Lecture Five
Hydrostatic Forces

1- Hydrostatic Forces on Submerged Plane Surface.

Any hydro structure design required a computation of the hydrostatic forces on various solid surfaces contact with fluid. We wish to determine the direction, location and magnitude of the resultant force acting on one side of the surface area due to the liquid in contact. The force acting on ***dA*** (differential area) is $dF = \gamma h\, dA$ which is perpendicular to the surface as shown in Fig. 1 . The magnitude of the resultant force can be found by summing or integrating this differential force over the entire surface

$$F_R = \int_A \gamma\, h\, dA = \int_A \gamma\, y \sin\theta\, dA$$

Where h =y*sinθ, for constant γ and θ

$$F_R = \gamma\, sin\theta \int_A y\, dA \qquad (1)$$

The integral appearing in above equation is the first moment of the area with respect to the x-axis, so we can write

$$\int_A y dA = y_{cg} A$$

y_{cg} is the y- coordinate of the centroid was measured from the x-axis passes through ***cg*** (*the center of gravity*)

$F_R = \gamma\, A\, y_{cg}\, sin\theta$

Or more simply as

$$F_R = \gamma\, h_{cg}\, A \qquad (2)$$

Note, the magnitude of the force is independent of (θ) and depends only on the specific weight, the total area, and the centroid of the area.

Where (h_{cg}) is the vertical distance from the fluid surface to the centroid of the area, but the resultant force is not actually pass through the centroid area. Its line of action passes through the *center of pressure* (***cp***).

The y-coordinate, (y_{cp}), of the resultant force can be determined by summation of moments around the x-axis, that is, the moment of the resultant force must equal the moment of the distributed pressure force or

$F_R\, y_{cp} = \int_A y\, dF = \int_A \gamma\, sin\theta\, y^2 dA$

And therefore, since $F_R = \gamma\, A\, y_{cg}\, sin\theta$

$$y_{cp} = \frac{\int_A y^2\, dA}{y_{cg}\, A}$$

The integral in the numerator is the second moment of the area (moment of inertia), I_x, thus, we can write

$$y_{cp} = \frac{I_x}{y_{cg} A} = \frac{I_x\, sin\theta}{h_{cg}\, A} \qquad (3)$$

From the parallel axis theorem can express I_x, as $I_x = I_{xcg} + A y_{cg}^2$, where I_{xcg} is the second moment of the area with respect to an axis passing through its centroid and parallel to the x-axis. Thus, from Eq. 3

$$y_{cp} = \frac{I_{xcg}}{y_{cg} A} + y_{cg} = \frac{I_{xcg} sin\theta}{h_{cg} A} + y_{cg} \qquad (4)$$

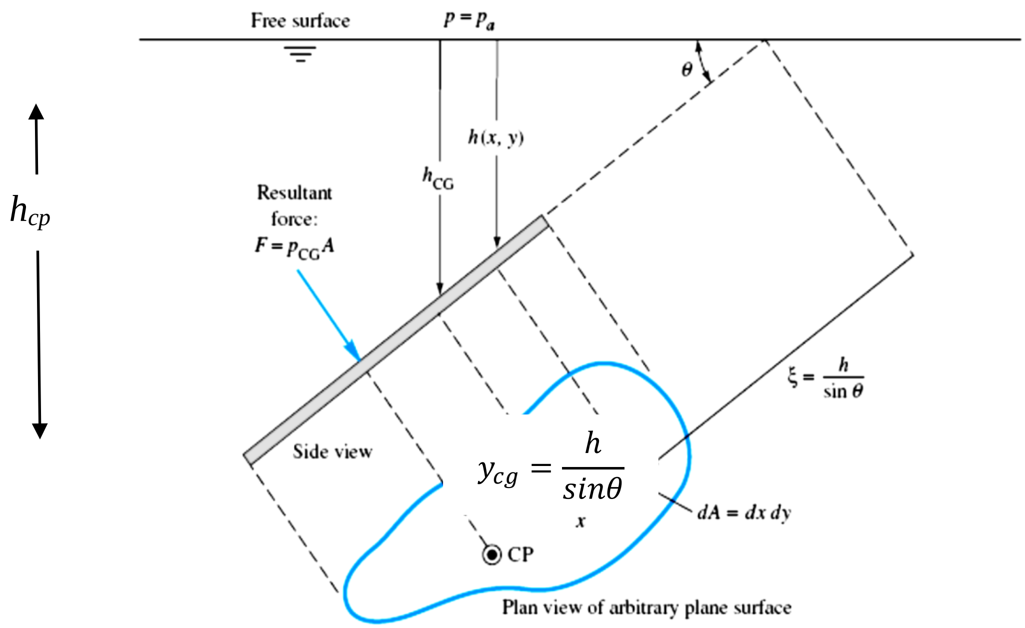

Figure 1: Notation for hydro static force on an inclined plane surface of arbitrary shape.

Since $sin\,\theta = \frac{h_{cg}}{y_{cg}}$. From Eq. 2.33 $\frac{I_{xcg}}{y_{cg}A} > 0$, then the resultant force does not pass through the centroid but is always below it.

From Fig. 1 $h_{cp} = y_{cp}sin\theta$. Know, from Eq. 4

$$h_{cp} = \frac{I_{xcg}sin\theta^2}{h_{cg}A} + h_{cg} \qquad (5)$$

If $\theta=90°$, then $sin\theta^2=1$. *i.e.* the surface is vertical. The center of pressure for immersed vertical surface becomes as follows,

$$h_{cp} = \frac{I_{xcg}}{h_{cg}A} + h_{cg} \qquad (6)$$

The x-coordinate, $\boldsymbol{x_{cp}}$, for the resultant force can be determined as follows

$$F_R x_{cp} = \int_A x\,dF = \int \gamma \sin\theta\, xy\, dA$$

Since $dF = \gamma\sin\theta\, y\, dA$, $F_R = \gamma A y_{cg}\sin\theta$

Therefore $x_{cp} = \frac{\int_A xy\,dA}{y_{cg}A} = \frac{I_{xy}}{y_{cg}A}$ (7)

Where $I_{xy} = I_{xycg} + A\, x_{cg}y_{cg}$ is the product of inertia with respect to the ***x&y*** axes. From parallel-theorem

$$x_{cp} = \frac{I_{xyc}}{y_{cg}A} + x_{cg} \qquad (8)$$

Summary:-

To find net hydrostatic force on a plane surface:

1- Find area in contact with fluid
2- Locate controid (cg) of that area.
3- Find hydrostatic pressure p_{cg} at controid = $\gamma\, h_{cg}$
4- Find force $F = p_{cg}.A$

5- Location will not be at (c.g.) but at a distance y_{cp} below centroid.

Ex.1

The 4m-diameter circular gate is located in the inclined wall of a large reservoir containing water ***(γ = 9.81 kN/m³)*** as show in Fig.2. The gate is mounted on a shaft along its horizontal diameter. For water depth of (***10 m***) above the shaft determine:

a) The magnitude and location of the resultant force exerted on the gate by the water.
b) The moment that would have to be applied to the shaft to open the gate.

Sol.

The resultant force

$F_R = \gamma h_{cg} A$

$F_R = (9.81 * 10^3) \quad (10)(4\pi) = 1230 * 10^3 \; N = 1.23 \; MN$

The location of F_R is at the center of pressure

$x_{cp} = \frac{I_{xyc}}{y_c A} + x_c$

$x_{cp} = 0$ since the area is symmetrical and the (c.p.) must lie along the (A-A) to obtain y_{cp}

$y_{cp} = \frac{I_{xc}}{y_{cg} A} + y_{cg}$

$I_{xc} = \frac{\pi R^4}{4} \quad , \quad y_{cg} = \frac{hcg}{sin60} \, , \quad y_{cg} = \frac{10}{sin60}$

$y_{cp} = \frac{\left(\frac{\pi}{4}\right)(2m)^4}{\left(\frac{10}{sin60}\right)(4\pi)} + \frac{10}{sin60} = 0.0866 + 11.547 = 11.63m$

The distance below the shaft (along the gate) to the (***cp***)

$y_{cp} - y_{cg} = 0.0866 \, m \quad Ans(a)$

O_x & O_y are the horizontal and vertical reactions of the shaft on the gate, from the sum moments about the shaft

$\sum M_c = 0$

$M = F_R \left(y_{cp} - y_{cg}\right)$

$=(1230*10^3)(0.0866) = 1.07 * 10^5$ N.m

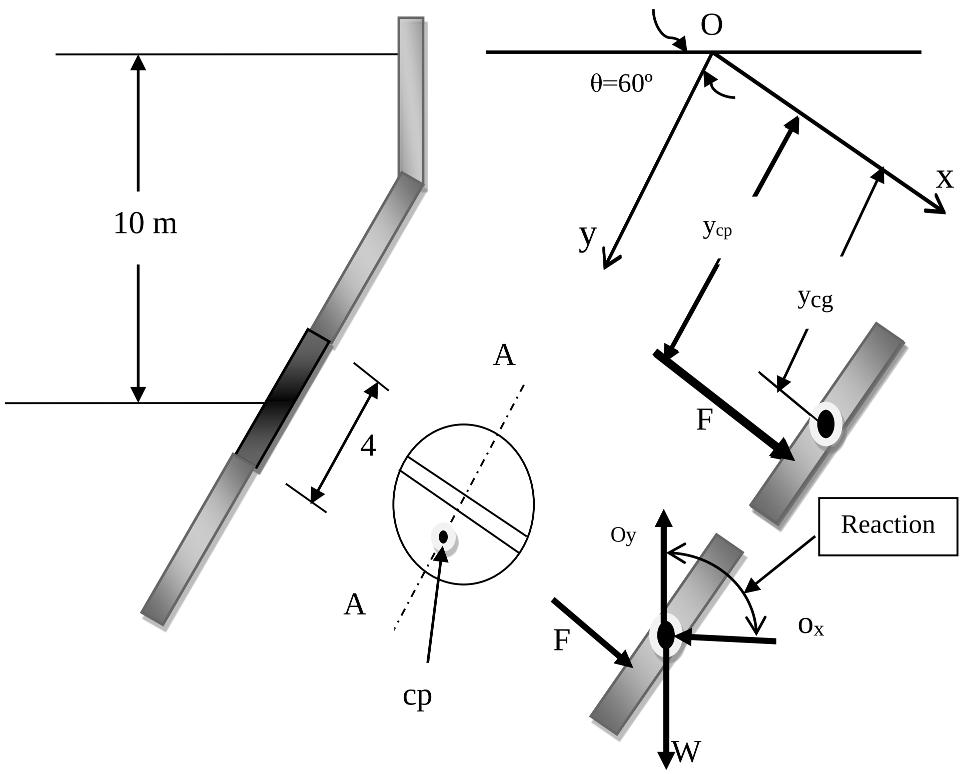

Figure 2

2- *Hydrostatic Forces on Curved Surface.*

Consider the curved section ***ab*** of the open tank as in Fig. 3. We wish to find the resultant force acting on this section with unit length perpendicular to the plane of the paper. The horizontal plane surface ***bc*** and the vertical plane surface ***ac*** are the projection areas of the curved surface ***ab***. F_h & F_v are the forces components that the tank exerts on the fluid. γ is the specific weight of the fluid times the enclosed volume acts through (***cg***), then,

i) Vertical forces F_v: the vertical force on a curved surface is given by the weight of the liquid enclosed by the surface and the vertical force acts on horizontal free surface of the liquid. The force acts along the center of gravity of the volume.

ii) Horizontal forces F_h: the horizontal force equals the force on the projected area of the curved surface and acts at the center of pressure of the projected area.

$F_h = F_2$

$F_v = F_1 + W$

The magnitude of the resultant is obtained from the following equation

$$F_R = \sqrt{(F_h)^2 + (F_v)^2} \qquad (2.38)$$

The direction of F_R is obtained from the following relation

$$\theta = \tan^{-1}\left(\frac{F_h}{F_v}\right) \qquad (2.39)$$

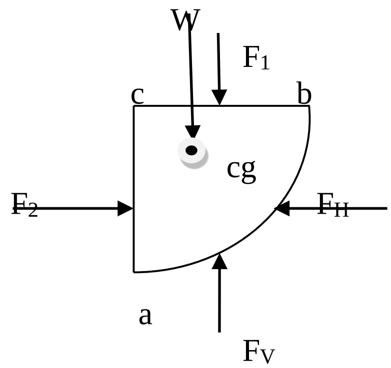

Figure 3: Hydrostatic forces on a curved surface

Ex.2

Determine the resultant force exerted by sea water ***S.G.=1.025*** on the curved ***AB*** of an oil tanker as shown in Fig.4. Also determine the direction of action of the force. Consider 1m width perpendicular to paper.

Sol.

$F_h = \gamma * A * h_{cg} = (1025 * 9.81) * (4 * 1)(15 + 4/2) = 683757\ N$

From Eq. 2.35, Line of action of horizontal force

$h_{cp} = \frac{I_{xc} sin\theta}{h_{cg} A} + h_{cg} = \left[\frac{1\times4^3}{12}\right]\left[\frac{1}{(17\times4\times1)}\right] + 17 = 17.0784m$ from top towards left, The vertical force is due to the volume of sea water displaced

$F_v = [\forall_{BCDE} + \forall_{ABE}]\gamma = [(15 \times 4 \times 1) + (4^2 \times \pi \times 1/4)][1025 \times 9.81] = 729673N$ acts upwards.

To find the location of vertical force which acts at x_{cg} in x-direction;

$\boldsymbol{x_{cg1}}$ of column area *BCDE* is in the vertical plane (*2 m*) from edge.

$\boldsymbol{x_{cg2}}$ of the area *ABE* =(4-4R/3π) = 2.302 m from edge. Taking moments of the area about the edge, the line of action of vertical force is

$X_{cg} =[(x_{cg1}*A_1)+(x_{cg2}*A_2)]/[A_1+A_2]=[2*(15*4) + (2.302*4^2*\pi/4)] /[(15*4)+(4^2\pi/4)]$ = ***2.0523 m*** from the edge.

The resultant force is

$F_R = \sqrt{(F_h)^2 + (F_v)^2} = \sqrt{(683757)^2 + (729673)^2} = 999973\ N$

The direction of action to the vertical is,

$\tan\theta = \frac{F_h}{F_v} = \frac{683757}{729673} = 0.937 \qquad \therefore\ \theta = 43.14°$

Figure 4