix} = \begin{Bmatrix} f_i^{'} \\ f_j^{'} \end{Bmatrix}$$

Augmenting this equation, we write

$$\frac{EA}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} u_i^{'} \\ v_i^{'} \\ u_j^{'} \\ v_j^{'} \end{Bmatrix} = \begin{Bmatrix} f_i^{'} \\ 0 \\ f_j^{'} \\ 0 \end{Bmatrix}$$

or,

$$\mathbf{k}^{'}\mathbf{u}^{'} = \mathbf{f}^{'}$$

Using transformations given in (29) and (30), we obtain

$$\mathbf{k}^{'}\mathbf{T}\mathbf{u} = \mathbf{T}\mathbf{f}$$

Multiplying both sides by $\mathbf{T}^T$ and noticing that $\mathbf{T}^T\mathbf{T} = \mathbf{I}$, we obtain

$$\mathbf{T}^T\mathbf{k}^{'}\mathbf{T}\mathbf{u} = \mathbf{f} \qquad (31)$$

Thus, the element stiffness matrix **k** in the global coordinate system is

$$\mathbf{k} = \mathbf{T}^T\mathbf{k}^{'}\mathbf{T} \qquad (32)$$

which is a 4×4 symmetric matrix.

Explicit form,

$$\mathbf{k} = \frac{EA}{L} \begin{array}{cccc} u_i & v_i & u_j & v_j \end{array}$$

$$\mathbf{k} = \frac{EA}{L}\begin{bmatrix} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -lm & -m^2 \\ -l^2 & -lm & l^2 & lm \\ -lm & -m^2 & lm & m^2 \end{bmatrix} \tag{33}$$

Calculation of the *directional cosines* l and m:

$$l = \cos\theta = \frac{X_j - X_i}{L}, \qquad m = \sin\theta = \frac{Y_j - Y_i}{L} \tag{34}$$

The structure stiffness matrix is assembled by using the element stiffness matrices in the usual way as in the 1-D case.

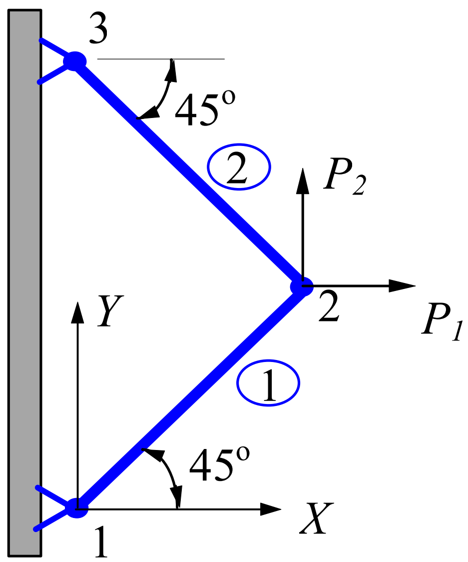
Element Stress

$$\sigma = E\varepsilon = E\mathbf{B}\begin{Bmatrix} u_i' \\ u_j' \end{Bmatrix} = E\begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix}\begin{bmatrix} l & m & 0 & 0 \\ 0 & 0 & l & m \end{bmatrix}\begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \end{Bmatrix}$$

That is,

$$\sigma = \frac{E}{L}\begin{bmatrix} -l & -m & l & m \end{bmatrix}\begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \end{Bmatrix} \tag{35}$$

Example 2.3



A simple plane truss is made of two identical bars (with *E, A,* and *L*), and loaded as shown in the figure. Find

1) displacement of node 2;

2) stress in each bar.

Solution:

This simple structure is used here to demonstrate the assembly and solution process using the bar element in 2-D space.

In local coordinate systems, we have

$$\mathbf{k}_1' = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \mathbf{k}_2'$$

These two matrices cannot be assembled together, because they are in different coordinate systems. We need to convert them to global coordinate system *OXY*.

Element 1:

$$\theta = 45^o, \quad l = m = \frac{\sqrt{2}}{2}$$

Using formula (32) or (33), we obtain the stiffness matrix in the global system

$$\begin{array}{c} \\ \mathbf{k}_1 = \mathbf{T}_1^T \mathbf{k}_1' \mathbf{T}_1 = \dfrac{EA}{2L} \end{array} \begin{array}{c} \begin{array}{cccc} u_1 & v_1 & u_2 & v_2 \end{array} \\ \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix} \end{array}$$

Element 2:

$$\theta = 135^o, \; l = -\frac{\sqrt{2}}{2}, \quad m = \frac{\sqrt{2}}{2}$$

We have,

$$\begin{array}{c} \\ \mathbf{k}_2 = \mathbf{T}_2^T \mathbf{k}_2' \mathbf{T}_2 = \dfrac{EA}{2L} \end{array} \begin{array}{c} \begin{array}{cccc} u_2 & v_2 & u_3 & v_3 \end{array} \\ \begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \end{array}$$

