Chapter 4a – Development of Beam Equations

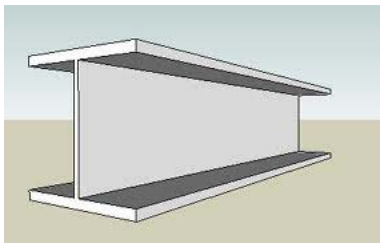
Learning Objectives

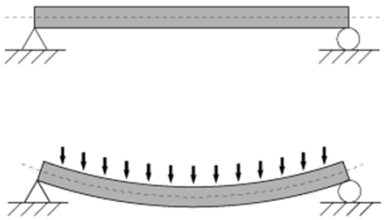
- To review the basic concepts of beam bending
- To derive the stiffness matrix for a beam element
- To demonstrate beam analysis using the direct stiffness method
- To illustrate the effects of shear deformation in shorter beams
- To introduce the work-equivalence method for replacing distributed loading by a set of discrete loads
- To introduce the general formulation for solving beam problems with distributed loading acting on them
- To analyze beams with distributed loading acting on them
- To compare the finite element solution to an exact solution for a beam
- To derive the stiffness matrix for the beam element with nodal hinge
- To show how the potential energy method can be used to derive the beam element equations
- To apply Galerkin's residual method for deriving the beam element equations

Development of Beam Equations

In this section, we will develop the stiffness matrix for a beam element, the most common of all structural elements.

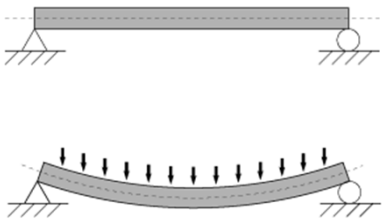
The ***beam*** element is considered to be straight and to have constant cross-sectional area.



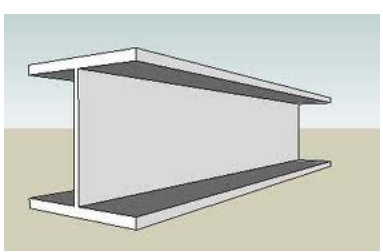


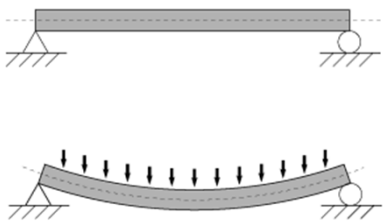
We will derive the beam element stiffness matrix by using the principles of simple beam theory.

The degrees of freedom associated with a node of a beam element are a transverse displacement and a rotation.



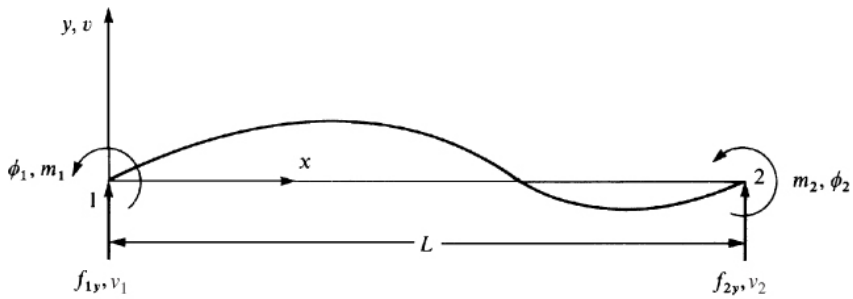
- We will discuss procedures for handling distributed loading and concentrated nodal loading.
- We will include the nodal shear forces and bending moments and the resulting shear force and bending moment diagrams as part of the total solution.
- We will develop the beam bending element equations using the potential energy approach.
- Finally, the Galerkin residual method is applied to derive the beam element equations





Beam Stiffness

Consider the beam element shown below.

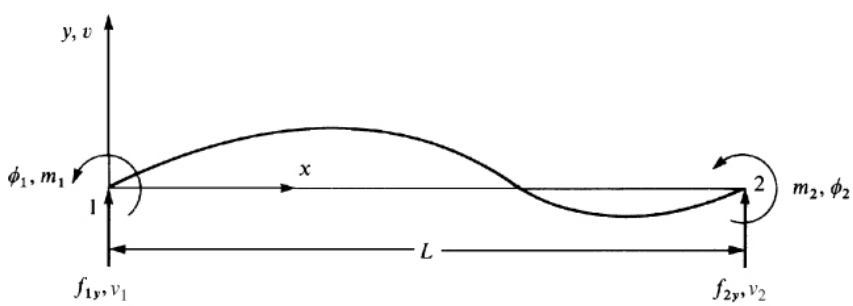


The beam is of length L with axial local coordinate x and transverse local coordinate y.

The local transverse nodal displacements are given by v_i and the rotations by ϕ_i. The local nodal forces are given by f_{iy} and the bending moments by m_i.

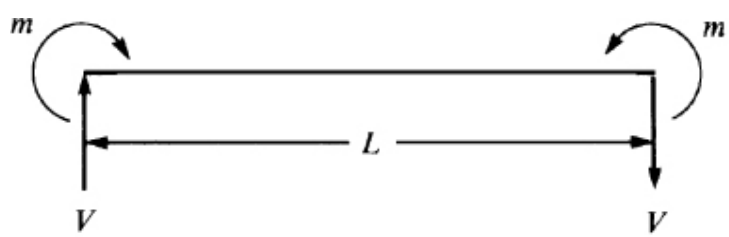
At all nodes, the following sign conventions are used on the element level:

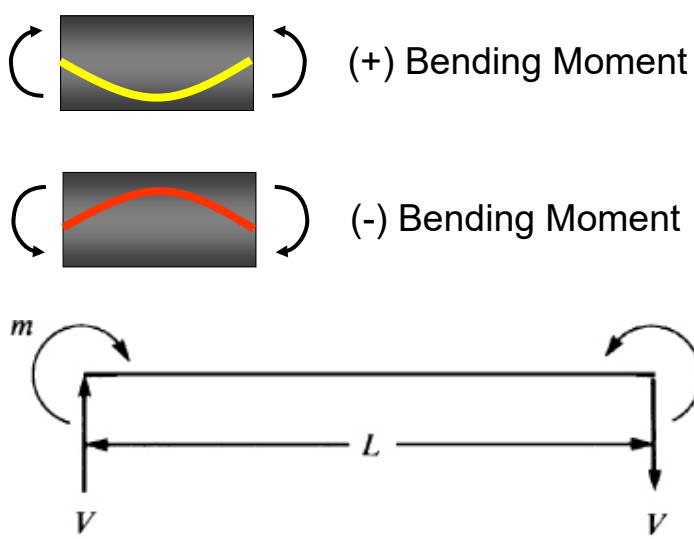
1. **Moments** are positive in the counterclockwise direction.
2. **Rotations** are positive in the counterclockwise direction.
3. **Forces** are positive in the positive y direction.
4. **Displacements** are positive in the positive y direction.

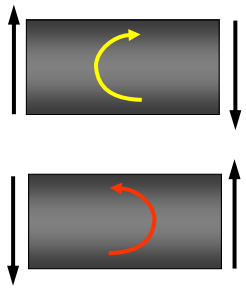


At all nodes, the following sign conventions are used on the global level:

1. **Bending moments *m*** are positive if they cause the beam to bend concave up.
2. **Shear forces *V*** are positive is the cause the beam to rotate clockwise.

