NATURAL LOGARITHMS

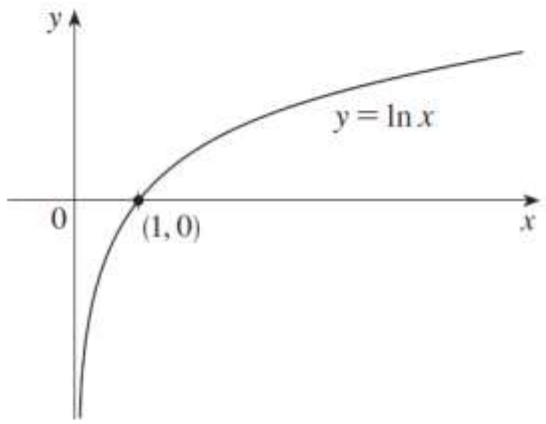
The logarithm with base is called the **natural logarithm** and has a special notation:

8 $$\ln x = y \iff e^y = x$$

9 $$\ln(e^x) = x \qquad x \in \mathbb{R}$$
$$e^{\ln x} = x \qquad x > 0$$

In particular, if we set $x = 1$, we get

$$\ln e = 1$$



x	y= lnx
0	∞
1	0
2	0.693
3	1.098
4	1.386
5	1.609
-1	∞
0.9	-0.105
0.5	-0.693
0.2	-1.609
0.1	-2.302

EXAMPLE 7 Find x if $\ln x = 5$.

SOLUTION 1 From (8) we see that

$$\ln x = 5 \qquad \text{means} \qquad e^5 = x$$

Therefore $x = e^5$.

(If you have trouble working with the "ln" notation, just replace it by $\log_e$. Then the equation becomes $\log_e x = 5$; so, by the definition of logarithm, $e^5 = x$.)

SOLUTION 2 Start with the equation

$$\ln x = 5$$

and apply the exponential function to both sides of the equation:

$$e^{\ln x} = e^5$$

But the second cancellation equation in (9) says that $e^{\ln x} = x$. Therefore $x = e^5$. ■

EXAMPLE 8 Solve the equation $e^{5-3x} = 10$.

SOLUTION We take natural logarithms of both sides of the equation and use (9):

$$\ln(e^{5-3x}) = \ln 10$$

$$5 - 3x = \ln 10$$

$$3x = 5 - \ln 10$$

$$x = \tfrac{1}{3}(5 - \ln 10)$$

Since the natural logarithm is found on scientific calculators, we can approximate the solution: to four decimal places, $x \approx 0.8991$. ■

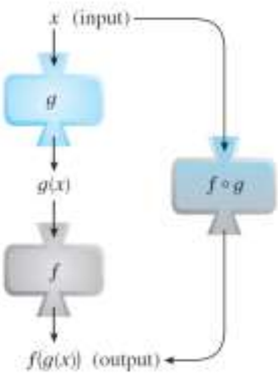
2. Algebra of functions

Let f is a fynction of x then we get f(x) and g is a function of x also we get g(x)

Df is the domain of f(x)
Dg is the domain of g(x)

Then:

f+g = f(x) + g(x) and Df ∩ Dg
f – g = f(x) - g(x)
f.g = f(x) . g(x)



and the domain is as same before

if f/g then Df ∩ Dg but g(x) ≠ 0
if g/f then Dg ∩ Df but f(x) ≠ 0

and Df_og = {x: x ∈ Dg, g(x) ∈ Df}
where

$f_og(x) = f(g(x))$ also called the composition of f and g

Example 8: Find f_og and g_of if $f_{(x)} = \sqrt{1-x}$ and $g_{(x)} = \sqrt{5+x}$

Solution

$(f_og)x = f(g(x)) = f(\sqrt{5+x}) = \sqrt{1-\sqrt{5+x}}$
$(1-x) \geq 0$ then $x \leq 1$ Df: $x \leq 1$
$5+x \geq 0$ then $x \geq -5$ Dg: $x \geq -5$
$D\ f_og = \{x: x \geq -5, \sqrt{5+x} \leq 1\} = \{x: -5 \leq x \leq -4\}$

Example 11: Given $F(x) = \cos^2(x+9)$, find functions f, g, and h such that $F = f \circ g \circ h$.

