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Finite Element Method

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Typical units for heat transfer

Variable	SI	U.S. Customary
Thermal conductivity, K	kW/(m · °C)	Btu/(h-ft-°F)
Temperature, T	°C or K	°F or °R
Internal heat source, Q	kW/m^3	Btu/(h-ft^3)
Heat flux, q	kW/m^2	Btu/(h-ft^2)
Convection coefficient, h	kW/(m^2 · °C)	Btu/(h-ft^2-°F)
Energy, E	kW · h	Btu
Specific heat, c	(kW · h)/(kg · °C)	Btu/(slug-°F)
Mass density, ρ	kg/m^3	slug/ft^3

Example 1:

Determine the temperature distribution along the length of the rod shown in Figure (1) with an insulated perimeter. The temperature at the left end is a constant 100 °F and the free-stream temperature is 10 °F. Let h = 10 Btu/(h-ft2-_F) and K_{xx} = 20 Btu/(h-ft-°F). The value of h is typical for forced air convection h=10 Btu/(h-ft^2-°F) and the value of K_{xx} is 20 Btu/(h-ft-°F)   for carbon steel (Tables 13–2 and 13–3).

$$\frac{AK_{xx}}{L} = \frac{\pi(1 \text{ in.})^2[20 \text{ Btu}/(\text{h-ft-°F})](1 \text{ ft}^2)}{\left(\frac{10 \text{ in.}}{12 \text{ in./ft}}\right)(144 \text{ in}^2)}$$

$$= 0.5236 \text{ Btu}/(\text{h-°F})$$

$$\frac{hPL}{6} = \frac{[10 \text{ Btu}/(\text{h-ft}^2\text{-°F})](2\pi)}{6}\left(\frac{1 \text{ in.}}{12 \text{ in./ft}}\right)\left(\frac{10 \text{ in.}}{12 \text{ in./ft}}\right)$$

$$= 0.7272 \text{ Btu}/(\text{h-°F})$$

$$hT_\infty PL = [10 \text{ Btu}/(\text{h-ft}^2\text{-°F})](10\,°\text{F})(2\pi)\left(\frac{1 \text{ in.}}{12 \text{ in./ft}}\right)\left(\frac{10 \text{ in.}}{12 \text{ in./ft}}\right)$$
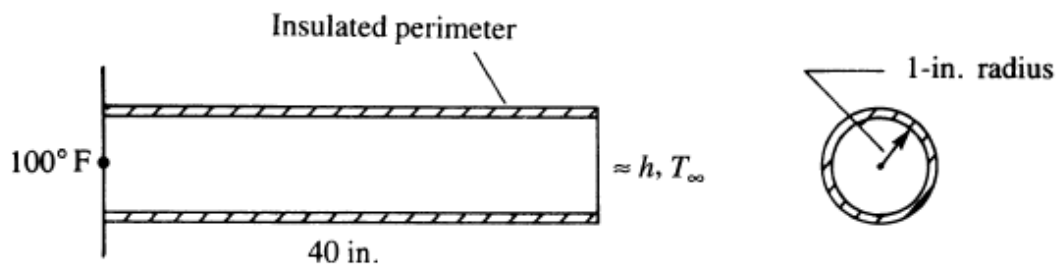
$$= 43.63 \text{ Btu/h}$$

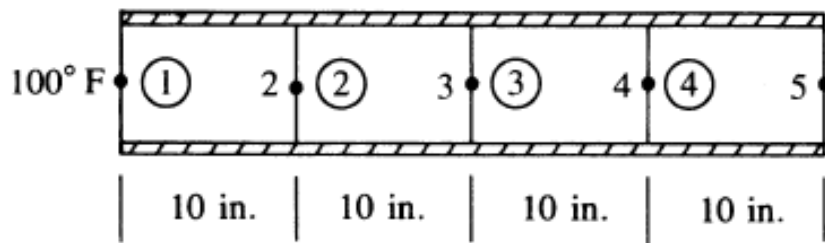**Figure 1: One-dimensional rod subjected to temperature variation**

Figure 2: Finite element discretized rod

In general, from Eqs. (13.4.22) and (13.4.27), we have

$$[k] = \frac{AK_{xx}}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{hPL}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} + \iint\limits_{S_{end}} h[N]^T[N]\,dS \qquad (13.4.34)$$