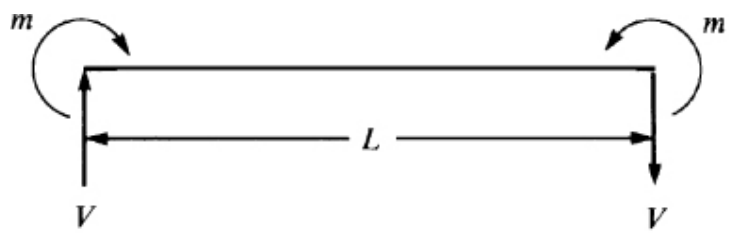




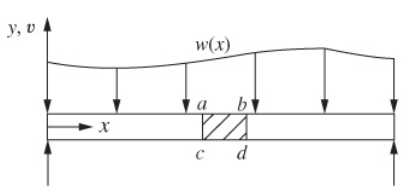


(+) Shear Force

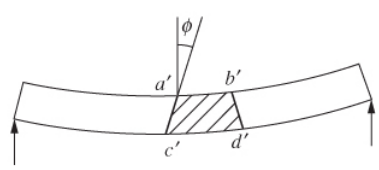
(-) Shear Force



The differential equation governing simple linear-elastic beam behavior can be derived as follows. Consider the beam shown below.

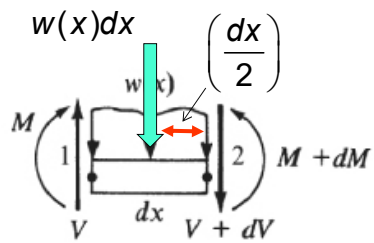


(a) Undeformed beam under load $w(x)$



(b) Deformed beam due to applied loading

The differential equation governing simple linear-elastic beam behavior can be derived as follows. Consider the beam shown below.



Write the equations of equilibrium for the differential element:

$$\circlearrowleft^{+}\sum M_{right-side} = 0 \quad = -\cancel{M} + (\cancel{M} + dM) - Vdx + w(x)dx\underbrace{\cancel{\left(\frac{dx}{2}\right)}}_{0}$$

$$^{+}\uparrow \sum F_y = 0 = \cancel{V} - (\cancel{V} + dV) - w(x)dx$$

From force and moment equilibrium of a differential beam element, we get:

$$\sum M_{right-side} = 0 \quad \Rightarrow \quad -Vdx + dM = 0 \quad \text{or} \quad V = \frac{dM}{dx}$$

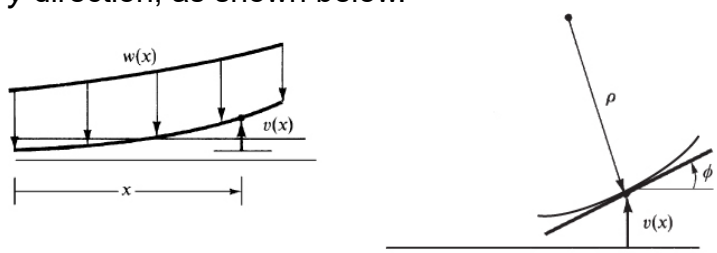
$$\sum F_y = 0 \quad \Rightarrow \quad -wdx - dV = 0 \quad \text{or} \quad w = -\frac{dV}{dx}$$

$$w = -\frac{d}{dx}\left(\frac{dM}{dx}\right)$$

The curvature κ of the beam is related to the moment by:

$$\kappa = \frac{1}{\rho} = \frac{M}{EI}$$

where ρ is the radius of the deflected curve, v is the transverse displacement function in the y direction, E is the modulus of elasticity, and I is the principle moment of inertia about y direction, as shown below.



(a) Portion of deflected curve of beam (b) Radius of deflected curve at $v(x)$

The curvature, κ for small slopes $\phi = \frac{dv}{dx}$ is given as:

$$\kappa = \frac{d^2v}{dx^2}$$

Therefore: $$\frac{d^2v}{dx^2} = \frac{M}{EI} \quad \Rightarrow \quad M = EI\frac{d^2v}{dx^2}$$

Substituting the moment expression into the moment-load equations gives:

$$\frac{d^2}{dx^2}\left(EI\frac{d^2v}{dx^2}\right) = -w(x)$$

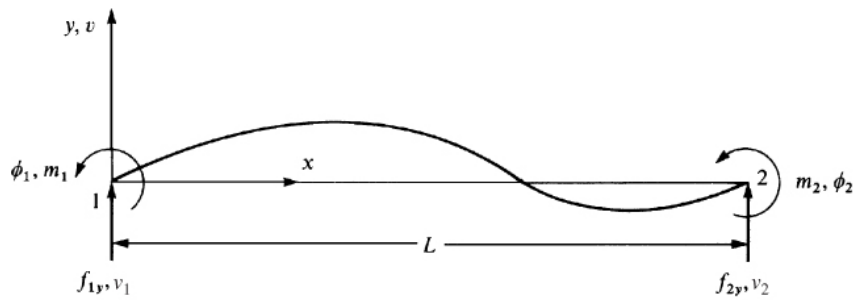
For constant values of EI, the above equation reduces to:

$$EI\left(\frac{d^4v}{dx^4}\right) = -w(x)$$

Beam Stiffness Formulation

Step 1 - Select Element Type

We will consider the linear-elastic beam element shown below.



Step 2 - Select a Displacement Function

Assume the transverse displacement function v is:

$$v = a_1 x^3 + a_2 x^2 + a_3 x + a_4$$

The number of coefficients in the displacement function a_i is equal to the total number of degrees of freedom associated with the element (displacement and rotation at each node). The boundary conditions are:

$$v(x=0) = v_1 \qquad v(x=L) = v_2$$

$$\left.\frac{dv}{dx}\right|_{x=0} = \phi_1 \qquad \left.\frac{dv}{dx}\right|_{x=L} = \phi_2$$

Applying the boundary conditions and solving for the unknown coefficients gives:

$$v(0) = v_1 = a_4 \qquad v(L) = v_2 = a_1L^3 + a_2L^2 + a_3L + a_4$$

$$\frac{dv(0)}{dx} = \phi_1 = a_3 \qquad \frac{dv(L)}{dx} = \phi_2 = 3a_1L^2 + 2a_2L + a_3$$

Solving these equations for a_1, a_2, a_3, and a_4 gives:

$$v = \left[\frac{2}{L^3}(v_1 - v_2) + \frac{1}{L^2}(\phi_1 + \phi_2)\right]x^3 + \left[-\frac{3}{L^2}(v_1 - v_2) - \frac{1}{L}(2\phi_1 + \phi_2)\right]x^2 + \phi_1 x + v_1$$

In matrix form the above equations are: $v = [N]\{d\}$

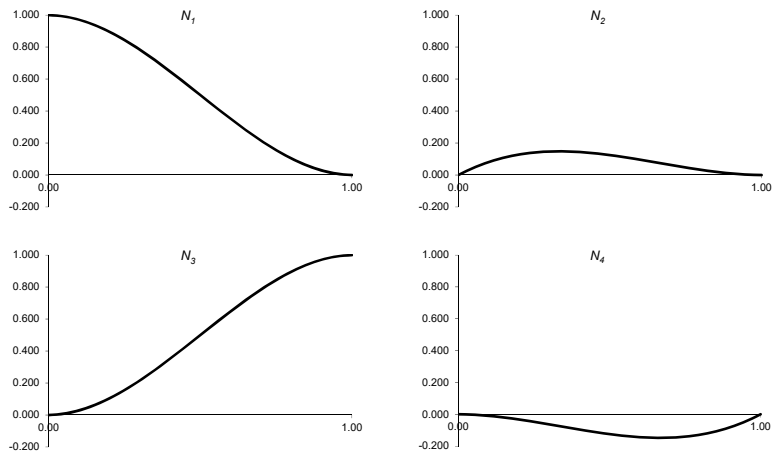
$$\{d\} = \begin{Bmatrix} v_1 \\ \phi_1 \\ v_2 \\ \phi_2 \end{Bmatrix} \qquad [N] = \begin{bmatrix} N_1 & N_2 & N_3 & N_4 \end{bmatrix}$$

where

$$N_1 = \frac{1}{L^3}\left(2x^3 - 3x^2L + L^3\right) \qquad N_2 = \frac{1}{L^3}\left(x^3L - 2x^2L^2 + xL^3\right)$$

$$N_3 = \frac{1}{L^3}\left(-2x^3 + 3x^2L\right) \qquad N_4 = \frac{1}{L^3}\left(x^3L - x^2L^2\right)$$

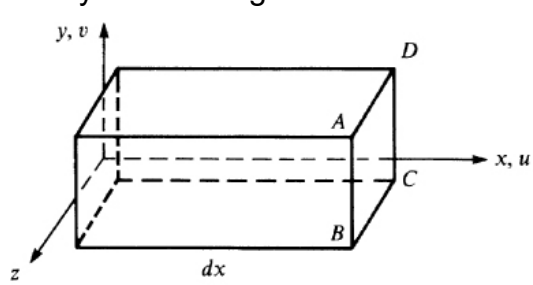
N_1, N_2, N_3, and N_4 are called the ***interpolation functions*** for a beam element.