Solution Since $F(x) = [\cos(x+9)]^2$, the formula for F says: First add 9, then take the cosine of the result, and finally square. So we let

$$h(x) = x + 9 \qquad g(x) = \cos x \qquad f(x) = x^2$$

Then

$$(f \circ g \circ h)(x) = f(g(h(x))) = f(g(x+9)) = f(\cos(x+9)) = [\cos(x+9)]^2 = F(x)$$ □

Example 9: If $f_{(x)} = \sqrt{x}$ and $g_{(x)} = \sqrt{1-x}$
Find:
f+g, f-g, g-f, f_og, f/g, g/f then graph f_og and also f+g.

Solution

$f_{(x)} = \sqrt{x}$ domain $x \geq 0$

$g_{(x)} = \sqrt{1-x}$ domain x ≤1

f+g = (f+g)x = $\sqrt{x} + \sqrt{1-x}$ domain 0 ≤x ≤1 or [0,1]

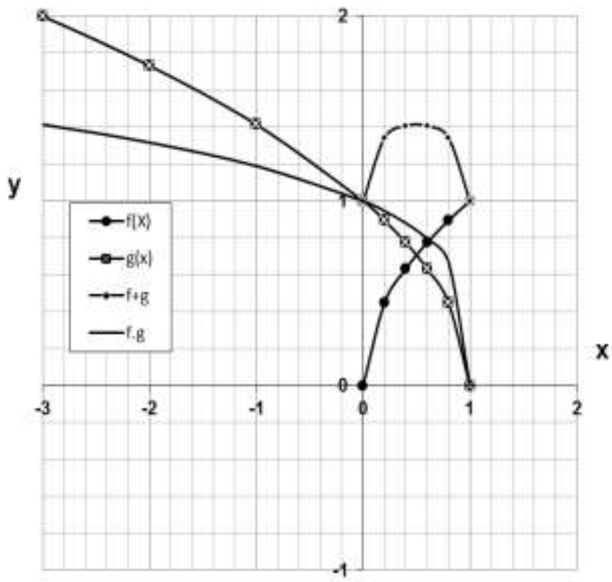
f-g = $\sqrt{x}$ - $\sqrt{1-x}$ domain 0 ≤x ≤1

g-f = $\sqrt{1-x}$ - $\sqrt{x}$ domain 0 ≤x ≤1

$f_o g$ = f(x) g(x) = f(g(x)) =f($\sqrt{1-x}$) = $\sqrt{\sqrt{1-x}}$ = $\sqrt[4]{1-x}$ domain (-∞, 1] (why?)

f/g = f(x)/g(x) = $\sqrt{\frac{x}{1-x}}$ domain (-∞, 1]

g/f = g(x)/f(x) = $\sqrt{\frac{1-x}{x}}$ domain (0, 1]



Inverse functions

A function that undoes, or inverts, the effect of a function f is called the inverse of f. Many common functions, though not all, are paired with an inverse. In this section we present the natural logarithmic function y = ln x as the inverse of the exponential function y = e^x, and we also give examples of several inverse trigonometric functions.

DEFINITION Suppose that f is a one-to-one function on a domain D with range R. The **inverse function** f^{-1} is defined by

$$f^{-1}(b) = a \quad \text{if} \quad f(a) = b.$$

The domain of f^{-1} is R and the range of f^{-1} is D.

Example 10:

Suppose a one-to-one function $y = f(x)$ is given by a table of values

x	1	2	3	4	5	6	7	8
$f(x)$	3	4.5	7	10.5	15	20.5	27	34.5

A table for the values of $x = f^{-1}(y)$ can then be obtained by simply interchanging the values in the columns (or rows) of the table for f:

y	3	4.5	7	10.5	15	20.5	27	34.5
$f^{-1}(y)$	1	2	3	4	5	6	7	8

■

Note:

Only a one-to-one function can have an inverse

Q: What is the one to one function ?

DEFINITION A function $f(x)$ is **one-to-one** on a domain D if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$ in D.

Note

$$\text{domain of } f^{-1} = \text{range of } f$$

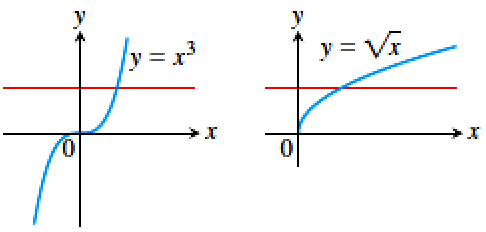
$$\text{range of } f^{-1} = \text{domain of } f$$

two functions have the same values on the smaller domain, so the original function is an extension of the restricted function from its smaller domain to the larger domain.

