Lecture Seven
Fluid in Rigid-Body Motion.

1- Acceleration on a Straight Path.

The general equation of motion for fluid that acts as a rigid body (no shear stresses) is determined to be ***Rigid-Body*** motion of fluids.

$\vec{\nabla} p + \rho g \vec{k} = -\rho \vec{a}$ From Eq. 2.12 in [1] where the viscous term vanishes identically, and p depends only upon the terms $\boldsymbol{\rho g}$ & $\boldsymbol{\rho a}$.

Resolving the vectors into their components,

$$\frac{\partial p}{\partial x}\vec{i} + \frac{\partial p}{\partial y}\vec{j} + \frac{\partial p}{\partial z}\vec{k} + \rho g \vec{k} = -\rho\left(a_x \vec{i} + a_y \vec{j} + a_z \vec{k}\right)$$

Accelerating fluids

$$\frac{\partial p}{\partial x} = -\rho a_x; \quad \frac{\partial p}{\partial y} = -\rho a_y; \quad \frac{\partial p}{\partial z} = -\rho(g + a_z) \qquad (1)$$

When the fluid is at rest

$$\frac{\partial p}{\partial x} = 0; \quad \frac{\partial p}{\partial y} = 0; \quad \frac{\partial p}{\partial z} = -\rho g$$

Considering the container is moving on a straight path with constant acceleration as shown in Fig.1, the ***x&z*** components of acceleration are $\boldsymbol{a_x}$&$\boldsymbol{a_z}$, there is no movement in the y-direction and $\boldsymbol{a_y}$=0. Then the equation of motion for accelerating fluids Eq.1 reduce to

$$\frac{\partial p}{\partial x} = -\rho a_x, \quad \frac{\partial p}{\partial y} = 0, and \quad \frac{\partial p}{\partial z} = -\rho(g + a_z)$$

Then p=p(x,z), which is

$$dp = \left(\frac{\partial p}{\partial x}\right) dx + \left(\frac{\partial p}{\partial z}\right) dz$$

$$dp = -\rho a_x dx - \rho(g + a_z) dz \qquad (2)$$

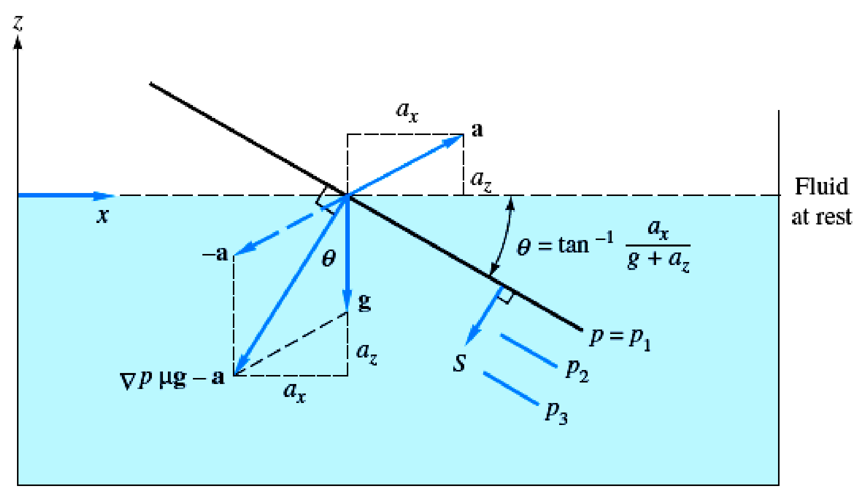


Figure 1: Liquid in rigid –body acceleration with constant pressure surface.

By integration Eq. 2 along a line of constant pressure between pressure point 1 and 2 when ρ is constant as in Fig. 2.

$$p_2 - p_1 = -\rho a_x (x_2 - x_1) - \rho(g + a_z)(z_2 - z_2) \qquad (3)$$

Taking point ***1*** to be origin where x=0 and z=0, since the pressure is atmosphere pressure p_o. The vertical rise or drop of the free surface at point ***2*** relative to point ***1*** can be determine by choosing both ***(1&2)*** on the free surface (so that $p_1=p_2$). Solving Eq. 3 for (z_2-z_1).