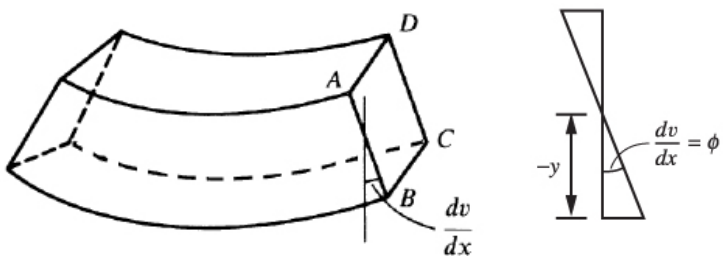
The stress-displacement relationship is: $\varepsilon_x(x,y) = \dfrac{du}{dx}$

where u is the axial displacement function.

We can relate the axial displacement to the transverse displacement by considering the beam element shown below:



Step 3 - Define the Strain/Displacement and Stress/Strain Relationships



$$u = -y\frac{dv}{dx}$$

One of the basic assumptions in simple beam theory is that planes remain planar after deformation, therefore:

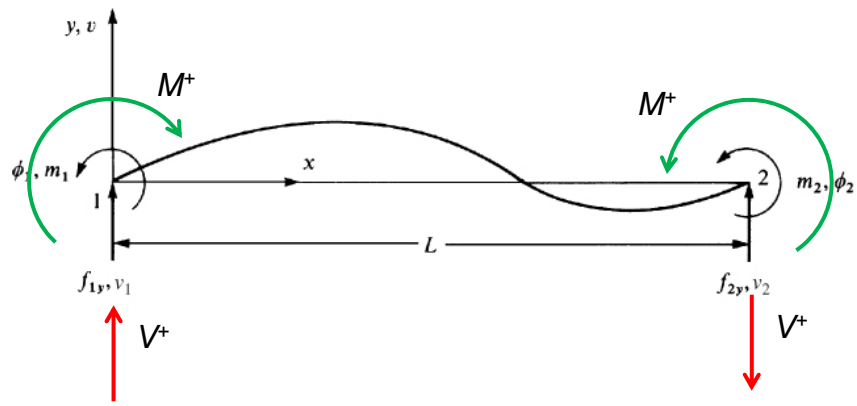
$$\varepsilon_x(x,y) = \frac{du}{dx} = -y\left(\frac{d^2v}{dx^2}\right)$$

Moments and shears are related to the transverse displacement as:

$$m(x) = EI\left(\frac{d^2v}{dx^2}\right) \qquad V(x) = EI\left(\frac{d^3v}{dx^3}\right)$$

Step 4 - Derive the Element Stiffness Matrix and Equations

Use beam theory sign convention for shear force and bending moment.



Using beam theory sign convention for shear force and bending moment we obtain the following equations:

$$f_{1y} = V = EI\left.\frac{d^3v}{dx^3}\right|_{x=0} = \frac{EI}{L^3}\left(12v_1 + 6L\phi_1 - 12v_2 + 6L\phi_2\right)$$

$$f_{2y} = -V = -EI\left.\frac{d^3v}{dx^3}\right|_{x=L} = \frac{EI}{L^3}\left(-12v_1 - 6L\phi_1 + 12v_2 - 6L\phi_2\right)$$

$$m_1 = -m = -EI\left.\frac{d^2v}{dx^2}\right|_{x=0} = \frac{EI}{L^3}\left(6Lv_1 + 4L^2\phi_1 - 6Lv_2 + 2L^2\phi_2\right)$$

$$m_2 = m = EI\left.\frac{d^2v}{dx^2}\right|_{x=L} = \frac{EI}{L^3}\left(6Lv_1 + 2L^2\phi_1 - 6Lv_2 + 4L^2\phi_2\right)$$

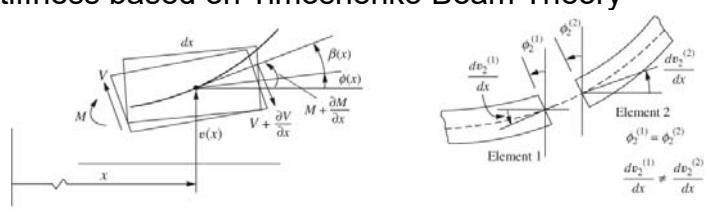
In matrix form the above equations are:

$$\begin{Bmatrix} f_{1y} \\ m_1 \\ f_{2y} \\ m_2 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} \boxed{12} & \boxed{6L} & -12 & 6L \\ 6L & \boxed{4L^2} & -6L & \boxed{2L^2} \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} v_1 \\ \phi_1 \\ v_2 \\ \phi_2 \end{Bmatrix} \qquad \begin{Bmatrix} f_{1y} \\ m_1 \\ f_{2y} \\ m_2 \end{Bmatrix} = k\begin{Bmatrix} v_1 \\ \phi_1 \\ v_2 \\ \phi_2 \end{Bmatrix}$$

where the stiffness matrix is:

$$\mathbf{k} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

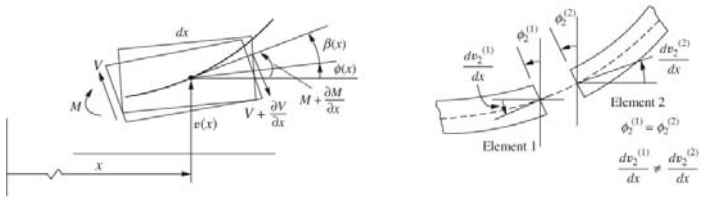
Beam stiffness based on Timoshenko Beam Theory



The total deflection of the beam at a point x consists of two parts, one caused by bending and one by shear force. The slope of the deflected curve at a point x is:

$$\frac{dv}{dx} = \phi(x) + \beta(x)$$

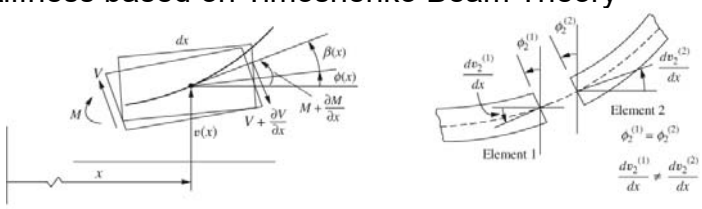
Beam stiffness based on Timoshenko Beam Theory



The relationship between bending moment and bending deformation is:

$$M(x) = EI\frac{d\phi(x)}{dx}$$

Beam stiffness based on Timoshenko Beam Theory

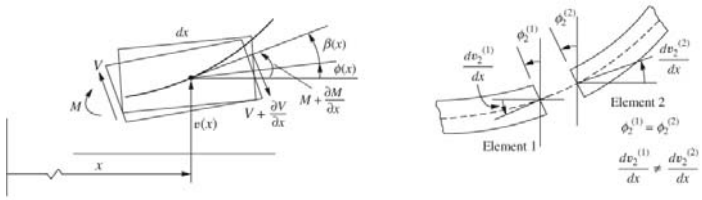


The relationship between shear force and shear deformation is:

$$V(x) = k_s AG\beta(x)$$

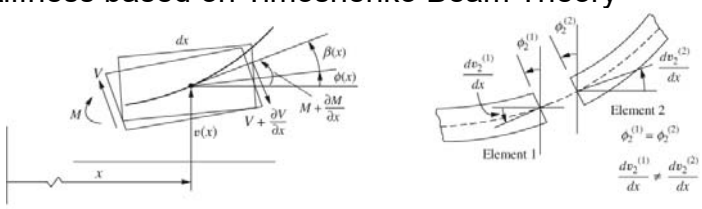
where k_sA is the shear area.