Assemble the structure FE equation,

$$\frac{EA}{2L} \begin{array}{c} \begin{array}{cccccc} u_1 & v_1 & u_2 & v_2 & u_3 & v_3 \end{array} \\ \begin{bmatrix} 1 & 1 & -1 & -1 & 0 & 0 \\ 1 & 1 & -1 & -1 & 0 & 0 \\ -1 & -1 & 2 & 0 & -1 & 1 \\ -1 & -1 & 0 & 2 & 1 & -1 \\ 0 & 0 & -1 & 1 & 1 & -1 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{bmatrix} \end{array} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} F_{1X} \\ F_{1Y} \\ F_{2X} \\ F_{2Y} \\ F_{3X} \\ F_{3Y} \end{Bmatrix}$$

Load and boundary conditions (BC):

$$u_1 = v_1 = u_3 = v_3 = 0, \qquad F_{2X} = P_1, \quad F_{2Y} = P_2$$

Condensed FE equation,

$$\frac{EA}{2L}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{Bmatrix} u_2 \\ v_2 \end{Bmatrix} = \begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix}$$

Solving this, we obtain the displacement of node 2,

$$\begin{Bmatrix} u_2 \\ v_2 \end{Bmatrix} = \frac{L}{EA}\begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix}$$

Using formula (35), we calculate the stresses in the two bars,

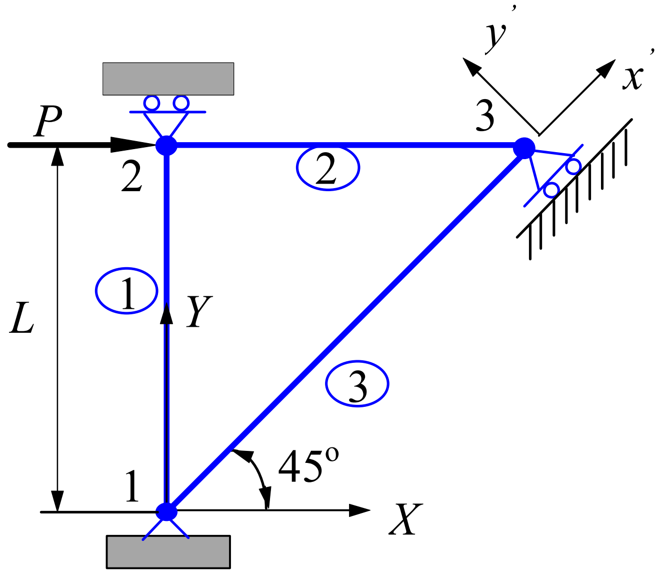
$$\sigma_1 = \frac{E}{L}\frac{\sqrt{2}}{2}\begin{bmatrix} -1 & -1 & 1 & 1 \end{bmatrix}\frac{L}{EA}\begin{Bmatrix} 0 \\ 0 \\ P_1 \\ P_2 \end{Bmatrix} = \frac{\sqrt{2}}{2A}\left(P_1 + P_2\right)$$

$$\sigma_2 = \frac{E}{L}\frac{\sqrt{2}}{2}\begin{bmatrix} 1 & -1 & -1 & 1 \end{bmatrix}\frac{L}{EA}\begin{Bmatrix} P_1 \\ P_2 \\ 0 \\ 0 \end{Bmatrix} = \frac{\sqrt{2}}{2A}\left(P_1 - P_2\right)$$

Check the results:

Look for the equilibrium conditions, symmetry, antisymmetry, etc.

Example 2.4 (*Multipoint Constraint*)



For the plane truss shown above,

$$P = 1000 \text{ kN}, \quad L = 1m, \quad E = 210\,GPa,$$

$$A = 6.0 \times 10^{-4} m^2 \quad \text{for elements 1 and 2,}$$

$$A = 6\sqrt{2} \times 10^{-4} m^2 \quad \text{for element 3.}$$

Determine the displacements and reaction forces.

Solution:

We have an inclined roller at node 3, which needs special attention in the FE solution. We first assemble the global FE equation for the truss.