Substituting Eqs. (13.4.33) into Eq. (13.4.34) for element 1, we have

$$[k^{(1)}] = 0.5236\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \text{ Btu/(h-°F)} \qquad (13.4.35)$$

where the second and third terms on the right side of Eq. (13.4.34) are zero because there are no convection terms associated with element 1. Similarly, for elements 2 and 3, we have

$$[k^{(2)}] = [k^{(3)}] = [k^{(1)}] \qquad (13.4.36)$$

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However, element 4 has an additional (convection) term owing to heat loss from the flat surface at its right end. Hence, using Eq. (13.4.28), we have

$$[k^{(4)}] = [k^{(1)}] + hA \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= 0.5236 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + [10 \text{ Btu}/(\text{h-ft}^2\text{-}°\text{F})]\pi \left(\frac{1 \text{ in.}}{12 \text{ in./ft}}\right)^2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.5236 & -0.5236 \\ -0.5236 & 0.7418 \end{bmatrix} \text{Btu}/(\text{h-}°\text{F}) \qquad (13.4.37)$$

In general, we would use Eqs. (13.4.23)–(13.4.25), and (13.4.29) to obtain the element force matrices. However, in this example, $Q = 0$ (no heat source), $q^* = 0$ (no heat flux), and there is no convection except from the right end. Therefore,

$$\{f^{(1)}\} = \{f^{(2)}\} = \{f^{(3)}\} = 0 \qquad (13.4.38)$$

and

$$\{f^{(4)}\} = hT_\infty A \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$$

$$= [10 \text{ Btu}/(\text{h-ft}^2\text{-}°\text{F})](10 \, °\text{F})\pi \left(\frac{1 \text{ in.}}{12 \text{ in./ft}}\right)^2 \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$$

$$= 2.182 \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \text{Btu/h} \qquad (13.4.39)$$

The assembly of the element stiffness matrices [Eqs. (13.4.35)–(13.4.37)] and the element force matrices [Eqs. (13.4.38) and (13.4.39)], using the direct stiffness method, produces the following system of equations:

$$\begin{bmatrix} 0.5236 & -0.5236 & 0 & 0 & 0 \\ -0.5236 & 1.0472 & -0.5236 & 0 & 0 \\ 0 & -0.5236 & 1.0472 & -0.5236 & 0 \\ 0 & 0 & -0.5236 & 1.0472 & -0.5236 \\ 0 & 0 & 0 & -0.5236 & 0.7418 \end{bmatrix} \begin{Bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ 0 \\ 0 \\ 0 \\ 2.182 \end{Bmatrix}$$

$$(13.4.40)$$

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where F_1 corresponds to an unknown rate of heat flow at node 1 (analogous to an un-known support force in the stress analysis problem). We have a known nodal temper-ature boundary condition of $t_1 = 100\,°F$. This nonhomogeneous boundary condition must be treated in the same manner as was described for the stress analysis problem (see Section 2.5 and Appendix B.4). We modify the stiffness (conduction) matrix and force matrix as follows:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1.0472 & -0.5236 & 0 & 0 \\ 0 & -0.5236 & 1.0472 & -0.5236 & 0 \\ 0 & 0 & -0.5236 & 1.0472 & -0.5236 \\ 0 & 0 & 0 & -0.5236 & 0.7418 \end{bmatrix} \begin{Bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \end{Bmatrix} = \begin{Bmatrix} 100 \\ 52.36 \\ 0 \\ 0 \\ 2.182 \end{Bmatrix} \quad (13.4.41)$$

where the terms in the first row and column of the stiffness matrix corresponding to the known temperature condition, $t_1 = 100\,°F$, have been set equal to 0 except for the main diagonal, which has been set equal to 1, and the first row of the force matrix has been set equal to the known nodal temperature at node 1. Also, the term $(-0.5236) \times (100\,°F) = -52.36$ on the left side of the second equation of Eq. (13.4.40) has been transposed to the right side in the second row (as $+52.36$) of Eq. (13.4.41). The second through fifth equations of Eq. (13.4.41) corresponding to the rows of un-known nodal temperatures can now be solved (typically by Gaussian elimination). The resulting solution is given by

$$t_2 = 85.93\,°F \qquad t_3 = 71.87\,°F \qquad t_4 = 57.81\,°F \qquad t_5 = 43.75\,°F \qquad (13.4.42)$$

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Example 2: To illustrate more fully the use of the equations developed in Section 13.4, we will now solve the heat-transfer problem shown in Figure 13–11. For the one-dimensional rod, determine the temperatures at 3-in. increments along the length of the rod and the rate of heat flow through element 1. Let K=3 Btu/(h-in.-°F), h=1.0 Btu/(h-in² -F),and T_∞= 0 F. The temperature at the left end of the rod is constant at 200 °F.
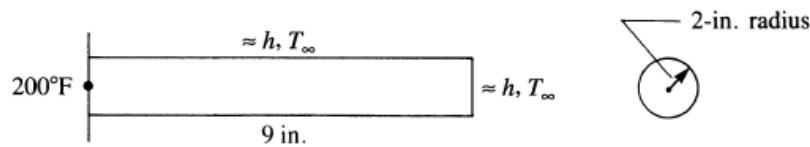

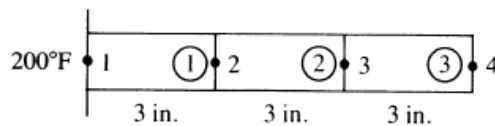**Figure 13–11   One-dimensional rod subjected to temperature variation**

Figure 13–12   Finite element discretized rod of Figure 13–11

The finite element discretization is shown in Figure 3. Three elements are sufficient to enable us to determine temperatures at the four points along the rod,although more elements would yield answers more closely approximating the analytical solution obtained by solving the differential equation such as Eq. (13.2.3) with the partial derivative with respect to time equal to zero. There will be convective heat loss over the perimeter and the right end of the rod. The left end will not have convective heat loss. Using Eqs. (13.4.22) and (13.4.28), we calculate the stiffness matrices for the elements as follows:

$$\frac{AK_{xx}}{L} = \frac{(4\pi)(3)}{3} = 4\pi \ \text{Btu}/(\text{h-}°\text{F})$$

$$\frac{hPL}{6} = \frac{(1)(4\pi)(3)}{6} = 2\pi \ \text{Btu}/(\text{h-}°\text{F}) \tag{13.4.43}$$

$$hA = (1)(4\pi) = 4\pi \ \text{Btu}/(\text{h-}°\text{F})$$

Substituting the results of Eqs. (13.4.43) into Eq. (13.4.22), we obtain the stiffness matrix for element 1 as

$$[k^{(1)}] = 4\pi \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + 2\pi \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$= 4\pi \begin{bmatrix} 2 & -\frac{1}{2} \\ -\frac{1}{2} & 2 \end{bmatrix} \ \text{Btu}/(\text{h-}°\text{F}) \tag{13.4.44}$$

Because there is no convection across the ends of element 1 (its left end has a known temperature and its right end is inside the whole rod and thus not exposed to fluid motion), the contribution to the stiffness matrix owing to convection from an end of the element, such as given by Eq. (13.4.28), is zero. Similarly,

$$[k^{(2)}] = [k^{(1)}] = 4\pi \begin{bmatrix} 2 & -\frac{1}{2} \\ -\frac{1}{2} & 2 \end{bmatrix} \text{Btu/(h-°F)} \qquad (13.4.45)$$

However, element 3 has an additional (convection) term owing to heat loss from the exposed surface at its right end. Therefore, Eq. (13.4.28) yields a contribution to the element 3 stiffness matrix, which is then given by

$$[k^{(3)}] = [k^{(1)}] + hA \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = 4\pi \begin{bmatrix} 2 & -\frac{1}{2} \\ -\frac{1}{2} & 2 \end{bmatrix} + 4\pi \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= 4\pi \begin{bmatrix} 2 & -\frac{1}{2} \\ -\frac{1}{2} & 3 \end{bmatrix} \text{Btu/(h-°F)} \qquad (13.4.46)$$

In general, we calculate the force matrices by using Eqs. (13.4.26) and (13.4.29). Because $Q = 0$, $q^* = 0$, and $T_\infty = 0°F$, all force terms are equal to zero.

The assembly of the element matrices, Eqs. (13.4.44)–(13.4.46), using the direct stiffness method, produces the following system of equations:

$$4\pi \begin{bmatrix} 2 & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 4 & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 4 & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & 3 \end{bmatrix} \begin{Bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \qquad (13.4.47)$$

We have a known nodal temperature boundary condition of $t_1 = 200°F$. As in Example 13.1, we modify the conduction matrix and force matrix as follows:

$$4\pi \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 4 & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & 3 \end{bmatrix} \begin{Bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{Bmatrix} = \begin{Bmatrix} 800\pi \\ 400\pi \\ 0 \\ 0 \end{Bmatrix} \qquad (13.4.48)$$

where the terms in the first row and column of the conduction matrix corresponding to the known temperature condition, $t_1 = 200°F$, have been set equal to zero except for the main diagonal, which has been set to equal one, and the row of the force matrix has been set equal to the known nodal temperature at node 1. That is, the first row force is $(200)(4\pi) = 800\pi$, as we have left the 4π term as a multiplier of the elements inside the stiffness matrix. Also, the term $(-1/2)(200)(4\pi) = -400\pi$ on the left side of the second equation of Eq. (13.4.47) has been transposed to the right side in the second row (as $+400\pi$) of Eq. (13.4.48). The second through fourth equations of Eq. (13.4.48), corresponding to the rows of unknown nodal temperatures, can now be solved. The resulting solution is given by

$$t_2 = 25.4°F \qquad t_3 = 3.24°F \qquad t_4 = 0.54°F \qquad (13.4.49)$$

Next, we determine the heat flux for element 1 by using Eqs. (13.4.6) in (13.4.8) as

$$q^{(1)} = -K_{xx}[B]\{t\} \qquad (13.4.50)$$

Using Eq. (13.4.7) in Eq. (13.4.50), we have

$$q^{(1)} = -K_{xx}\left[-\frac{1}{L} \quad \frac{1}{L}\right]\left\{\begin{matrix} t_1 \\ t_2 \end{matrix}\right\} \qquad (13.4.51)$$

Substituting the numerical values into Eq. (13.4.51), we obtain

$$q^{(1)} = -3\left[-\frac{1}{3} \quad \frac{1}{3}\right]\left\{\begin{matrix} 200 \\ 25.4 \end{matrix}\right\}$$

or

$$q^{(1)} = 174.6 \text{ Btu/(h-in}^2) \qquad (13.4.52)$$

We then determine the rate of heat flow $\bar{q}$ by multiplying Eq. (13.4.52) by the cross-sectional area over which q acts. Therefore,

$$\bar{q}^{(1)} = 174.6(4\pi) = 2194 \text{ Btu/h} \qquad (13.4.53)$$

Here positive heat flow indicates heat flow from node 1 to node 2 (to the right). ∎

Example 3:

The plane wall shown in Figure 13–13 is 1 m thick. The left surface of the wall $(x = 0)$ is maintained at a constant temperature of 200 °C, and the right surface $(x = L = 1 \text{ m})$ is insulated. The thermal conductivity is $K_{xx} = 25$ W/(m · °C) and there is a uniform generation of heat inside the wall of $Q = 400$ W/m^3. Determine the temperature distribution through the wall thickness.
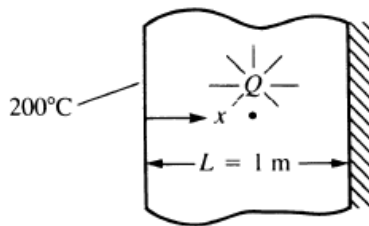

Figure 13–13 Conduction in a plane wall subjected to uniform heat generation
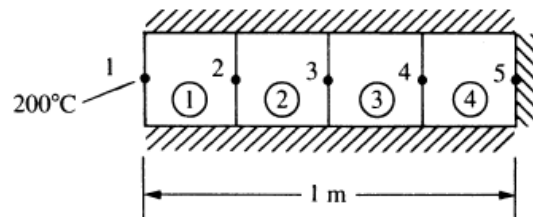
Figure 13–14 Discretized model of Figure 13–13

This problem is assumed to be approximated as a one-dimensional heat-transfer problem. The discretized model of the wall is shown in Figure 13–14. For simplicity, we use four equal-length elements all with unit cross-sectional area $(A = 1 \text{ m}^2)$. The unit area represents a typical cross section of the wall. The perimeter of the wall model is then insulated to obtain the correct conditions.

Using Eqs. (13.4.22) and (13.4.28), we calculate the element stiffness matrices as follows:

$$\frac{AK_{xx}}{L} = \frac{(1 \text{ m}^2)[25 \text{ W}/(\text{m} \cdot °\text{C})]}{0.25 \text{ m}} = 100 \text{ W}/°\text{C}$$

For each identical element, we have

$$[k] = 100 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \text{ W}/°\text{C} \tag{13.4.54}$$

Because no convection occurs, h is equal to zero; therefore, there is no convection contribution to $\underline{k}$.