Beam stiffness based on Timoshenko Beam Theory



You can review the details in your book, but by including the effects of shear deformations into the relationship between forces and nodal displacements a modified elemental stiffness can be developed.

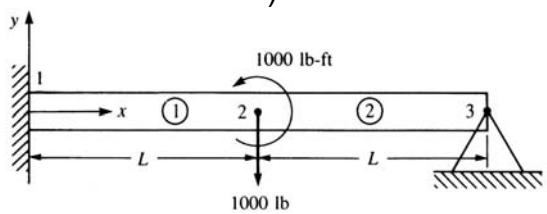
Beam stiffness based on Timoshenko Beam Theory



$$\mathbf{k} = \frac{EI}{L^3(1+\varphi)}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & (4+\varphi)L^2 & -6L & (2-\varphi)L^2 \\ -12 & -6L & 12 & -6L \\ 6L & (2-\varphi)L^2 & -6L & (4+\varphi)L^2 \end{bmatrix} \qquad \varphi = \frac{12EI}{k_s AGL^2}$$

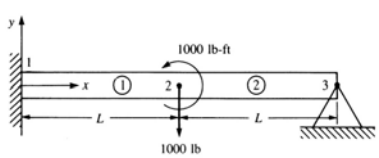
Step 5 - Assemble the Element Equations and Introduce Boundary Conditions

Consider a beam modeled by two beam elements (do not include shear deformations):



Assume the *EI* to be constant throughout the beam. A force of 1,000 *lb* and moment of 1,000 *lb-ft* are applied to the mid-point of the beam.

The beam element stiffness matrices are:



$$\mathbf{k}^{(1)} = \frac{EI}{L^3}\begin{array}{c} \begin{array}{cccc} v_1 & \phi_1 & v_2 & \phi_2 \end{array} \\ \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \end{array}$$

$$\mathbf{k}^{(2)} = \frac{EI}{L^3}\begin{array}{c} \begin{array}{cccc} v_2 & \phi_2 & v_3 & \phi_3 \end{array} \\ \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \end{array}$$

In this example, the local coordinates coincide with the global coordinates of the whole beam (therefore there is no transformation required for this problem).

The total stiffness matrix can be assembled as:

$$\begin{Bmatrix} F_{1y} \\ M_1 \\ F_{2y} \\ M_2 \\ F_{3y} \\ M_3 \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 \\ 6L & 4L^2 & -6L & 2L^2 & 0 & 0 \\ -12 & -6L & 12+12 & -6L+6L & -12 & 6L \\ 6L & 2L^2 & -6L+6L & 4L^2+4L^2 & -6L & 2L^2 \\ 0 & 0 & -12 & -6L & 12 & -6L \\ 0 & 0 & 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \begin{Bmatrix} v_1 \\ \phi_1 \\ v_2 \\ \phi_2 \\ v_3 \\ \phi_3 \end{Bmatrix}$$

Element 1 Element 2

The boundary conditions are: $v_1 = \phi_1 = v_3 = 0$

$$\begin{Bmatrix} F_{1y} \\ M_1 \\ F_{2y} \\ M_2 \\ F_{3y} \\ M_3 \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 \\ 6L & 4L^2 & -6L & 2L^2 & 0 & 0 \\ -12 & -6L & 12+12 & -6L+6L & -12 & 6L \\ 6L & 2L^2 & -6L+6L & 4L^2+4L^2 & -6L & 2L^2 \\ 0 & 0 & -12 & -6L & 12 & -6L \\ 0 & 0 & 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ v_2 \\ \phi_2 \\ 0 \\ \phi_3 \end{Bmatrix}$$

By applying the boundary conditions the beam equations reduce to:

$$\begin{Bmatrix} -1{,}000\ lb \\ 1{,}000\ lb\,ft \\ 0 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 24 & 0 & 6L \\ 0 & 8L^2 & 2L^2 \\ 6L & 2L^2 & 4L^2 \end{bmatrix}\begin{Bmatrix} v_2 \\ \phi_2 \\ \phi_3 \end{Bmatrix}$$

Step 6 - Solve for the Unknown Degrees of Freedom

Solving the above equations gives:

$$v_2 = \frac{-875L^3 + 375L^2}{12EI}\ in \qquad \phi_2 = \frac{-125L^2 + 625L}{4EI}\ rad \qquad \phi_3 = \frac{125L^2 - 125L}{EI}\ rad$$

Step 7 - Solve for the Element Strains and Stresses

$$m(x) = EI\left(\frac{d^2v}{dx^2}\right) = EI\left(\frac{d^2N}{dx^2}\right)\begin{Bmatrix} v_1 \\ \phi_1 \\ v_2 \\ \phi_2 \end{Bmatrix}$$

The second derivative of N is linear; therefore $m(x)$ is linear.

Solving the above equations gives:

$$v_2 = -\frac{875L^3 - 375L^2}{12EI}\ in \qquad \phi_2 = -\frac{125L^2 - 625L}{4EI}\ rad \qquad \phi_3 = -\frac{125L^2 - 125L}{EI}\ rad$$

Step 7 - Solve for the Element Strains and Stresses

$$V(x) = EI\left(\frac{d^3v}{dx^3}\right) = EI\left(\frac{d^3N}{dx^2}\right)\begin{Bmatrix} v_1 \\ \phi_1 \\ v_2 \\ \phi_2 \end{Bmatrix}$$

The third derivative of N is a constant; therefore $V(x)$ is constant.

Assume L = 120 *in*, E = 29x10^6 *psi*, and I = 100 *in*^{4}:

$$v_2 = -0.0433\ in \qquad \phi_2 = -7.758\times10^{-5}\ rad \qquad \phi_3 = 5.586\times10^{-4}\ rad$$

Element #1:

$$m(x) = EI\left(\frac{d^2v}{dx^2}\right) = EI\left(\frac{d^2N}{dx^2}\right)\begin{Bmatrix} v_1 \\ \phi_1 \\ v_2 \\ \phi_2 \end{Bmatrix}$$

$$m_1 = \frac{EI}{L^3}\left(6Lv_1 + 4L^2\phi_1 - 6Lv_2 + 2L^2\phi_2\right) = 3{,}875\ lb\cdot ft$$

$$m_2 = \frac{EI}{L^3}\left(6Lv_1 + 2L^2\phi_1 - 6Lv_2 + 4L^2\phi_2\right) = 3{,}562.5\ lb\cdot ft$$

Assume L = 120 *in*, E = 29x10^6 *psi*, and I = 100 *in*^{4}:

$$v_2 = -0.0433\ in \qquad \phi_2 = -7.758\times10^{-5}\ rad \qquad \phi_3 = 5.586\times10^{-4}\ rad$$

Element #2:

$$m(x) = EI\left(\frac{d^2v}{dx^2}\right) = EI\left(\frac{d^2N}{dx^2}\right)\begin{Bmatrix} v_1 \\ \phi_1 \\ v_2 \\ \phi_2 \end{Bmatrix}$$

$$m_2 = \frac{EI}{L^3}\left(6Lv_2 + 4L^2\phi_2 - 6Lv_3 + 2L^2\phi_3\right) = -2{,}562.5\,lb\cdot ft$$

$$m_3 = \frac{EI}{L^3}\left(6Lv_2 + 2L^2\phi_2 - 6Lv_3 + 4L^2\phi_3\right) = 0$$

Assume L = 120 *in*, E = 29x10^6 *psi*, and I = 100 *in*^{4}:

$$v_2 = -0.0433\ in \qquad \phi_2 = -7.758\times10^{-5}\ rad \qquad \phi_3 = 5.586\times10^{-4}\ rad$$