Element 1:

$$\theta = 90^o, \quad l = 0, \quad m = 1$$

$$\begin{array}{c} \\ \mathbf{k}_1 = \dfrac{(210\times 10^9)(6.0\times 10^{-4})}{1} \end{array} \begin{array}{c} \begin{array}{cccc} u_1 & v_1 & u_2 & v_2 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \end{array} (\mathrm{N/m})$$

Element 2:

$$\theta = 0^o, \quad l = 1, \quad m = 0$$

$$\begin{array}{c} \\ \mathbf{k}_2 = \dfrac{(210\times 10^9)(6.0\times 10^{-4})}{1} \end{array} \begin{array}{c} \begin{array}{cccc} u_2 & v_2 & u_3 & v_3 \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} (\mathrm{N/m})$$

Element 3:

$$\theta = 45^o, \quad l = \frac{1}{\sqrt{2}}, \quad m = \frac{1}{\sqrt{2}}$$

$$\begin{array}{c} \\ \mathbf{k}_3 = \dfrac{(210\times 10^9)(6\sqrt{2}\times 10^{-4})}{\sqrt{2}} \end{array} \begin{array}{c} \begin{array}{cccc} u_1 & v_1 & u_3 & v_3 \end{array} \\ \begin{bmatrix} 0.5 & 0.5 & -0.5 & -0.5 \\ 0.5 & 0.5 & -0.5 & -0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \end{bmatrix} \end{array}$$

$$(\mathrm{N/m})$$

The global FE equation is,

$$1260\times10^5\begin{bmatrix}0.5 & 0.5 & 0 & 0 & -0.5 & -0.5\\ & 1.5 & 0 & -1 & -0.5 & -0.5\\ & & 1 & 0 & -1 & 0\\ & & & 1 & 0 & 0\\ & & & & 1.5 & 0.5\\ Sym. & & & & & 0.5\end{bmatrix}\begin{Bmatrix}u_1\\ v_1\\ u_2\\ v_2\\ u_3\\ v_3\end{Bmatrix}=\begin{Bmatrix}F_{1X}\\ F_{1Y}\\ F_{2X}\\ F_{2Y}\\ F_{3X}\\ F_{3Y}\end{Bmatrix}$$

Load and boundary conditions (BC):

$$u_1 = v_1 = v_2 = 0, \quad \text{and} \quad v_3' = 0,$$
$$F_{2X} = P, \quad F_{3x'} = 0.$$

From the transformation relation and the BC, we have

$$v_3' = \begin{bmatrix}-\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}\end{bmatrix}\begin{Bmatrix}u_3\\ v_3\end{Bmatrix} = \frac{\sqrt{2}}{2}(-u_3 + v_3) = 0,$$

that is,

$$u_3 - v_3 = 0$$

This is a *multipoint constraint* (MPC).

Similarly, we have a relation for the force at node 3,

$$F_{3x'} = \begin{bmatrix}\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}\end{bmatrix}\begin{Bmatrix}F_{3X}\\ F_{3Y}\end{Bmatrix} = \frac{\sqrt{2}}{2}(F_{3X} + F_{3Y}) = 0,$$

that is,

$$F_{3X} + F_{3Y} = 0$$

Applying the load and BC's in the structure FE equation by 'deleting' 1st, 2nd and 4th rows and columns, we have

$$1260\times 10^5\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1.5 & 0.5 \\ 0 & 0.5 & 0.5 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} P \\ F_{3X} \\ F_{3Y} \end{Bmatrix}$$

Further, from the MPC and the force relation at node 3, the equation becomes,

$$1260\times 10^5\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1.5 & 0.5 \\ 0 & 0.5 & 0.5 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} P \\ F_{3X} \\ -F_{3X} \end{Bmatrix}$$

which is

$$1260\times 10^5\begin{bmatrix} 1 & -1 \\ -1 & 2 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} P \\ F_{3X} \\ -F_{3X} \end{Bmatrix}$$

The 3rd equation yields,

$$F_{3X} = -1260\times 10^5 u_3$$

Substituting this into the 2nd equation and rearranging, we have

$$1260\times 10^5\begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} P \\ 0 \end{Bmatrix}$$

Solving this, we obtain the displacements,

$$\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \frac{1}{2520 \times 10^5} \begin{Bmatrix} 3P \\ P \end{Bmatrix} = \begin{Bmatrix} 0.01191 \\ 0.003968 \end{Bmatrix} \quad \text{(m)}$$

From the global FE equation, we can calculate the reaction forces,

$$\begin{Bmatrix} F_{1X} \\ F_{1Y} \\ F_{2Y} \\ F_{3X} \\ F_{3Y} \end{Bmatrix} = 1260 \times 10^5 \begin{bmatrix} 0 & -0.5 & -0.5 \\ 0 & -0.5 & -0.5 \\ 0 & 0 & 0 \\ -1 & 1.5 & 0.5 \\ 0 & 0.5 & 0.5 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} -500 \\ -500 \\ 0.0 \\ -500 \\ 500 \end{Bmatrix} \text{(kN)}$$

Check the results!

A general *multipoint constraint* (MPC) can be described as,

$$\sum_j A_j u_j = 0$$

where A_j's are constants and u_j's are nodal displacement components. In the FE software, such as *MSC/NASTRAN*, users only need to specify this relation to the software. The software will take care of the solution.

Penalty Approach for Handling BC's and MPC's