The element force matrices are given by Eq. (13.4.26). With $Q = 400$ W/m^3, $q = 0$, and $h = 0$, Eq. (13.4.26) becomes

$$\{f\} = \frac{QAL}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \tag{13.4.55}$$

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Evaluating Eq. (13.4.55) for a typical element, such as element 1, we obtain

$$\left\{ \begin{matrix} f_{1x} \\ f_{2x} \end{matrix} \right\} = \frac{(400 \text{ W/m}^3)(1 \text{ m}^2)(0.25 \text{ m})}{2} \left\{ \begin{matrix} 1 \\ 1 \end{matrix} \right\} = \left\{ \begin{matrix} 50 \\ 50 \end{matrix} \right\} \text{ W} \qquad (13.4.56)$$

The force matrices for all other elements are equal to Eq. (13.4.56).

The assemblage of the element matrices, Eqs. (13.4.54) and (13.4.56) and the other force matrices similar to Eq. (13.4.56), yields

$$100 \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \left\{ \begin{matrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \end{matrix} \right\} = \left\{ \begin{matrix} F_1 + 50 \\ 100 \\ 100 \\ 100 \\ 50 \end{matrix} \right\} \qquad (13.4.57)$$

Substituting the known temperature $t_1 = 200\,°C$ into Eq. (13.4.57), dividing both sides of Eq. (13.4.57) by 100, and transposing known terms to the right side, we have

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \left\{ \begin{matrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \end{matrix} \right\} = \left\{ \begin{matrix} 200\,°C \\ 201 \\ 1 \\ 1 \\ 0.5 \end{matrix} \right\} \qquad (13.4.58)$$

The second through fifth equations of Eq. (13.4.58) can now be solved simultaneously to yield

$$t_2 = 203.5\,°C \qquad t_3 = 206\,°C \qquad t_4 = 207.5\,°C \qquad t_5 = 208\,°C \qquad (13.4.59)$$

Using the first of Eqs. (13.4.57) yields the rate of heat flow out the left end:

$$F_1 = 100(t_1 - t_2) - 50$$

$$F_1 = 100(200 - 203.5) - 50$$

$$F_1 = -400 \text{ W}$$

The closed-form solution of the differential equation for conduction, Eq. (13.1.9), with the left-end boundary condition given by Eq. (13.1.10) and the right-end boundary condition given by Eq. (13.1.11), and with $q_x^* = 0$, is shown in Reference [2] to yield a parabolic temperature distribution through the wall. Evaluating the expression for the temperature function given in Reference [2] for values of x corresponding to the node points of the finite element model, we obtain

$$t_2 = 203.5\,°C \qquad t_3 = 206\,°C \qquad t_4 = 207.5\,°C \qquad t_5 = 208\,°C \qquad (13.4.60)$$

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Figure 13–15 is a plot of the closed-form solution and the finite element solution for the temperature variation through the wall. The finite element nodal values and the closed-form values are equal, because the consistent equivalent force matrix has been used. (This was also discussed in Sections 3.10 and 3.11 for the axial bar subjected to distributed loading, and in Section 4.5 for the beam subjected to distributed loading.) However, recall that the finite element model predicts a linear temperature distribution within each element as indicated by the straight lines connecting the nodal temperature values in Figure 13–15.
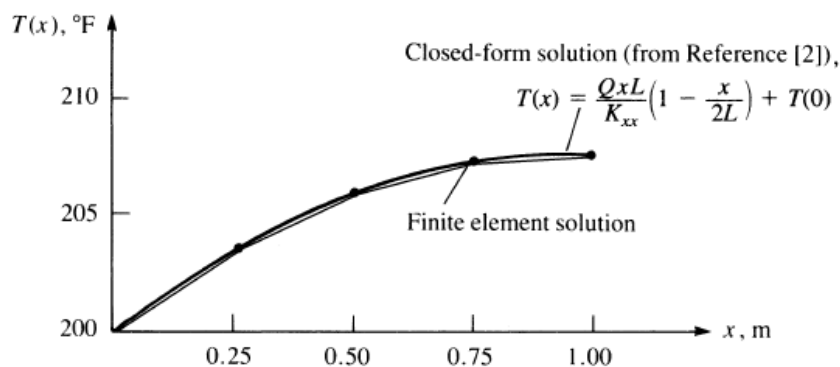

Closed-form solution (from Reference [2]),

$$T(x) = \frac{QxL}{K_{xx}}\left(1 - \frac{x}{2L}\right) + T(0)$$

Finite element solution

Figure 13–15 Comparison of the finite element and closed-form solutions for Example 13.3

∎

[2] Kreith, F., and Black, W. Z., *Basic Heat Transfer*, Harper & Row, New York, 1980.