Element #1:

$$V(x) = EI\left(\frac{d^3v}{dx^3}\right) = EI\left(\frac{d^3N}{dx^2}\right)\begin{Bmatrix} v_1 \\ \phi_1 \\ v_2 \\ \phi_2 \end{Bmatrix}$$

$$f_{1y} = \frac{EI}{L^3}\left(12v_1 + 6L\phi_1 - 12v_2 + 6L\phi_2\right) = 743.75\,lb$$

$$f_{2y} = \frac{EI}{L^3}\left(-12v_1 - 6L\phi_1 + 12v_2 - 6L\phi_2\right) = -743.75\,lb$$

Assume L = 120 *in*, E = 29x10^6 *psi*, and I = 100 *in*^{4}:

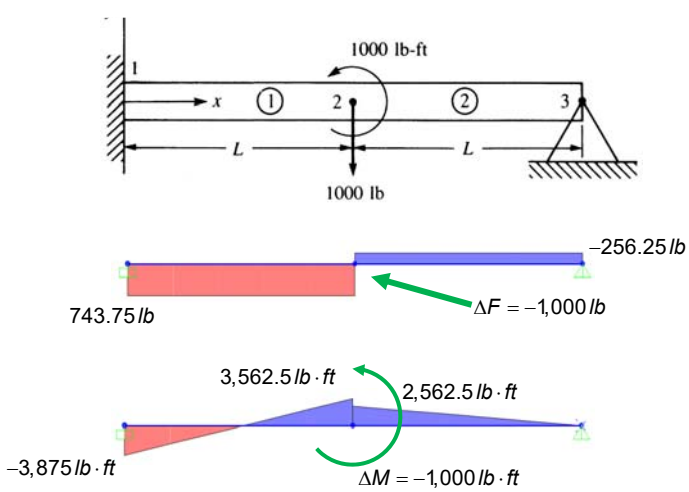
$$v_2 = -0.0433\ in \qquad \phi_2 = -7.758\times10^{-5}\ rad \qquad \phi_3 = 5.586\times10^{-4}\ rad$$

Element #2:

$$V(x) = EI\left(\frac{d^3v}{dx^3}\right) = EI\left(\frac{d^3N}{dx^2}\right)\begin{Bmatrix} v_1 \\ \phi_1 \\ v_2 \\ \phi_2 \end{Bmatrix}$$

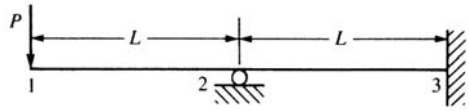
$$f_{2y} = \frac{EI}{L^3}\left(12v_2 + 6L\phi_2 - 12v_3 + 6L\phi_3\right) = -256.25\,lb$$

$$f_{3y} = \frac{EI}{L^3}\left(-12v_2 - 6L\phi_2 + 12v_3 - 6L\phi_3\right) = 256.25\,lb$$



Example 1 - Beam Problem

Consider the beam shown below. Assume that EI is constant and the length is $2L$ (no shear deformation).



The beam element stiffness matrices are:

$$\mathbf{k}^{(1)} = \frac{EI}{L^3}\begin{array}{c} \begin{array}{cccc} v_1 & \phi_1 & v_2 & \phi_2 \end{array} \\ \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \end{array} \qquad \mathbf{k}^{(2)} = \frac{EI}{L^3}\begin{array}{c} \begin{array}{cccc} v_2 & \phi_2 & v_3 & \phi_3 \end{array} \\ \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \end{array}$$

Example 1 - Beam Problem

The local coordinates coincide with the global coordinates of the whole beam (therefore there is no transformation required for this problem).

The total stiffness matrix can be assembled as:

$$\begin{Bmatrix} F_{1y} \\ M_1 \\ F_{2y} \\ M_2 \\ F_{3y} \\ M_3 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 \\ 6L & 4L^2 & -6L & 2L^2 & 0 & 0 \\ -12 & -6L & 24 & 0 & -12 & 6L \\ 6L & 2L^2 & 0 & 8L^2 & -6L & 2L^2 \\ 0 & 0 & -12 & -6L & 12 & -6L \\ 0 & 0 & 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} v_1 \\ \phi_1 \\ v_2 \\ \phi_2 \\ v_3 \\ \phi_3 \end{Bmatrix}$$

Element 1 Element 2

Example 1 - Beam Problem

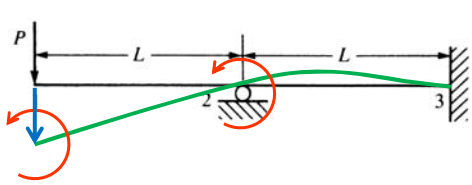
The boundary conditions are: $v_2 = v_3 = \phi_3 = 0$

$$\begin{Bmatrix} F_{1y} \\ M_1 \\ F_{2y} \\ M_2 \\ F_{3y} \\ M_3 \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 \\ 6L & 4L^2 & -6L & 2L^2 & 0 & 0 \\ -12 & -6L & 24 & 0 & -12 & 6L \\ 6L & 2L^2 & 0 & 8L^2 & -6L & 2L^2 \\ 0 & 0 & -12 & -6L & 12 & -6L \\ 0 & 0 & 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \begin{Bmatrix} v_1 \\ \phi_1 \\ 0 \\ \phi_2 \\ 0 \\ 0 \end{Bmatrix}$$

By applying the boundary conditions the beam equations reduce to:

$$\begin{Bmatrix} -P \\ 0 \\ 0 \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & 6L \\ 6L & 4L^2 & 2L^2 \\ 6L & 2L^2 & 8L^2 \end{bmatrix} \begin{Bmatrix} v_1 \\ \phi_1 \\ \phi_2 \end{Bmatrix}$$

Solving the above equations gives:



$$\begin{Bmatrix} v_1 \\ \phi_1 \\ \phi_2 \end{Bmatrix} = \frac{PL^2}{4EI} \begin{Bmatrix} -\frac{7L}{3} \\ 3 \\ 1 \end{Bmatrix}$$

Example 1 - Beam Problem

The positive signs for the rotations indicate that both are in the counterclockwise direction.

The negative sign on the displacement indicates a deformation in the *-y* direction.

$$\begin{Bmatrix} F_{1y} \\ M_1 \\ F_{2y} \\ M_2 \\ F_{3y} \\ M_3 \end{Bmatrix} = \frac{P}{4L} \begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 \\ 6L & 4L^2 & -6L & 2L^2 & 0 & 0 \\ -12 & -6L & 24 & 0 & -12 & 6L \\ 6L & 2L^2 & 0 & 8L^2 & -6L & 2L^2 \\ 0 & 0 & -12 & -6L & 12 & -6L \\ 0 & 0 & 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \begin{Bmatrix} -7L/3 \\ 3 \\ 0 \\ 1 \\ 0 \\ 0 \end{Bmatrix}$$

Example 1 - Beam Problem

The local nodal forces for element 1:

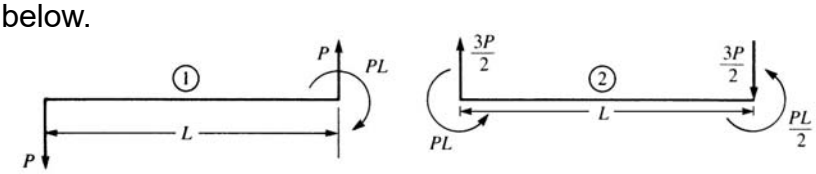
$$\begin{Bmatrix} f_{1y} \\ m_1 \\ f_{2y} \\ m_2 \end{Bmatrix} = \frac{P}{4L} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \begin{Bmatrix} -7L/3 \\ 3 \\ 0 \\ 1 \end{Bmatrix} = \begin{Bmatrix} -P \\ 0 \\ P \\ -PL \end{Bmatrix}$$

The local nodal forces for element 2:

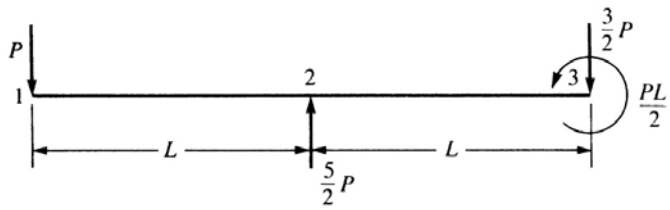
$$\begin{Bmatrix} f_{2y} \\ m_2 \\ f_{3y} \\ m_3 \end{Bmatrix} = \frac{P}{4L} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \begin{Bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 1.5P \\ PL \\ -1.5P \\ 0.5PL \end{Bmatrix}$$

Example 1 - Beam Problem

The free-body diagrams for the each element are shown below.