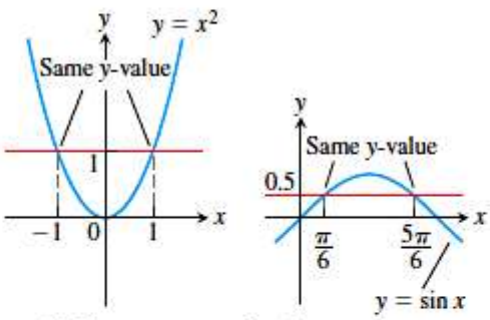
(a) $f(x) = \sqrt{x}$ is one-to-one on any domain of nonnegative numbers because $\sqrt{x_1} \neq \sqrt{x_2}$ whenever $x_1 \neq x_2$.

(b) $g(x) = \sin x$ is *not* one-to-one on the interval $[0, \pi]$ because $\sin(\pi/6) = \sin(5\pi/6)$. In fact, for each element x_1 in the subinterval $[0, \pi/2)$ there is a corresponding element x_2 in the subinterval $(\pi/2, \pi]$ satisfying $\sin x_1 = \sin x_2$, so distinct elements in the domain are assigned to the same value in the range. The sine function *is* one-to-one on $[0, \pi/2]$, however, because it is an increasing function on $[0, \pi/2]$ giving distinct outputs for distinct inputs. ■

The graph of a one-to-one function $y = f(x)$ can intersect a given horizontal line at most once. If the function intersects the line more than once, it assumes the same y-value for at least two different x-values and is therefore not one-to-one



(a) One-to-one: Graph meets each horizontal line at most once.



(b) Not one-to-one: Graph meets one or more horizontal lines more than once.

5 How to Find the Inverse Function of a One-to-One Function f

STEP 1 Write $y = f(x)$.

STEP 2 Solve this equation for x in terms of y (if possible).

STEP 3 To express f^{-1} as a function of x, interchange x and y. The resulting equation is $y = f^{-1}(x)$.

Example 11:

Find the inverse function of $f(x) = x^3 + 2$.

SOLUTION According to (5) we first write

$$y = x^3 + 2$$

Then we solve this equation for x:

$$x^3 = y - 2$$

$$x = \sqrt[3]{y - 2}$$

Finally, we interchange x and y:

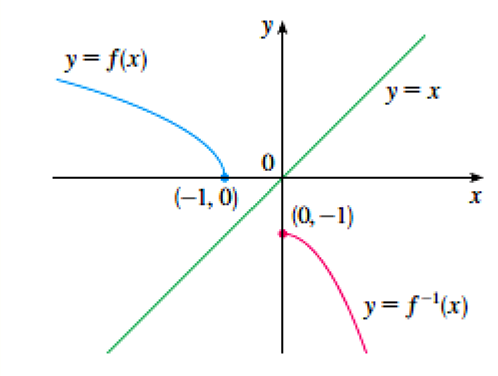
$$y = \sqrt[3]{x - 2}$$

Therefore the inverse function is $f^{-1}(x) = \sqrt[3]{x - 2}$.

Example 12:

Sketch the graphs of $f(x) = \sqrt{-1 - x}$ and its inverse function using the same coordinate axes.

