2 Limits & Continuity

In this chapter, we'll define how limit of function values are defined and calculated.

Definition: the limit of f(x) as x tends to a is defined as the value of f(x) as x approaches closer and closer to a without actually reaching it and denoted by:

$\lim_{x \to a} f_{(x)} = L$ L is a single finite real number

It's important to know

1. We don't evaluate the limit by actually substituting x = a in f(x) in general, although in some cases its possible.

2. The value of the limit can depend on which side its approach

3. The limit may not exist at all.

Example 1: to explain the concept of limit, take the function f(x) = 2x – 4 if the

$\lim_{x \to 1} f_{(x)} = 2*1-4=-2$

But the following table express many values of x can be expressed close to 1.

x	0.5	0.8	0.9	0.99	0.999	1.001	1.01	1.1	1.2
f(x)	-3	-2.4	-2.2	-2.02	-2.002	-1.998	-1.98	-1.8	-1.6

Question: Why we take values approaches to 2 in example 1 instead we take x = 1 directly?

Solution: the answer about this question can be expressed in the following example:

$$f_{(x)} = 3^{-\frac{1}{x^2}} + 1$$

If x = 0 then 1/0 = ∞

So..

x	±0.2	±0.5	etc..
f(x)	1.00000	1.012345679	

In limits we avoid ∞

Definition

We say that the limit of *f(x)* is *L* as *x* approaches *a* and write this as

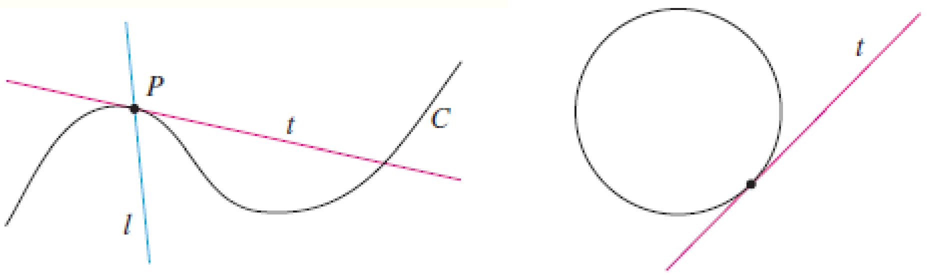
$$\lim_{x \to a} f(x) = L$$

provided we can make *f(x)* as close to *L* as we want for all *x* sufficiently close to *a*, from both sides, without actually letting *x* be *a*.

THE TANGENT PROBLEM

The word tangent is derived from the Latin word tangens, which means “touching.” Thus a tangent to a curve is a line that touches the curve. In other words, a tangent line should have the same direction as the curve at the point of contact. How can this idea be made precise?

For a circle we could simply follow Euclid and say that a tangent is a line that intersects the circle once and only once as in Figure (a). For more complicated curves this definition is inadequate as shown in Figure (b)



Example 2: Find an eq. of the tangent line to the parabola $y = x^2$ at point (1,1)?

Solution

We will be able to find an equation of the tangent line t as soon as we know its slope m. The difficulty is that we know only one point, P, on t, whereas we need two points to compute the slope. But observe that we can compute an approximation to m by choosing a nearby point $Q(x, x^2)$ on the parabola (as in Figure) and computing the slope m_{PQ} of the secant line PQ.

We choose $x \neq 1$ so that $Q \neq P$. Then

$$m_{PQ} = \frac{x^2 - 1}{x - 1}$$

For instance, for the point $Q(1.5, 2.25)$ we have

$$m_{PQ} = \frac{2.25 - 1}{1.5 - 1} = \frac{1.25}{0.5} = 2.5$$

The tables in the margin show the values of m_{PQ} for several values of x close to 1. The closer Q is to P, the closer x is to 1 and, it appears from the tables, the closer m_{PQ} is to 2. This suggests that the slope of the tangent line t should be $m = 2$.

We say that the slope of the tangent line is the *limit* of the slopes of the secant lines, and we express this symbolically by writing

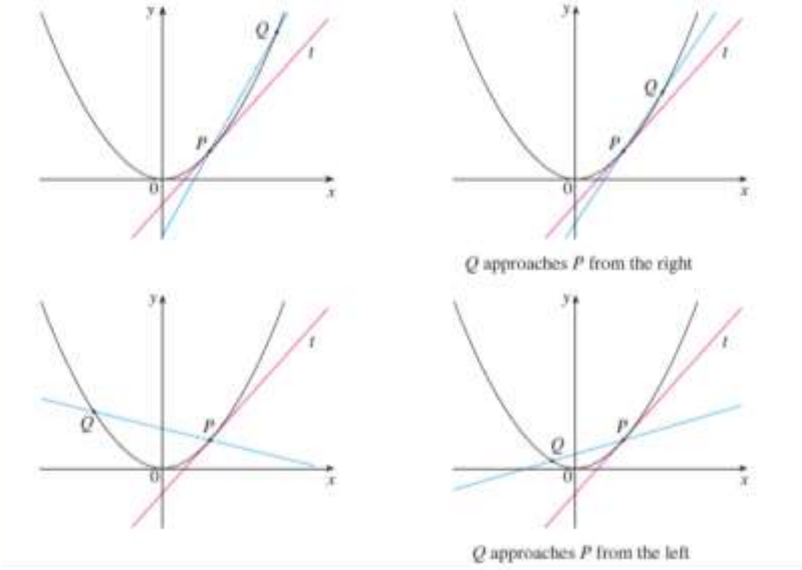
$$\lim_{Q \to P} m_{PQ} = m \qquad \text{and} \qquad \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2$$