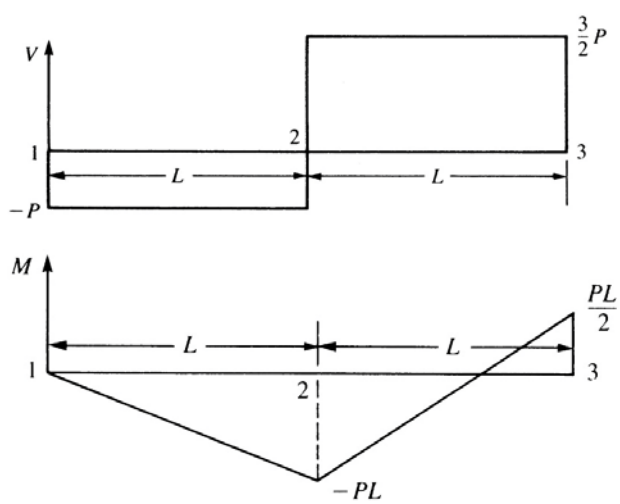


Combining the elements gives the forces and moments for the original beam.



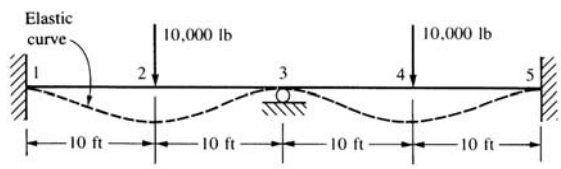
Example 1 - Beam Problem

Therefore, the shear force and bending moment diagrams are:



Example 2 - Beam Problem

Consider the beam shown below. Assume E = 30 x 10^6 *psi* and I = 500 in^4 are constant throughout the beam. Use four elements of equal length to model the beam.



The beam element stiffness matrices are:

$$\mathbf{k}^{(i)} = \frac{EI}{L^3}\begin{array}{c} \begin{array}{cccc} v_i & \phi_i & v_{(i+1)} & \phi_{(i+1)} \end{array} \\ \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \end{array}$$

Using the direct stiffness method, the four beam element stiffness matrices are superimposed to produce the global stiffness matrix.

Element 1 Element 2

$$\begin{Bmatrix} F_{1y} \\ M_1 \\ F_{2y} \\ M_2 \\ F_{3y} \\ M_3 \\ F_{4y} \\ M_4 \\ F_{5y} \\ M_5 \end{Bmatrix} = \frac{EI}{L^3}\begin{array}{c} \begin{array}{cccccccccc} v_1 & \phi_1 & v_2 & \phi_2 & v_3 & \phi_3 & v_4 & \phi_4 & v_5 & \phi_5 \end{array} \\ \begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 & 0 & 0 & 0 & 0 \\ 6L & 4L^2 & -6L & 2L^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -12 & -6L & 12+12 & -6L+6L & -12 & 6L & 0 & 0 & 0 & 0 \\ 6L & 2L^2 & -6L+6L & 4L^2+4L^2 & -6L & 2L^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -12 & -6L & 12+12 & -6L+6L & -12 & 6L & 0 & 0 \\ 0 & 0 & 6L & 2L^2 & -6L+6L & 4L^2+4L^2 & -6L & 2L^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -12 & -6L & 12+12 & -6L+6L & -12 & 6L \\ 0 & 0 & 0 & 0 & 6L & 2L^2 & -6L+6L & 4L^2+4L^2 & -6L & 2L^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & -12 & -6L & 12 & -6L \\ 0 & 0 & 0 & 0 & 0 & 0 & 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \end{array} \begin{Bmatrix} v_1 \\ \phi_1 \\ v_2 \\ \phi_2 \\ v_3 \\ \phi_3 \\ v_4 \\ \phi_4 \\ v_5 \\ \phi_5 \end{Bmatrix}$$

Element 3

Element 4

The boundary conditions for this problem are:

$$v_1 = \phi_1 = v_3 = v_5 = \phi_5 = 0$$

$$\begin{Bmatrix} F_{1y} \\ M_1 \\ F_{2y} \\ M_2 \\ F_{3y} \\ M_3 \\ F_{4y} \\ M_4 \\ F_{5y} \\ M_5 \end{Bmatrix} = \frac{EI}{L^3} \begin{array}{c} \begin{array}{cccccccccc} v_1 & \phi_1 & v_2 & \phi_2 & v_3 & \phi_3 & v_4 & \phi_4 & v_5 & \phi_5 \end{array} \\ \begin{bmatrix} & & & & & & & & & \\ & & 12+12 & -6L+6L & & 6L & 0 & 0 & & \\ & & -6L+6L & 4L^2+4L^2 & & 2L^2 & 0 & 0 & & \\ & & 6L & 2L^2 & & 4L^2+4L^2 & -6L & 2L^2 & & \\ & & 0 & 0 & & -6L & 12+12 & -6L+6L & & \\ & & 0 & 0 & & 2L^2 & -6L+6L & 4L^2+4L^2 & & \\ & & & & & & & & & \end{bmatrix} \end{array} \begin{Bmatrix} v_1 \\ \phi_1 \\ v_2 \\ \phi_2 \\ v_3 \\ \phi_3 \\ v_4 \\ \phi_4 \\ v_5 \\ \phi_5 \end{Bmatrix}$$

The boundary conditions for this problem are:

$$v_1 = \phi_1 = v_3 = v_5 = \phi_5 = 0$$

After applying the boundary conditions the global beam equations reduce to:

$$\frac{EI}{L^3} \begin{bmatrix} 24 & 0 & 6L & 0 & 0 \\ 0 & 8L^2 & 2L^2 & 0 & 0 \\ 6L & 2L^2 & 8L^2 & -6L & 2L^2 \\ 0 & 0 & -6L & 24 & 0 \\ 0 & 0 & 2L^2 & 0 & 8L^2 \end{bmatrix} \begin{Bmatrix} v_2 \\ \phi_2 \\ \phi_3 \\ v_4 \\ \phi_4 \end{Bmatrix} = \begin{Bmatrix} -10{,}000\ lb \\ 0 \\ 0 \\ -10{,}000\ lb \\ 0 \end{Bmatrix}$$

Substituting L = 120 *in*, E = 30 x 10^6 *psi*, and I = 500 in^4 into the above equations and solving for the unknowns gives:

$$v_2 = v_4 = -0.048\ in \qquad \phi_2 = \phi_3 = \phi_4 = 0$$

The global forces and moments can be determined as:

$$\begin{aligned} F_{1y} &= 5\ kips & M_1 &= 25\ kips{\cdot}ft \\ F_{2y} &= -10\ kips & M_2 &= 0 \\ F_{3y} &= 10\ kips & M_3 &= 0 \\ F_{4y} &= -10\ kips & M_4 &= 0 \\ F_{5y} &= 5\ kips & M_5 &= -25\ kips{\cdot}ft \end{aligned}$$

The local nodal forces for element 1:

$$\begin{Bmatrix} f_{1y} \\ m_1 \\ f_{2y} \\ m_2 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ -0.048 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 5\ kips \\ 25\ k{\cdot}ft \\ -5\ kips \\ 25\ k{\cdot}ft \end{Bmatrix}$$