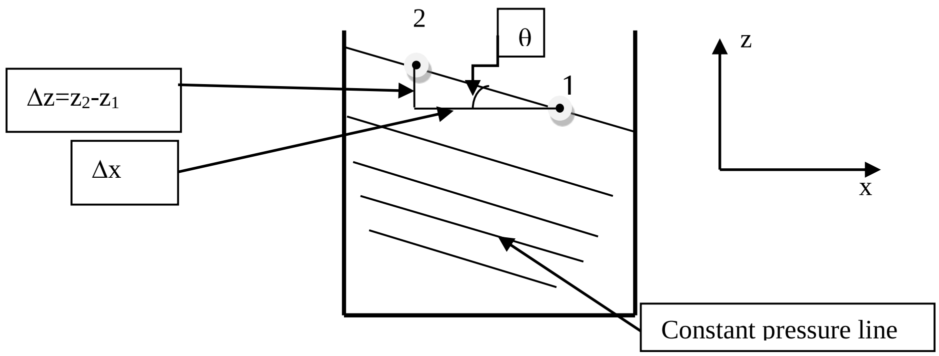


Figure 2: Linear acceleration of a liquid with a free surface.

$$\Delta z = z_2 - z_1 = -\frac{a_x}{g+a_z}(x_2 - x_1) \qquad (4)$$

Eq. 4 is the equation of constant pressure line, called (isobars) obtained from Eq. 2. From Fig. 2, the line of constant pressure

$$\frac{dz}{dx} = -\frac{a_x}{g+a_z} = constant$$

Slop of isobars is $slop = \frac{dz}{dx} = -\frac{a_x}{g+a_z} = -tan\theta$

$$tan\theta = \frac{a_x}{g+a_z} \qquad (5)$$

If $\boldsymbol{a_z=0}$ then $\tan\theta = \frac{a_x}{g}$

Ex.1

An ***80 cm*** high fish tank of cross section (***2m*0.6m***) which is initially filled with water is to be transported on the back of a track. The track accelerates from ***0 to 90 km/hr*** in ***10 s***. If it's desired that no water spills during acceleration, determine the allowable initial water height in the tank. Would you recommend the tank to be aligned with the long or short side parallel to the direction of motion?

Sol.

The road is horizontal during acceleration $\therefore a_z = 0$

$$a_x = \frac{\Delta V}{\Delta t} = \left(\frac{90-0}{10}\right)\left(\frac{1}{3.6}\right) = 2.5\ m/s^2$$

$$\tan\theta = \frac{a_x}{g+a_z} = \frac{2.5}{9.81+0} = 0.255 \qquad \therefore \theta = 14.3°$$

Then the vertical rise at the back of the tank relative to the mid-plane for two possible orientations as in figure becomes

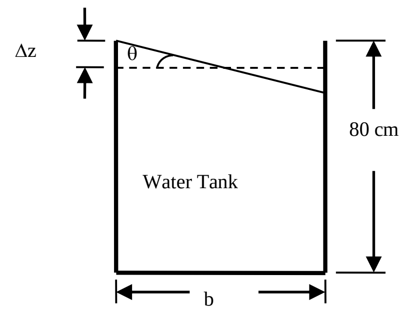
Case-1: The long side is parallel to the direction of motion

$$\Delta z_1 = \left(\frac{b_1}{2}\right)\tan\theta = \left(\frac{2}{2}\right) \times 0.255 = 0.255m = 25.5cm$$

Case-2: The short side is parallel to the direction of motion

$$\Delta z_2 = \left(\frac{b_2}{2}\right)\tan\theta = \left(\frac{0.6}{2}\right) \times 0.255 = 0.076m = 7.6cm$$

The tank should definitely be oriented such that its short side is parallel to the direction of motion, the tank such that its free surface level drop just 7.6 cm, then the initial high becomes = 80-7.6=***72.4cm.***



2- Rotation in a Cylindrical Container.

This problem is best analyzed in cylindrical coordinates ***(r,θ,z)***.the centripetal acceleration of a fluid particle rotating with a constant ***ω*** at a distance ***r*** from the axis of rotation is $a_r = -r\omega^2$ (Directed radially toward the axis of rotation) symmetry about z-axis (axis of rotation) and thus there is no ***θ*** dependence as shown in Fig. 3.

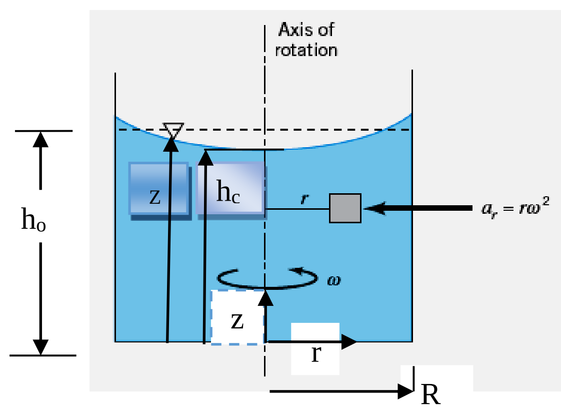


Figure 3: Rigid-body rotation of a liquid in a tank.

p=p(r,z); a_θ=0 and ***a_z=0*** (no motion in z-direction). Equation of motion for rotating fluids reduce to

$$\frac{\partial p}{\partial r} = -\rho a_r = \rho r\omega^2,\ \frac{\partial p}{\partial \theta} = 0\ \ and\ \ \frac{\partial p}{\partial z} = -\rho g \tag{6}$$

Then the total differential of ***p=p(r,z)***, which is

$$dp = \left(\frac{\partial p}{\partial r}\right) dr + \left(\frac{\partial p}{\partial z}\right) dz$$, becomes

$$dp = \rho r\omega^2 dr - \rho g dz \tag{7}$$

The equation for surfaces of constant pressure is obtained by setting ***dp=0*** and replace ***z*** by ***z_{isobar}***, which is the value of the surface as function of ***r***, it gives

$$\frac{dz_{isobar}}{dr} = \frac{r\omega^2}{g}$$

By integration the equation for the surface of constant pressure is determined to be

$$z_{isobar} = \frac{\omega^2}{2g} r^2 + C_1 \tag{8}$$

Eq. 8 is the equation of parabola. For each isobar surface there is C_1 different. For free surface setting ***r=0*** gives ***$z_{isobar}(0)=C_1=h_c$***. Then Eq. 8 for free surface becomes

$$z_s = \frac{\omega^2}{2g} r^2 + h_c \tag{9}$$

Where $\boldsymbol{z_s}$ is the distance of the free surface. The volume of a cylindrical shell element of radius ***r***, height z_s and thickness ***dr*** is

$$d\forall = 2\pi r z_s dr$$

Then the volume of the parabola formed by the free surface is

$$\forall = \int_{r=0}^{R} 2\pi r z_s dr = 2\pi \int_{r=0}^{R} \left(\frac{\omega^2}{2g} r^2 + h_c\right) r dr$$

$$\forall = \pi R^2 \left(\frac{\omega^2}{4g} R^2 + h_c\right) \qquad (10)$$

Original volume in the container is

$$\forall_o = \pi R^2 h_o$$

Where $\boldsymbol{h_o}$ is the original height, setting these two volumes equal to each other,

$$\pi R^2 h_o = \pi R^2 \left(\frac{\omega^2 R^2}{4g} + h_c\right)$$

$$h_c = h_o - \frac{\omega^2 R^2}{4g}$$

Then Eq. 9 for free surface becomes

$$z_s = h_o - \frac{\omega^2}{4g}(R^2 - 2r^2) \qquad (11)$$

The maximum height difference is $\Delta z_{s\ max.}$ is at r=R & r=0

$$\Delta z_{s,max} = z_s(R) - z_s(0) = \left(h_o - \frac{\omega^2}{4g}(-R^2)\right) - \left(h_o - \frac{\omega^2}{4g} R^2\right)$$

$$\Delta z_{s,max} = \frac{\omega^2}{4g} R^2 + \frac{\omega^2}{4g} R^2 = \frac{\omega^2}{2g} R^2 \qquad (12)$$

For ρ is constant, the pressure difference is determined by integration Eq. 7 between two point (1&2) as follows

$$p_2 - p_1 = \frac{\rho\omega^2}{2}(r_2^2 - r_1^2) - \rho g(z_2 - z_1)$$

Taking point ***1*** to be the origin ***(r=0, z=0)*** where the pressure is $\boldsymbol{p_o}$ and point ***2*** to be any point in the fluid, the pressure variation is given by

$$p = p_o + \frac{\rho\omega^2}{2} r^2 - \rho g z \qquad (13)$$

The pressure is linear in ***z*** and parabolic in ***r***.

<u>Ex.2</u>

A ***20 cm*** diameter, ***60 cm*** high vertical cylinder container shown in figure, is partially filled with ***50 cm*** high liquid whose density is ***850 kg/m³***. Now the cylinder is rotated at a constant speed. Determine the rotational speed at which the liquid will start spilling from the edges of the container.

<u>Sol.</u>

From Eq. 11

$$z_s = h_o - \frac{\omega^2}{4g}(R^2 - 2r^2)$$

Taking ***z = 0*** at ***r = 0,*** then the vertical height of the liquid at the edge of container at ***r = R*** becomes

$$z_s(R) = h_o + \frac{\omega^2}{4g} R^2\ ;\ h_o = 0.5m$$

The height of the liquid at edge of the container equals the height of the container, and thus $z_s(R) = 0.6m$ solving for $\boldsymbol{\omega}$

$$\omega = \sqrt{\frac{4g[z_s(R) - h_o]}{R^2}} = \sqrt{\frac{4 \times 9.81 \times (0.6 - 0.5)}{0.1^2}} = 19.8\, rad/s$$

$\omega = \frac{2\pi n}{60} \quad \rightarrow n = \frac{\omega \times 60}{2\pi} = 189\, rpm.$

The liquid height at the center is

$z_s(0) = h_o - \frac{\omega^2}{4g} R^2 = 0.4\, m$