Assuming that the slope of the tangent line is indeed 2, we use the point-slope form of the equation of a line to write the equation of the tangent line

$$y - 1 = 2(x - 1) \qquad \text{or} \qquad y = 2x - 1$$

x	m_{PQ}
2	3
1.5	2.5
1.1	2.1
1.01	2.01
1.001	2.001

x	m_{PQ}
0	1
0.5	1.5
0.9	1.9
0.99	1.99
0.999	1.999



THE VELOCITY PROBLEM

If you watch the speedometer of a car as you travel in city traffic, you see that the needle doesn't stay still for very long; that is, the velocity of the car is not constant. We assume from watching the speedometer that the car has a definite velocity at each moment, but how is the "instantaneous" velocity defined? Let's investigate the example of a falling ball.

Suppose that a ball is dropped from the upper observation deck of the CN Tower in Toronto, 450 m above the ground. Find the velocity of the ball after 5 seconds.

Through experiments carried out four centuries ago, Galileo discovered that the distance fallen by any freely falling body is proportional to the square of the time it has been falling. (This model for free fall neglects air resistance.) If the distance fallen after t seconds is denoted by $s(t)$ and measured in meters, then Galileo's law is expressed by the equation

$$s(t) = 4.9t^2$$

The difficulty in finding the velocity after 5 s is that we are dealing with a single instant of time ($t = 5$), so no time interval is involved. However, we can approximate the desired quantity by computing the average velocity over the brief time interval of a tenth of a second from $t = 5$ to $t = 5.1$:

$$\text{average velocity} = \frac{\text{change in position}}{\text{time elapsed}}$$

$$= \frac{s(5.1) - s(5)}{0.1}$$

The following table shows the results of similar calculations of the average velocity over successively smaller time periods.

Time interval	Average velocity (m/s)
$5 \leq t \leq 6$	53.9
$5 \leq t \leq 5.1$	49.49
$5 \leq t \leq 5.05$	49.245
$5 \leq t \leq 5.01$	49.049
$5 \leq t \leq 5.001$	49.0049

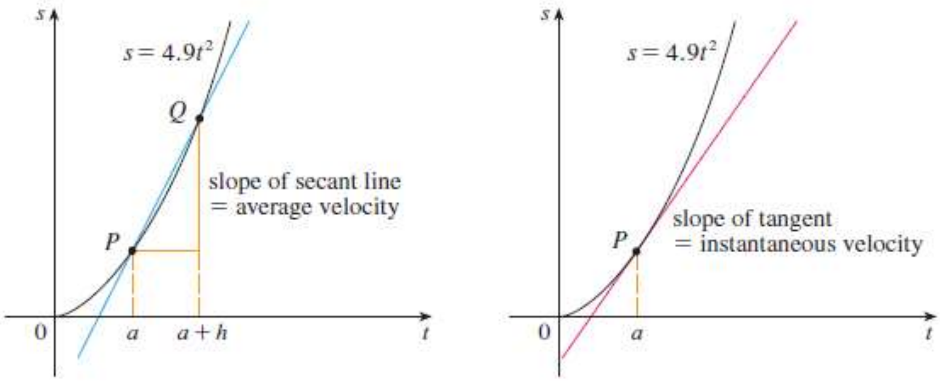
It appears that as we shorten the time period, the average velocity is becoming closer to 49 m/s. The **instantaneous velocity** when $t = 5$ is defined to be the limiting value of these average velocities over shorter and shorter time periods that start at $t = 5$. Thus the (instantaneous) velocity after 5 s is

$$v = 49 \text{ m/s}$$ □

You may have the feeling that the calculations used in solving this problem are very similar to those used earlier in this section to find tangents. In fact, there is a close connection between the tangent problem and the problem of finding velocities. If we draw the graph of the distance function of the ball (as in Figure 5) and we consider the points $P(a, 4.9a^2)$ and $Q(a + h, 4.9(a + h)^2)$ on the graph, then the slope of the secant line PQ is

$$m_{PQ} = \frac{4.9(a + h)^2 - 4.9a^2}{(a + h) - a}$$

which is the same as the average velocity over the time interval $[a, a + h]$. Therefore, the velocity at time $t = a$ (the limit of these average velocities as h approaches 0) must be equal to the slope of the tangent line at P (the limit of the slopes of the secant lines).