The local nodal forces for element 2:

$$\begin{Bmatrix} f_{2y} \\ m_2 \\ f_{3y} \\ m_3 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} -0.048 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -5\ kips \\ -25\ k{\cdot}ft \\ 5\ kips \\ -25\ k{\cdot}ft \end{Bmatrix}$$

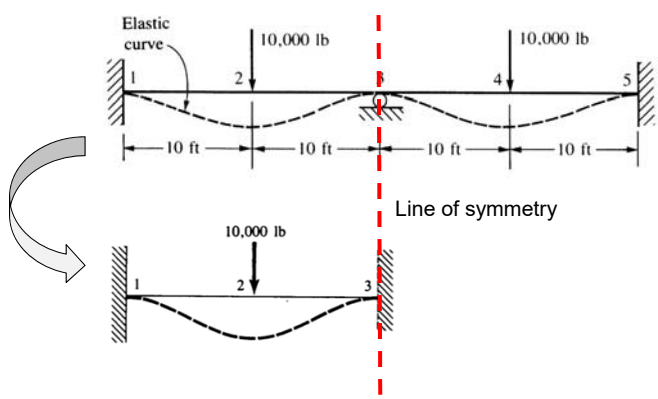
The local nodal forces for element 3:

$$\begin{Bmatrix} f_{3y} \\ m_3 \\ f_{4y} \\ m_4 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ -0.048 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 5\ kips \\ 25\ k{\cdot}ft \\ -5\ kips \\ 25\ k{\cdot}ft \end{Bmatrix}$$

The local nodal forces for element 4:

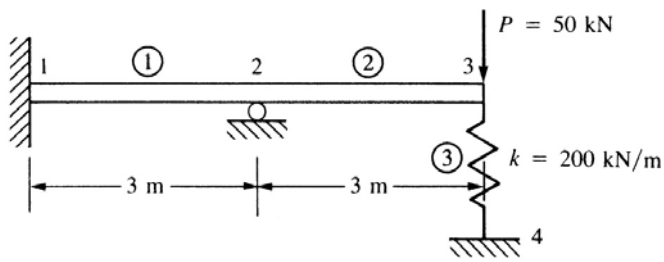
$$\begin{Bmatrix} f_{4y} \\ m_4 \\ f_{5y} \\ m_5 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} -0.048 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -5\ kips \\ -25\ k{\cdot}ft \\ 5\ kips \\ -25\ k{\cdot}ft \end{Bmatrix}$$

Note: Due to symmetry about the vertical plane at node 3, we could have worked just half the beam, as shown below.



Example 3 - Beam Problem

Consider the beam shown below. Assume E = 210 GPa and I = 2 x 10^{-4} m^4 are constant throughout the beam and the spring constant k = 200 kN/m. Use two beam elements of equal length and one spring element to model the structure.



The beam element stiffness matrices are:

$$\mathbf{k}^{(1)} = \frac{EI}{L^3}\begin{array}{c} \begin{array}{cccc} v_1 & \phi_1 & v_2 & \phi_2 \end{array} \\ \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \end{array} \qquad \mathbf{k}^{(2)} = \frac{EI}{L^3}\begin{array}{c} \begin{array}{cccc} v_2 & \phi_2 & v_3 & \phi_3 \end{array} \\ \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \end{array}$$

The spring element stiffness matrix is:

$$\mathbf{k}^{(3)} = \begin{array}{c} \begin{array}{cc} v_3 & v_4 \end{array} \\ \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \end{array} \quad \Rightarrow \quad \mathbf{k}^{(3)} = \begin{array}{c} \begin{array}{ccc} v_3 & \phi_3 & v_4 \end{array} \\ \begin{bmatrix} k & 0 & -k \\ 0 & 0 & 0 \\ -k & 0 & k \end{bmatrix} \end{array}$$

Using the direct stiffness method and superposition gives the global beam equations.

Element 1 Element 2

$$\begin{Bmatrix} F_{1y} \\ M_1 \\ F_{2y} \\ M_2 \\ F_{3y} \\ M_3 \\ F_{4y} \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 & 0 \\ 6L & 4L^2 & -6L & 2L^2 & 0 & 0 & 0 \\ -12 & -6L & 24 & 0 & -12 & 6L & 0 \\ 6L & 2L^2 & 0 & 8L^2 & -6L & 2L^2 & 0 \\ 0 & 0 & -12 & -6L & 12+k' & -6L & -k' \\ 0 & 0 & 6L & 2L^2 & -6L & 4L^2 & 0 \\ 0 & 0 & 0 & 0 & -k' & 0 & k' \end{bmatrix} \begin{Bmatrix} v_1 \\ \phi_1 \\ v_2 \\ \phi_2 \\ v_3 \\ \phi_3 \\ v_4 \end{Bmatrix} \qquad k' = \frac{kL^3}{EI}$$

Element 3

The boundary conditions for this problem are: $v_1 = \phi_1 = v_2 = v_4 = 0$

$$\begin{Bmatrix} F_{1y} \\ M_1 \\ F_{2y} \\ M_2 \\ F_{3y} \\ M_3 \\ F_{4y} \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 & 0 \\ 6L & 4L^2 & -6L & 2L^2 & 0 & 0 & 0 \\ -12 & -6L & 24 & 0 & -12 & 6L & 0 \\ 6L & 2L^2 & 0 & 8L^2 & -6L & 2L^2 & 0 \\ 0 & 0 & -12 & -6L & 12+k' & -6L & -k' \\ 0 & 0 & 6L & 2L^2 & -6L & 4L^2 & 0 \\ 0 & 0 & 0 & 0 & -k' & 0 & k' \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \phi_2 \\ v_3 \\ \phi_3 \\ 0 \end{Bmatrix} \qquad k' = \frac{kL^3}{EI}$$

After applying the boundary conditions the global beam equations reduce to:

$$\begin{Bmatrix} M_2 \\ F_{3y} \\ M_3 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 8L^2 & -6L & 2L^2 \\ -6L & 12+k' & -6L \\ 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} \phi_2 \\ v_3 \\ \phi_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -P \\ 0 \end{Bmatrix}$$

Solving the above equations gives:

$$\begin{Bmatrix} \phi_2 \\ v_3 \\ \phi_3 \end{Bmatrix} = \begin{Bmatrix} -\dfrac{3PL^2}{EI}\left(\dfrac{1}{12+7k'}\right) \\ -\dfrac{7PL^3}{EI}\left(\dfrac{1}{12+7k'}\right) \\ -\dfrac{9PL^2}{EI}\left(\dfrac{1}{12+7k'}\right) \end{Bmatrix} \qquad k' = \frac{kL^3}{EI}$$

Substituting L = 3 m, E = 210 GPa, I = 2 x 10^{-4} m^4, and k = 200 kN/m in the above equations gives:

$$v_3 = -0.0174\ m$$
$$\phi_2 = -0.00249\ rad$$
$$\phi_3 = -0.00747\ rad$$

Substituting the solution back into the global equations gives:

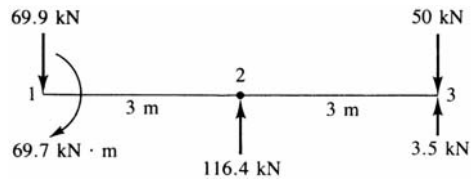
$$\begin{Bmatrix} F_{1y} \\ M_1 \\ F_{2y} \\ M_2 \\ F_{3y} \\ M_3 \\ F_{4y} \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 & 0 \\ 6L & 4L^2 & -6L & 2L^2 & 0 & 0 & 0 \\ -12 & -6L & 24 & 0 & -12 & 6L & 0 \\ 6L & 2L^2 & 0 & 8L^2 & -6L & 2L^2 & 0 \\ 0 & 0 & -12 & -6L & 12+k' & -6L & -k' \\ 0 & 0 & 6L & 2L^2 & -6L & 4L^2 & 0 \\ 0 & 0 & 0 & 0 & -k' & 0 & k' \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ 0 \\ -0.00249\ rad \\ -0.0174\ m \\ -0.00747\ rad \\ 0 \end{Bmatrix}$$