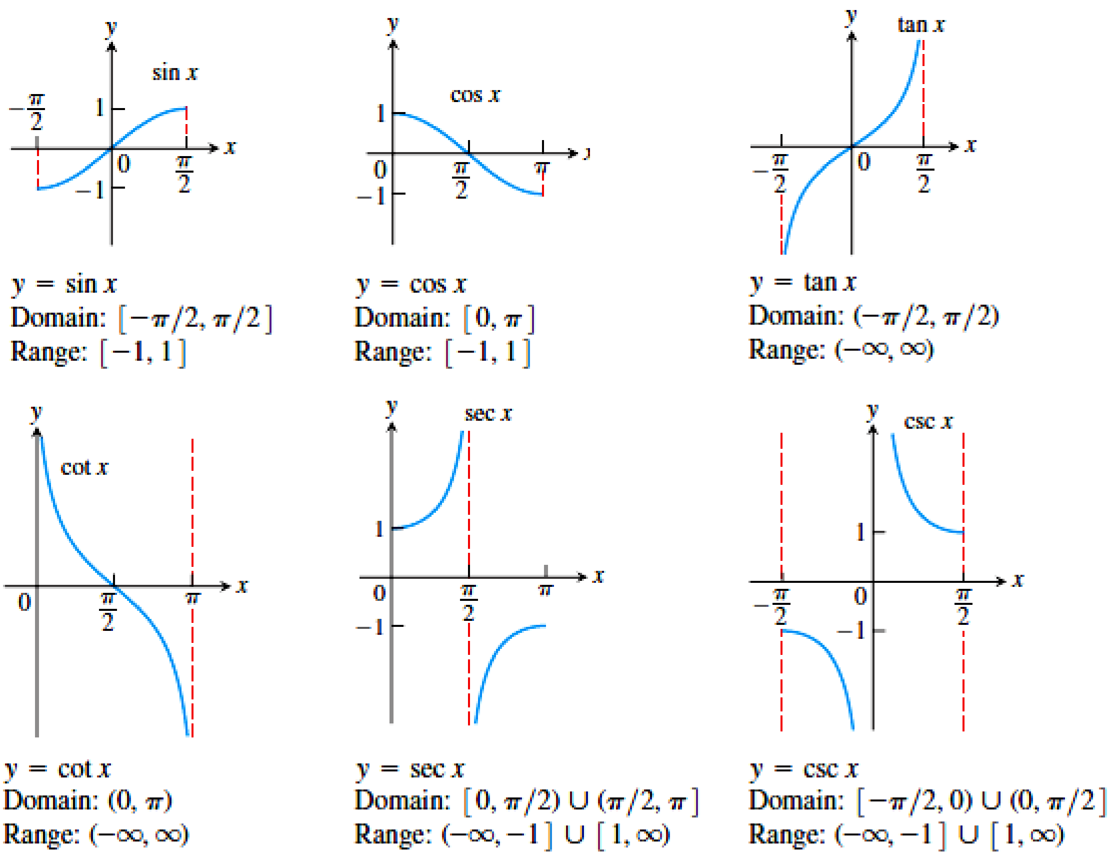
SOLUTION First we sketch the curve $y = \sqrt{-1 - x}$ (the top half of the parabola $y^2 = -1 - x$, or $x = -y^2 - 1$) and then we reflect about the line $y = x$ to get the graph of f^{-1}. (See Figure 10.) As a check on our graph, notice that the expression for f^{-1} is $f^{-1}(x) = -x^2 - 1$, $x \geqslant 0$. So the graph of f^{-1} is the right half of the parabola $y = -x^2 - 1$ and this seems reasonable from Figure ■



Inverse Trigonometric Functions

The six basic trigonometric functions of a general radian angle x were reviewed in Chapter 2. These functions are not one-to-one (their values repeat periodically).

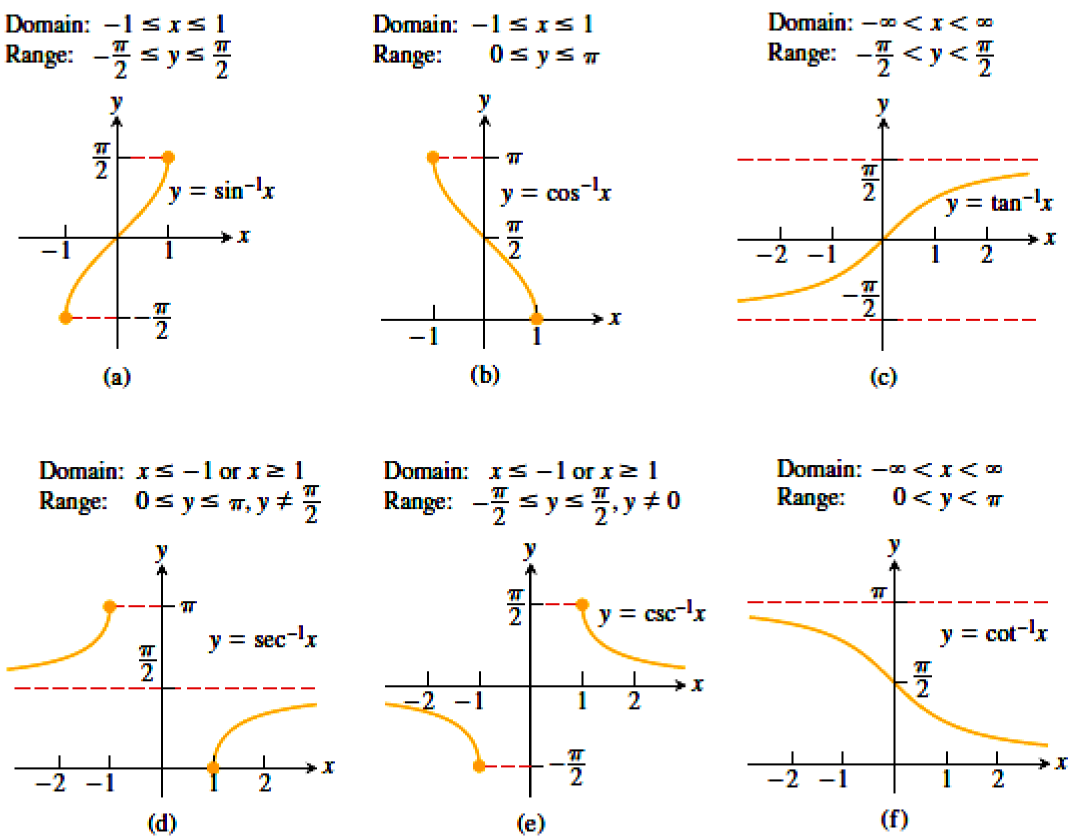
Domain restrictions that make the trigonometric functions one-to-one



Since these restricted functions are now one-to-one, they have inverses, which we denote by

$$
\begin{array}{lcl}
y = \sin^{-1} x & \text{or} & y = \arcsin x \\
y = \cos^{-1} x & \text{or} & y = \arccos x \\
y = \tan^{-1} x & \text{or} & y = \arctan x \\
y = \cot^{-1} x & \text{or} & y = \operatorname{arccot} x \\
y = \sec^{-1} x & \text{or} & y = \operatorname{arcsec} x \\
y = \csc^{-1} x & \text{or} & y = \operatorname{arccsc} x
\end{array}
$$

Graph of inverse trig functions



Notes:

To convert from degree to radian

$$\pi \text{ radians} = 180^\circ$$

and

$$1 \text{ radian} = \frac{180}{\pi} (\approx 57.3) \text{ degrees} \quad \text{or} \quad 1 \text{ degree} = \frac{\pi}{180} (\approx 0.017) \text{ radians.}$$

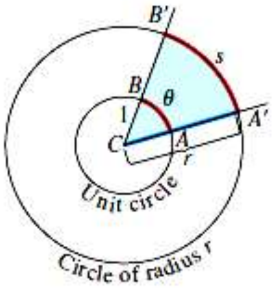


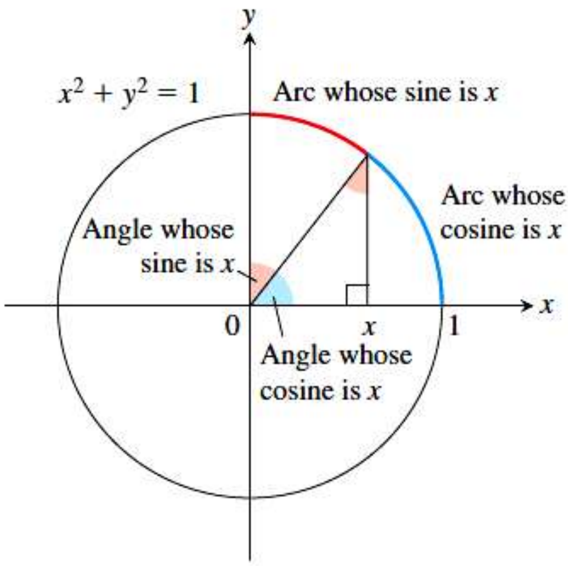
TABLE 1.1 Angles measured in degrees and radians

Degrees	−180	−135	−90	−45	0	30	45	60	90	120	135	150	180	270	360
θ (radians)	$-\pi$	$\frac{-3\pi}{4}$	$\frac{-\pi}{2}$	$\frac{-\pi}{4}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π

TABLE 1.2 Values of sin θ, cos θ, and tan θ for selected values of θ

Degrees	−180	−135	−90	−45	0	30	45	60	90	120	135	150	180	270	360
θ (radians)	$-\pi$	$\frac{-3\pi}{4}$	$\frac{-\pi}{2}$	$\frac{-\pi}{4}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π
sin θ	0	$\frac{-\sqrt{2}}{2}$	−1	$\frac{-\sqrt{2}}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	−1	0
cos θ	−1	$\frac{-\sqrt{2}}{2}$	0	$\frac{\sqrt{2}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{-\sqrt{2}}{2}$	$\frac{-\sqrt{3}}{2}$	−1	0	1
tan θ	0	1		−1	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$		$-\sqrt{3}$	−1	$\frac{-\sqrt{3}}{3}$	0		0

The "Arc" in Arcsine and Arccosine
For a unit circle and radian angles, the arc length equation $s = r\theta$ becomes $s = \theta$, so central angles and the arcs they subtend have the same measure. If $x = \sin y$, then, in addition to being the angle whose sine is x, y is also the length of arc on the unit circle that subtends an angle whose sine is x. So we call y "the arc whose sine is x."



Example 13:

Evaluate (a) $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$ and (b) $\cos^{-1}\left(-\frac{1}{2}\right)$.

Solution

(a) We see that

$$\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$$

because $\sin(\pi/3) = \sqrt{3}/2$ and $\pi/3$ belongs to the range $[-\pi/2, \pi/2]$ of the arcsine function. See Figure 1.68a.

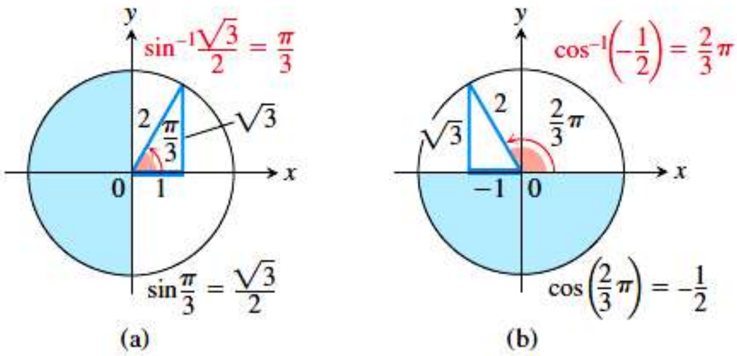
(b) We have

$$\cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$$

because $\cos(2\pi/3) = -1/2$ and $2\pi/3$ belongs to the range $[0, \pi]$ of the arccosine

We can create the following table of common values for the arcsine and arccosine functions

x	$\sin^{-1}x$	$\cos^{-1}x$
$\sqrt{3}/2$	$\pi/3$	$\pi/6$
$\sqrt{2}/2$	$\pi/4$	$\pi/4$
$1/2$	$\pi/6$	$\pi/3$
$-1/2$	$-\pi/6$	$2\pi/3$
$-\sqrt{2}/2$	$-\pi/4$	$3\pi/4$
$-\sqrt{3}/2$	$-\pi/3$	$5\pi/6$



Example 14:

Evaluate (a) $\sin^{-1}(\frac{1}{2})$ and (b) $\tan(\arcsin \frac{1}{3})$.

SOLUTION

(a) We have

$$\sin^{-1}(\tfrac{1}{2}) = \frac{\pi}{6}$$

because $\sin(\pi/6) = \frac{1}{2}$ and $\pi/6$ lies between $-\pi/2$ and $\pi/2$.

(b) Let $\theta = \arcsin \frac{1}{3}$, so $\sin \theta = \frac{1}{3}$. Then we can draw a right triangle with angle θ as in Figure and deduce from the Pythagorean Theorem that the third side has length $\sqrt{9-1} = 2\sqrt{2}$. This enables us to read from the triangle that