Example 3: Discuss the function $f_{(x)} = \frac{x^2 - 9}{x - 3}$

If (1) $x = 1, x = 2$

(2) $x = 3$

(3) $x \to 1, x \to 2$

(4) $x \to 3$

Solution:

$$f_{(x)} = \frac{x^2-9}{x-3} = \frac{(x-3)(x+3)}{(x-3)} = x+3$$ and x ≠ 3

Its equivalent to g(x) = x+3 and x ≠ 3, then:

f(1) = g(1) = 4

f(2) = g(2) = 5

if x → 1 then f(x) = 4 and $\lim_{x\to 1} f_{(x)} = 4$

if x = 3 then f(3) = 0/0 = ∞

if x → 3 then $\lim_{x\to 3} f_{(x)} = 6$

note: if f(x) is defined by two different forms before and after x = a then we must discuss the left limit and the right limit.

Properties of limits:

If $\lim_{x\to a} f_{(x)} = b \quad \lim_{x\to a} f_{(x)} = c$

Then:

1. $\lim_{x\to a} k\, f_{(x)} = kb$ for any constant k
2. $\lim_{x\to a} [\, f_{(x)} \pm g_{(x)}] = \lim_{x\to a} f_{(x)} \pm \lim_{x\to a} g_{(x)} = b + c$
3. $\lim_{x\to a} [\, f_{(x)} . g_{(x)}] = \lim_{x\to a} f_{(x)} . \lim_{x\to a} g_{(x)} = b.c$
4. $\lim_{x\to a} [\, f_{(x)} / g_{(x)}] = \lim_{x\to a} f_{(x)} / \lim_{x\to a} g_{(x)} = b / c \text{ if } c \neq 0$
5. $\lim_{x\to a} [\, f_{(x)}]^{1/n} == b^{1/n} \quad \text{real values only for } n$

The limit must exist before applying the above results.

Example 3: find the limits of the following functions:

1. $\lim\limits_{x\to 3} = \dfrac{x^2-9}{x-3}$ = $\lim\limits_{x\to 3} = \dfrac{(x-3)(x+3)}{(x-3)} = \lim\limits_{x\to 3}(x+3) = 3+3=6$

2. $\lim\limits_{x\to 2} = \dfrac{\sqrt{x+2}-2}{x-2} = \lim\limits_{x\to 2} = \dfrac{\sqrt{x+2}-2}{x-2} * \dfrac{\sqrt{x+2}+2}{\sqrt{x+2}+2} = \lim\limits_{x\to 2} = \dfrac{(x+2)-4}{x-2\sqrt{x+2}+2}$

$$= \frac{1}{\sqrt{2+2}+2} = \frac{1}{2+2} = \frac{1}{4}$$

3. $\lim\limits_{x\to\infty} = \dfrac{3x^2+2x+1}{5x^2+7x+1}$ ÷ x^2

$$\lim_{x\to\infty} = \frac{3+\frac{2}{x}+\frac{1}{x^2}}{5+\frac{7}{x}+\frac{4}{x^2}} = \frac{3}{5}$$

Note: $\lim\limits_{x\to\infty} = \dfrac{a_n x^n + a_{n-1}x^{n-1} + \ldots\ldots + a_o}{b_m x^m + a_{m-1}x^{m-1} + \ldots\ldots + b_o} = \begin{cases} 0 & n<m \\ \dfrac{a}{b} & n=m \\ \infty & n>m \end{cases}$

Example 4: find

$$\lim_{x\to 1} = \frac{x^2-1}{(x-1)\sqrt{x^2+2x+3}}$$

Solution:

$$= \lim_{x\to 1} = \frac{(x-1)(x+1)}{(x-1)\sqrt{x^2+2x+3}} = \lim_{x\to 1}(x+1) - \lim_{x\to 1}\sqrt{x^2+2x+3} = 2 \div \sqrt{\lim_{x\to 1}(x^2+2x+3)}$$

$$= 2 \div \sqrt{6} = \frac{2}{\sqrt{6}}$$

Theorem I If g(x) ≤ f(x) ≤ h(x) and $\lim_{x\to a} g_{(x)} = \lim_{x\to a} h_{(x)} = L$

Theorem II $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1 \quad or \quad \lim_{x\to a} \frac{\sin(x-a)}{(x-a)} = 1$

Example 5:

1. $\lim_{x\to 0} \frac{\sin 5x}{\sin 7x} = \frac{\frac{\sin 5x}{5x}.5x}{\frac{\sin 7x}{7x}.7x} = 5/7$

2. $\lim_{x\to 2} \frac{\sin(x-2)}{x^2-4} = \lim_{x\to 2} \frac{\sin(x-2)}{(x-2)(x+2)} = \frac{1}{(x+2)} = 1/4$