Numerical Analysis2

First Lecture

For solving a linear system of n equations in n variable. Such a system has the form

$$
\begin{aligned}
&E_1 : a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\
&E_2 : a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\
&. \quad . \quad \quad . \quad \quad \quad \quad \quad . \\
&. \quad . \quad \quad \quad \quad \quad \quad \quad \quad . \\
&. \quad . \quad \quad \quad \quad \quad \quad \quad \quad . \\
&E_n : a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = b_n
\end{aligned}
$$

In this system we are given the constants a_{ij} $for\, i, \mathrm{j} = 1, 2, \ldots, \mathrm{n}$ and $b_i = 1, 2, \ldots, n$ and we need for solving system of this type. Direct methods find the solution in a finite number of steps or iterative methods start with an arbitrary first approximation to x and then improve this estimate in an infinite but convergent sequence of steps.

1-Direct methods:

1-1Gauss elimination method:

Gauss elimination method is used to solve a system of linear equation by transforming it into an upper triangular system. (i.e. one in which all of the coefficients below the lading diagonal are zero) using elementary row operations. The solution of the upper triangular system. Then found using back substitution. We will describe the method in detail in the example down .

Example 1: The four equations

$$\begin{aligned}
&E_1: x_1 + x_2 + 0 + 3x_4 = 4 \\
&E_2: 2x_1 + x_2 - x_3 + x_4 = 1 \\
&E_3: 3x_1 - x_2 - x_3 + 2x_4 = -3 \\
&E_4: -x_1 + 2x_2 + 3x_3 - x_4 = -3
\end{aligned}$$

Will be solved for x_1, x_2, x_3 *and* x_4. We first use E_1 to eliminate the unknown E_2, E_3 *and* E_4 by performing $(E_2 - 2E_1) \to E_2, (E_3 - 3E_1) \to E_3$ *and* $(E_4 + E_1) \to E_4$. For example, in the second equation

$$(E_2 - 2E_1) \to E_2$$

Which simplifies to the result shown as E_2 in

$$\begin{aligned}
&E_1: x_1 + x_2 + 0 + 3x_n = 4 \\
&E_2: \quad -x_2 - x_3 - 5x_4 = -7 \\
&E_3: \quad -4x_2 - x_3 - 7x_4 = -15 \\
&E_4: \quad 3x_2 + 3x_3 + 2x_4 = 8
\end{aligned}$$

For simplicity, the new equations are again labelled E_1, E_2, E_3 *and* E_4 . In the new system, E_2 is used to eliminate the unknown x_2 from E_3 *and* E_4 by performing

$(E_3 - 4E_2) \to E_3$ *and* $(E_4 + 3E_2) \to E_4$ This results in

$$\begin{aligned}
&E_1: x_1 + x_2 + 0 + 3x_n = 4 \\
&E_2: \quad -x_2 - x_3 - 5x_4 = -7 \\
&E_3: \quad 3x_3 + 13x_4 = 13 \\
&E_4: \quad -13x_4 = -13
\end{aligned} \qquad (*)$$

The system (*) is now in triangular from and can be solved for the unknown by a backward – substitution process. Since E_4 implies $x_4 = 1$,we can solve

$$E_3 \ \ for \ \ x_3 \ \ to\, given \ \ x_3 = \frac{1}{3}(13-13x_4) = \frac{1}{3}(13-13) = 0$$

Continuing, E_2 given $x_2 = -(-7+5x_4+x_3) = -(-7+5+0) = 2$ and E_1 given

$x_1 = 4-3x_4-x_2 - = 4-3-2 = -1$.

The solution to system (*), and consequently to system (1) is therefor

$x_1 = -1, x_2 = 2,\ x_3 = 0 \ and \ \ x_4 = 1$

1-1-1 Gauss elimination method with partial pivoting.

To reduce round-off error, it is often necessary to perform row interchanges even when the pivot element is not zero. If one of the pivot element is $a_{ij}\ \ 0$. This row interchange has the form $E_i \leftrightarrow E_k$.

pivoting is performed by selecting an element a_{ij} with a larger magnitude as the pivot, and interchanging the kth and pth rows. This can be followed by the interchange of the kth and qth columns if necessary.

Example2:

Apply Gauss elimination to the system

$$E_1 : 0.003000x_1 + 59.14x_2 = 59.17$$
$$E_2 : 5.29x_1 - 6.130x_2 = 46.78$$

Using partial pivoting and four-digit arithmetic with rounding , and compare the results to the exact solution

$$x_1 = 10.00 \ and \ x_2 = 1.$$

Solution

The partial –pivoting procedure first requires finding max

$$\left\{ | a_{11}^{(1)} |, | a_{21}^{(1)} | \right\} = \max \left\{ | 0.00300 |, | 5.291 | \right\} = \left\{ | a_{21}^{(1)} | \right\}$$

This requires that the operation $(E_2) \leftrightarrow (E_1)$ be performed to produce the equivalent system

$$E_1 : 5.29x_1 - 6.130x_2 = 46.78$$
$$E_2 : 0.003000x_1 + 59.14x_2 = 59.17$$

The multiplier for this system is

$$m_{21} = \frac{a_{21}^{(1)}}{a_{11}^{(1)}} = 0.0005670$$

And the operation $(E_2 - m_{21}E_1) \rightarrow E_2$ reduces the system to

$$E_1 : 5.29x_1 - 6.130x_2 = 46.78$$
$$E_2 : \quad 0 \quad + 59.14x_2 = 59.17$$

Homework

Example 3

Use the G.E. method to solve the following linear system , if possible, and determined whether row interchanges are necessary.

(Exact solution

a) $x_1 = 1.1875, x_2 = 1.8125$ *and* $x_3 = 0.875$ with one row interchange required

b) $x_1 = -1, x_2 = 0$ *and* $x_3 = 1$ with no interchange required)

a)
$$\begin{aligned} E_1 &: x_1 - x_2 + 3x_3 = 2 \\ E_2 &: 3x_1 - 3x_2 + x_3 = -1 \\ E_3 &: x_1 + x_2 \qquad = 3 \end{aligned}$$

b)
$$\begin{aligned} E_1 &: 2x_1 - 1.5x_2 + 3x_3 = 1 \\ E_2 &: -x_1 \qquad + 2x_3 = 3 \\ E_3 &: 4x_1 - 4.5x_2 + 5x_3 = 1 \end{aligned}$$

Example 4

Use the G.E. method with backward substitution and two-digit rounding arithmetic to solve the following linear system . Don't reader the equation

(Exact solution

a) $x_1 = 1, x_2 = -1$ *and* $x_3 = 3$

a)
$$\begin{aligned} E_1 &: 4x_1 - x_2 + 3x_3 = 8 \\ E_2 &: 2x_1 + 5x_2 + 2x_3 = 3 \\ E_3 &: x_1 + 2x_2 + 4x_3 = 11 \end{aligned}$$

b)
$$\begin{aligned} E_1 &: 4x_1 + x_2 + 2x_3 = 9 \\ E_2 &: 2x_1 + 4x_2 - x_3 = -5 \\ E_3 &: x_1 + x_2 - 3x_3 = -9 \end{aligned}$$

Second Lecture

2-Gauss Jordan method:

The general idea of this method is to changes the coefficients down the diagonal to zero (called upper-triangular form), as well as change the coefficients above the diagonal to zero (called Lower-triangular form), by the same steps used in the G.E. method .At last, we obtain the solution of system without using process backward substitution.

Example 5

Use Gauss-Jorden method and two –digit chopping arithmetic to solve the linear system.

$$\begin{aligned} 2x_1 + 3x_2 - x_3 &= 5 \\ 4x_1 + 4x_2 - 3x_3 &= 3 \\ -x_1 + 3x_2 - x_3 &= 1 \end{aligned}$$

Solution

$$[A:B] = \left[\begin{array}{ccc|c} 2 & 3 & -1 & 5 \\ 4 & 4 & -3 & 3 \\ -2 & 3 & -1 & 1 \end{array}\right] \xrightarrow[r_1+r_2]{-2r_1+r_2} \left[\begin{array}{ccc|c} 2 & 3 & -1 & 5 \\ 0 & -2 & -1 & -7 \\ 0 & 6 & -2 & 6 \end{array}\right]$$

$$\xrightarrow{3r_2+2r_1} \left[\begin{array}{ccc|c} 4 & 0 & -5 & -11 \\ 0 & -2 & -1 & -7 \\ 0 & 0 & -5 & -15 \end{array}\right] \xrightarrow[-5r_2+r_3]{-r_3+r_1} \left[\begin{array}{ccc|c} 4 & 0 & 0 & 4 \\ 0 & 10 & 0 & 20 \\ 0 & 6 & -5 & -15 \end{array}\right]$$

$$\Rightarrow -5x_3 = -15 \rightarrow x_3 = 3, x_2 = 2 \ , x_1 = 1$$

3- Inverse Matrices method

Certain $n \times n$ matrices have the property that another $n \times n$ matrix , which we will denote A^{-1}

Exists with $AA^{-1} = A^{-1}A = I$. In this case A is said to be non-singular, or invertible, and the matrix A^{-1} is called the inverse to A. A matrix without an inverse is called singular or noninvertible.

Example 6:

Find the inverse of the following system

$$\begin{aligned} x_1 - x_2 + x_3 &= -4 \\ 5x_1 - 4x_2 + 3x_3 &= -12 \\ 2x_1 + x_2 + x_3 &= 11 \end{aligned}$$

Solution

$$\begin{bmatrix} 1 & -1 & 1 & 1 & 0 & 0 \\ 5 & -4 & 3 \,| & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{-5r_1 + r_2} \xrightarrow[-2r_1 + r_3]{} \begin{bmatrix} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -2 \,| & -5 & 1 & 0 \\ 0 & 3 & -1 & -2 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{3r_2 + r_3} \begin{bmatrix} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -2 \,| & -5 & 1 & 0 \\ 0 & 3 & 5 & 13 & -3 & 1 \end{bmatrix} \xrightarrow{-5r_1 + r_3} \xrightarrow[2r_3 + 5r_2]{} \begin{bmatrix} -5 & 5 & 0 & 8 & -3 & 1 \\ 0 & 5 & 0 \,| & 1 & -1 & 2 \\ 0 & 0 & 5 & 13 & -3 & 1 \end{bmatrix}$$

$$\xrightarrow{-r_1+r_2}\left[\begin{array}{ccc|ccc}5 & 0 & 0 & -7 & 2 & 1\\0 & 5 & 0 & 1 & -1 & 2\\0 & 0 & 5 & 13 & -3 & 1\end{array}\right]\xrightarrow{\div\frac{1}{5}}\left[\begin{array}{ccc|ccc}1 & 0 & 0 & -1.4 & 0.4 & 0.2\\0 & 1 & 0 & 0.2 & -0.2 & 0.4\\0 & 0 & 1 & 2.6 & 0.6 & 0.2\end{array}\right]$$