$$\tan(\arcsin \tfrac{1}{3}) = \tan \theta = \frac{1}{2\sqrt{2}}$$

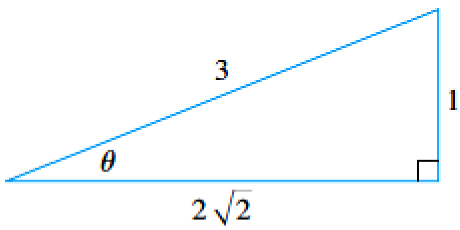
■

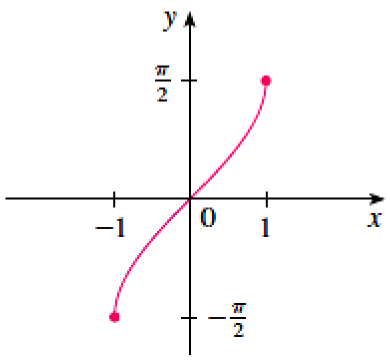
The cancellation equations for inverse functions become, in this case,

$$\sin^{-1}(\sin x) = x \quad \text{for } -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$$

$$\sin(\sin^{-1}x) = x \quad \text{for } -1 \leq x \leq 1$$

The inverse sine function, $\sin^{-1}$, has domain $[-1, 1]$ and range $[-\pi/2, \pi/2]$, and its graph, shown in Figure 20, is obtained from that of the restricted sine function by reflection about the line $y = x$.





$y = \sin^{-1}x = \arcsin x$

Example 15:

Simplify the expression $\cos(\tan^{-1}x)$.

SOLUTION 1 Let $y = \tan^{-1}x$. Then $\tan y = x$ and $-\pi/2 < y < \pi/2$. We want to find $\cos y$ but, since $\tan y$ is known, it is easier to find $\sec y$ first:

$$\sec^2 y = 1 + \tan^2 y = 1 + x^2$$

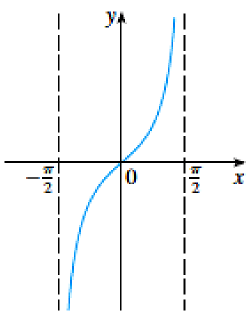
$$\sec y = \sqrt{1 + x^2} \qquad (\text{since } \sec y > 0 \text{ for } -\pi/2 < y < \pi/2)$$

SOLUTION 2 Instead of using trigonometric identities as in Solution 1, it is perhaps easier to use a diagram. If $y = \tan^{-1}x$, then $\tan y = x$, and we can read from Figure 24 (which illustrates the case $y > 0$) that

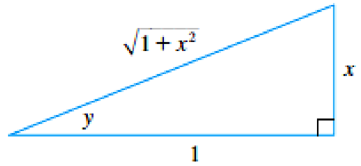
$$\cos(\tan^{-1}x) = \cos y = \frac{1}{\sqrt{1 + x^2}}$$ ■

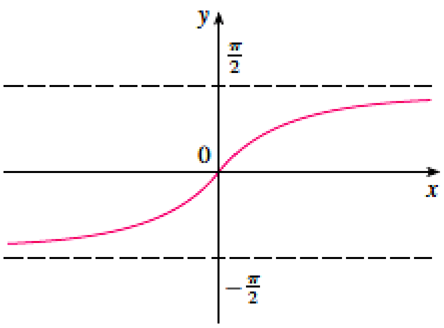
The inverse tangent function, $\tan^{-1} = \arctan$, has domain $\mathbb{R}$ and range $(-\pi/2, \pi/2)$. Its graph is shown in Figure .

$y = \cos^{-1}x = \arccos x$

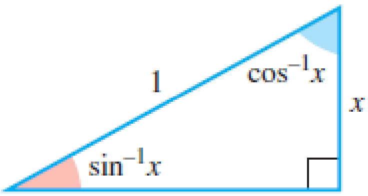


$y = \tan x, \ -\frac{\pi}{2} < x < \frac{\pi}{2}$



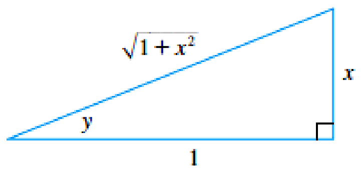


Identities of inverse trig functions

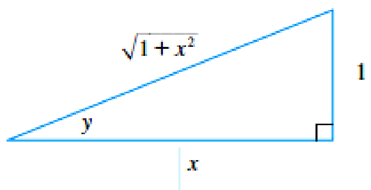


$$\sin^{-1}x + \cos^{-1}x = \pi/2.$$

$\sin^{-1}x$



$\cos^{-1}x$



and so on ..