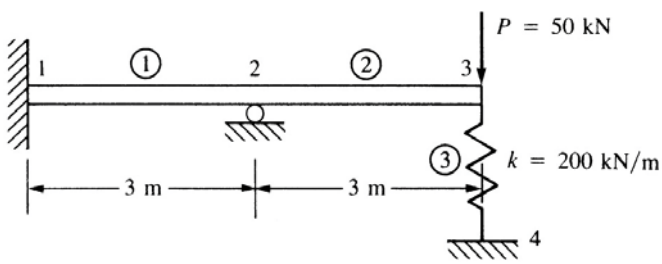
Substituting L = 3 m, E = 210 GPa, I = 2 x 10^{-4} m^4, and k = 200 kN/m in the above equations gives:

$$v_3 = -0.0174\ m$$
$$\phi_2 = -0.00249\ rad$$
$$\phi_3 = -0.00747\ rad$$

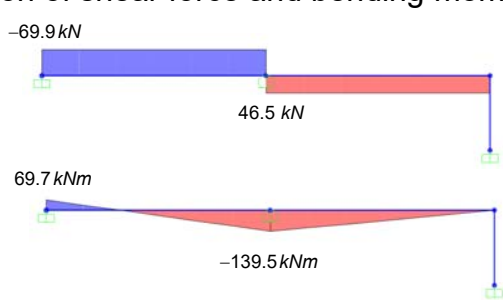
Substituting the solution back into the global equations gives:

$$\begin{Bmatrix} F_{1y} \\ M_1 \\ F_{2y} \\ M_2 \\ F_{3y} \\ M_3 \\ F_{4y} \end{Bmatrix} = \begin{Bmatrix} -69.9\ kN \\ -69.7\ kN \cdot m \\ 116.4\ kN \\ 0 \\ -50\ kN \\ 0 \\ 3.5\ kN \end{Bmatrix}$$





The variation of shear force and bending moment is:

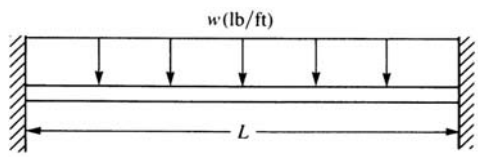


Distributed Loadings

Beam members can support distributed loading as well as concentrated nodal loading.

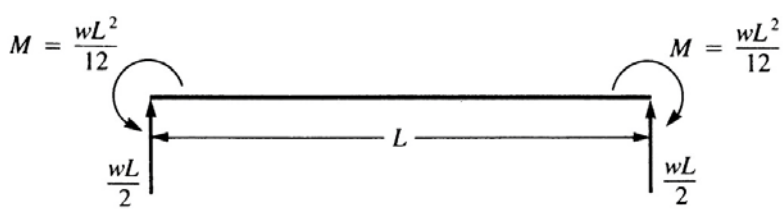
Therefore, we must be able to account for distributed loading.

Consider the fixed-fixed beam subjected to a uniformly distributed loading *w* shown the figure below.



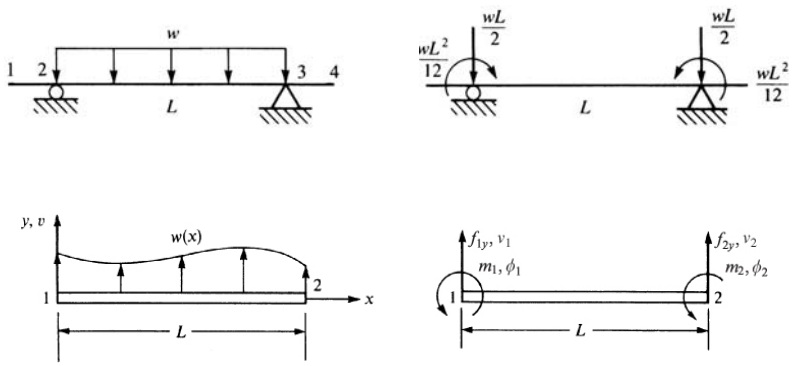
The reactions, determined from structural analysis theory, are called fixed-end reactions.

In general, fixed-end reactions are those reactions at the ends of an element if the ends of the element are assumed to be fixed (displacements and rotations are zero).



Therefore, guided by the results from structural analysis for the case of a uniformly distributed load, we replace the load by concentrated nodal forces and moments tending to have the same effect on the beam as the actual distributed load.

The figures below illustrates the idea of equivalent nodal loads for a general beam. We can replace the effects of a uniform load by a set of nodal forces and moments.



Beam Stiffness

Work Equivalence Method

This method is based on the concept that the work done by the distributed load is equal to the work done by the discrete nodal loads. The work done by the distributed load is:

$$W_{distributed} = \int_0^L w(x)\, v(x)\, dx$$

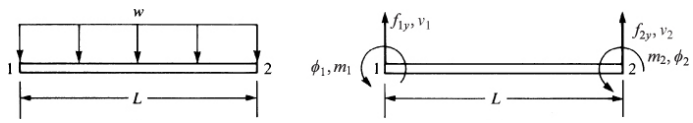
where $v(x)$ is the transverse displacement. The work done by the discrete nodal forces is:

$$W_{nodes} = m_1\phi_1 + m_2\phi_2 + f_{1y}v_1 + f_{2y}v_2$$

The nodal forces can be determined by setting $W_{distributed} = W_{nodes}$ for arbitrary displacements and rotations.

Example 4 - Load Replacement

Consider the beam, shown below, and determine the equivalent nodal forces for the given distributed load.



Using the work equivalence method or: $W_{distributed} = W_{nodes}$

$$\int_0^L w(x)\, v(x)\, dx = m_1\phi_1 + m_2\phi_2 + f_{1y}v_1 + f_{2y}v_2$$

Evaluating the left-hand-side of the above expression with:

$$w(x) = -w$$

$$v(x) = \left[\frac{2}{L^3}(v_1 - v_2) + \frac{1}{L^2}(\phi_1 + \phi_2)\right]x^3 + \left[-\frac{3}{L^2}(v_1 - v_2) - \frac{1}{L}(2\phi_1 + \phi_2)\right]x^2 + \phi_1 x + v_1$$

gives:

$$\int_0^L w\, v(x)\, dx = \frac{Lw}{2}(v_1 - v_2) - \frac{L^2 w}{4}(\phi_1 + \phi_2) - Lw(v_2 - v_1) + \frac{L^2 w}{3}(2\phi_1 + \phi_2) - \frac{L^2 w}{2}\phi_1 - wLv_1$$

Using a set of arbitrary nodal displacements, such as:

$$v_1 = v_2 = \phi_2 = 0 \qquad \phi_1 = 1$$

The resulting nodal equivalent force or moment is:

$$m_1\phi_1 + m_2\phi_2 + f_{1y}v_1 + f_{2y}v_2 = \int_0^L w(x)\, v(x)\, dx$$

$$m_1 = -\left(\frac{wL^2}{4} - \frac{2}{3}L^2 w + \frac{L^2}{2}w\right) = -\frac{wL^2}{12}$$

Using a set of arbitrary nodal displacements, such as:

$$v_1 = v_2 = \phi_1 = 0 \qquad \phi_2 = 1$$

The resulting nodal equivalent force or moment is:

$$m_1\phi_1 + m_2\phi_2 + f_{1y}v_1 + f_{2y}v_2 = \int_0^L w(x)\, v(x)\, dx$$

$$m_2 = -\left(\frac{wL^2}{4} - \frac{wL^2}{3}\right) = \frac{wL^2}{12}$$

Setting the nodal rotations equal zero except for the nodal displacements gives:

$$f_{1y} = -\frac{LW}{2} + Lw - Lw = -\frac{Lw}{2}$$

$$f_{2y} = \frac{LW}{2} - Lw = -\frac{Lw}{2}$$

Summarizing, the equivalent nodal forces and moments are: