Left and right – side limits

Right-handed limit

We say

$$\lim_{x \to a^+} f(x) = L$$

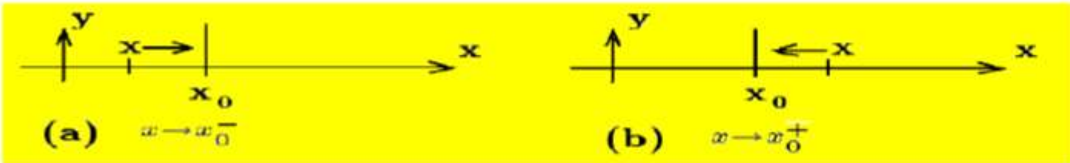
provided we can make *f(x)* as close to *L* as we want for all *x* sufficiently close to *a* and $x > a$ without actually letting *x* be *a*.

Left-handed limit

We say

$$\lim_{x \to a^-} f(x) = L$$

provided we can make *f(x)* as close to *L* as we want for all *x* sufficiently close to *a* and $x < a$ without actually letting *x* be *a*.



Example 6: Discuss the $\lim_{x \to 2} f(x)$ if $f(x) = \begin{cases} 3x+2 & x<2 \\ 4 & x=2 \\ 8-x & x>2 \end{cases}$

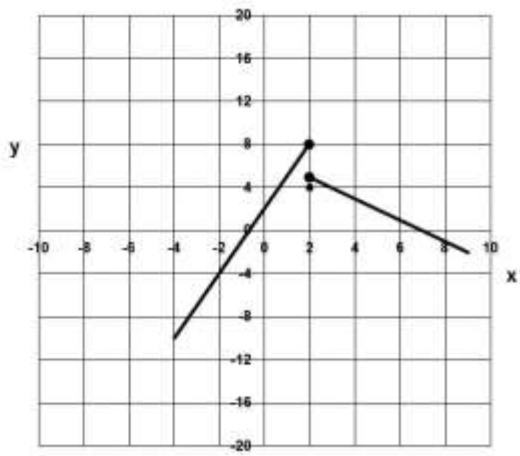
Solution:

If x > 2

Then f (2^+) = $\lim_{x \to 2} f(x) = \lim_{x \to 2} (8-x) = 8-2 = 6$

If x < 2 then

f(2^-) $\lim_{x \to 2} f(x) = \lim_{x \to 2} (3x+2) = 8$



Then right limit ≠ left limit at x = 2

Then, we say that $\lim_{x \to 2} f(x)$ doesn't exists

Fact

Given a function *f(x)* if,

$$\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = L$$

then the normal limit will exist and

$$\lim_{x \to a} f(x) = L$$

Likewise, if

$$\lim_{x \to a} f(x) = L$$

then,

$$\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = L$$

Example Evaluate the following limit.

$$\lim_{z \to 1} \frac{6 - 3z + 10z^2}{-2z^4 + 7z^3 + 1}$$

Solution

First notice that we can use property 4) to write the limit as,

$$\lim_{z \to 1} \frac{6 - 3z + 10z^2}{-2z^4 + 7z^3 + 1} = \frac{\lim_{z \to 1} 6 - 3z + 10z^2}{\lim_{z \to 1} -2z^4 + 7z^3 + 1}$$

Well, actually we should be a little careful. We can do that provided the limit of the denominator isn't zero. As we will see however, it isn't in this case so we're okay.

Now, both the numerator and denominator are polynomials so we can use the fact above to compute the limits of the numerator and the denominator and hence the limit itself.

$$\lim_{z \to 1} \frac{6 - 3z + 10z^2}{-2z^4 + 7z^3 + 1} = \frac{6 - 3(1) + 10(1)^2}{-2(1)^4 + 7(1)^3 + 1}$$

$$= \frac{13}{6}$$

Notice that the limit of the denominator wasn't zero and so our use of property 4 was legitimate.

Additional examples:

Example 1: $\lim_{x\to 1} \frac{x^2+1}{x}$

$= \frac{1+1}{1} = 2$

Example 2: $\lim_{x\to -1} \frac{x^2-1}{x+1}$

$= \lim_{x\to -1} \frac{(x-1)(x+1)}{(x+1)} = -1-1 = -2$

Example 3: $\lim_{x\to 1} \frac{x^3-1}{x-1}$

$= \lim_{x\to 1} \frac{(x-1)(x^2+x+1)}{(x-1)} = 1+1+1 = 3$

▶ **Example 8** For the function f graphed in Figure 1.1.18, find

(a) $\lim_{x\to -2^-} f(x)$ (b) $\lim_{x\to -2^+} f(x)$ (c) $\lim_{x\to 0^-} f(x)$ (d) $\lim_{x\to 0^+} f(x)$

(e) $\lim_{x\to 4^-} f(x)$ (f) $\lim_{x\to 4^+} f(x)$ (g) the vertical asymptotes of the graph of f.

Solution (a) and (b).

$$\lim_{x\to -2^-} f(x) = 1 = f(-2) \quad \text{and} \quad \lim_{x\to -2^+} f(x) = -2$$

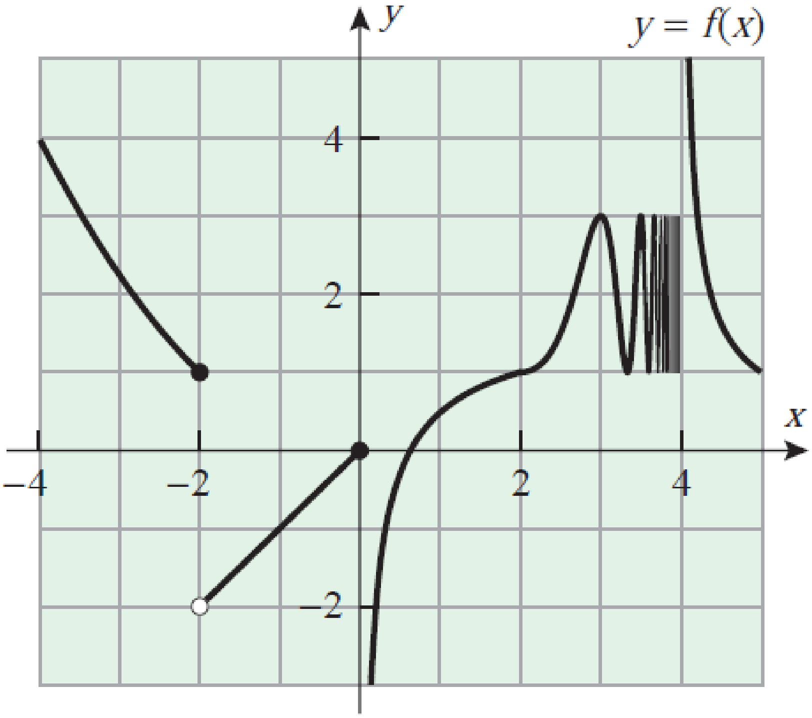
Solution (c) and (d).

$$\lim_{x\to 0^-} f(x) = 0 = f(0) \quad \text{and} \quad \lim_{x\to 0+} f(x) = -\infty$$

Solution (e) and (f).

$$\lim_{x\to 4^-} f(x) \text{ does not exist due to oscillation} \quad \text{and} \quad \lim_{x\to 4^+} f(x) = +\infty$$

Solution (g). The y-axis and the line $x = 4$ are vertical asymptotes for the graph of f. ◀


y
y = f(x)
4
2
−2
x
−4
−2
2
4

11. $\lim_{x\to 5}\dfrac{x^2-6x+5}{x-5}=\lim_{x\to 5}\dfrac{(x-5)(x-1)}{x-5}=\lim_{x\to 5}(x-1)=5-1=4$

12. $\lim_{x\to 4}\dfrac{x^2-4x}{x^2-3x-4}=\lim_{x\to 4}\dfrac{x(x-4)}{(x-4)(x+1)}=\lim_{x\to 4}\dfrac{x}{x+1}=\dfrac{4}{4+1}=\dfrac{4}{5}$

13. $\lim_{x\to 5}\dfrac{x^2-5x+6}{x-5}$ does not exist since $x-5\to 0$, but $x^2-5x+6\to 6$ as $x\to 5$.

14. $\lim_{x\to -1}\dfrac{x^2-4x}{x^2-3x-4}$ does not exist since $x^2-3x-4\to 0$ but $x^2-4x\to 5$ as $x\to -1$.

15. $\lim_{t\to -3}\dfrac{t^2-9}{2t^2+7t+3}=\lim_{t\to -3}\dfrac{(t+3)(t-3)}{(2t+1)(t+3)}=\lim_{t\to -3}\dfrac{t-3}{2t+1}=\dfrac{-3-3}{2(-3)+1}=\dfrac{-6}{-5}=\dfrac{6}{5}$

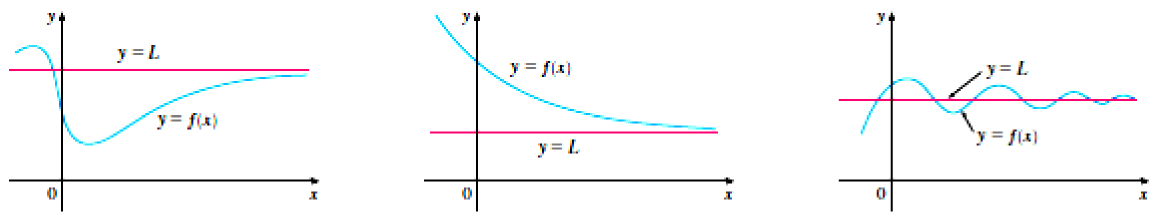
16. $\lim_{x\to -1}\dfrac{2x^2+3x+1}{x^2-2x-3}=\lim_{x\to -1}\dfrac{(2x+1)(x+1)}{(x-3)(x+1)}=\lim_{x\to -1}\dfrac{2x+1}{x-3}=\dfrac{2(-1)+1}{-1-3}=\dfrac{-1}{-4}=\dfrac{1}{4}$

Limits at Infinity: Horizontal Asymptote

1 DEFINITION Let f be a function defined on some interval (a,∞). Then

$$\lim_{x\to\infty} f(x)=L$$

means that the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large.



The line L is called horizontal asymptote of the graph of the function (f). If the value of f(x) increases without bound as x → +∞ or x → - ∞, then we write:

$$\lim_{x\to+\infty} f(x)=+\infty \quad \text{Or} \quad \lim_{x\to-\infty} f(x)=-\infty$$

If the value of f(x) decreases without bound as x + ∞ or x - ∞, then we write:

$\lim_{x\to+\infty} f(x) = -\infty$ Or $\lim_{x\to-\infty} f(x) = +\infty$

2 DEFINITION Let f be a function defined on some interval $(-\infty, a)$. Then

$$\lim_{x\to-\infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large negative.

3 DEFINITION The line $y = L$ is called a **horizontal asymptote** of the curve $y = f(x)$ if either

$$\lim_{x\to\infty} f(x) = L \qquad \text{or} \qquad \lim_{x\to-\infty} f(x) = L$$

Example 7: Find $\lim_{x\to\infty} \frac{1}{x}$ and $\lim_{x\to-\infty} \frac{1}{x}$

Solution:

Observe that when x is large, $1/x$ is small. For instance,

$$\frac{1}{100} = 0.01 \qquad \frac{1}{10{,}000} = 0.0001 \qquad \frac{1}{1{,}000{,}000} = 0.000001$$

In fact, by taking x large enough, we can make $1/x$ as close to 0 as we please. Therefore, according to Definition 1, we have

$$\lim_{x\to\infty} \frac{1}{x} = 0$$

Similar reasoning shows that when x is large negative, $1/x$ is small negative, so we also have

$$\lim_{x\to-\infty} \frac{1}{x} = 0$$

It follows that the line $y = 0$ (the x-axis) is a horizontal asymptote of the curve $y = 1/x$. (This is an equilateral hyperbola; see Figure 6.) □

5 THEOREM If $r > 0$ is a rational number, then

$$\lim_{x \to \infty} \frac{1}{x^r} = 0$$

If $r > 0$ is a rational number such that x^r is defined for all x, then

$$\lim_{x \to -\infty} \frac{1}{x^r} = 0$$

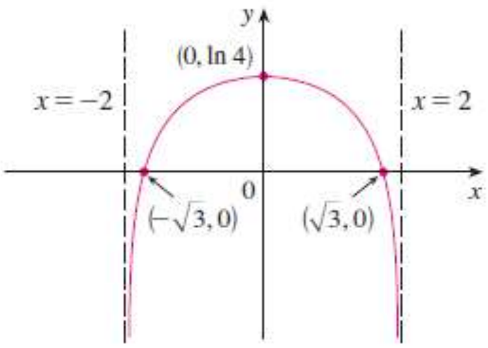
Infinite limits and Vertical Asymptotes

As the line x = a is a vertical asymptote if at least one of the following statements is true:

$$\lim_{x \to a^+} f(x) = \infty \qquad \lim_{x \to a^-} f(x) = \infty$$

$$\lim_{x \to a^+} f(x) = -\infty \qquad \lim_{x \to a^-} f(x) = -\infty$$

Example 8: $\lim_{x \to 2^-} \ln(4 - x^2) = -\infty \qquad \lim_{x \to -2^+} \ln(4 - x^2) = -\infty$



Example 9: Find the horizontal and vertical asymptotes of the graph of the function

$$f(x) = \frac{\sqrt{2x^2+1}}{3x-5}$$

SOLUTION Dividing both numerator and denominator by x and using the properties of limits, we have

$$\lim_{x\to\infty} \frac{\sqrt{2x^2+1}}{3x-5} = \lim_{x\to\infty} \frac{\dfrac{\sqrt{2x^2+1}}{x}}{\dfrac{3x-5}{x}} = \lim_{x\to\infty} \frac{\sqrt{\dfrac{2x^2+1}{x^2}}}{\dfrac{3x-5}{x}} \qquad (\text{since } \sqrt{x^2} = x \text{ for } x > 0)$$

$$= \frac{\displaystyle\lim_{x\to\infty}\sqrt{2+\frac{1}{x^2}}}{\displaystyle\lim_{x\to\infty}\left(3-\frac{5}{x}\right)} = \frac{\sqrt{\displaystyle\lim_{x\to\infty} 2 + \lim_{x\to\infty}\frac{1}{x^2}}}{\displaystyle\lim_{x\to\infty} 3 - 5\lim_{x\to\infty}\frac{1}{x}} = \frac{\sqrt{2+0}}{3-5\cdot 0} = \frac{\sqrt{2}}{3}$$

Therefore the line $y = \sqrt{2}/3$ is a horizontal asymptote of the graph of f.

In computing the limit as $x \to -\infty$, we must remember that for $x < 0$, we have $\sqrt{x^2} = |x| = -x$. So when we divide the numerator by x, for $x < 0$ we get

$$\frac{\sqrt{2x^2+1}}{x} = \frac{\sqrt{2x^2+1}}{-\sqrt{x^2}} = -\sqrt{\frac{2x^2+1}{x^2}} = -\sqrt{2+\frac{1}{x^2}}$$

Therefore

$$\lim_{x\to-\infty} \frac{\sqrt{2x^2+1}}{3x-5} = \lim_{x\to-\infty} \frac{-\sqrt{2+\dfrac{1}{x^2}}}{3-\dfrac{5}{x}} = \frac{-\sqrt{2+\displaystyle\lim_{x\to-\infty}\frac{1}{x^2}}}{3-5\displaystyle\lim_{x\to-\infty}\frac{1}{x}} = -\frac{\sqrt{2}}{3}$$

Thus the line $y = -\sqrt{2}/3$ is also a horizontal asymptote.

A vertical asymptote is likely to occur when the denominator, $3x - 5$, is 0, that is, when $x = \frac{5}{3}$. If x is close to $\frac{5}{3}$ and $x > \frac{5}{3}$, then the denominator is close to 0 and $3x - 5$ is positive. The numerator $\sqrt{2x^2 + 1}$ is always positive, so $f(x)$ is positive. Therefore

$$\lim_{x \to (5/3)^+} \frac{\sqrt{2x^2 + 1}}{3x - 5} = \infty$$

(Notice that the numerator does *not* approach 0 as $x \to 5/3$).

If x is close to $\frac{5}{3}$ but $x < \frac{5}{3}$, then $3x - 5 < 0$ and so $f(x)$ is large negative. Thus

$$\lim_{x \to (5/3)^-} \frac{\sqrt{2x^2 + 1}}{3x - 5} = -\infty$$

The vertical asymptote is $x = \frac{5}{3}$. All three asymptotes are shown in Figure 8. ■

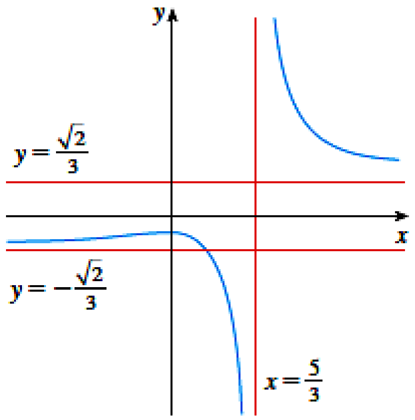


FIGURE 8

$$y = \frac{\sqrt{2x^2 + 1}}{3x - 5}$$

Limits at Infinity of exponential and logarithmic functions

Example 1 Evaluate each of the following limits.

$$\lim_{x\to\infty} e^{x} \qquad \lim_{x\to-\infty} e^{x} \qquad \lim_{x\to\infty} e^{-x} \qquad \lim_{x\to-\infty} e^{-x}$$

Solution

There are really just restatements of facts given in the basic exponential section of the review so we'll leave it to you to go back and verify these.

$$\lim_{x\to\infty} e^{x} = \infty \qquad \lim_{x\to-\infty} e^{x} = 0 \qquad \lim_{x\to\infty} e^{-x} = 0 \qquad \lim_{x\to-\infty} e^{-x} = \infty$$

Example 2 Evaluate each of the following limits.

(a) $\lim\limits_{x\to\infty} \mathbf{e}^{2-4x-8x^2}$ [Solution]

(b) $\lim\limits_{t\to-\infty} \mathbf{e}^{t^4-5t^2+1}$ [Solution]

(c) $\lim\limits_{z\to0^+} \mathbf{e}^{\frac{1}{z}}$ [Solution]

Solution

(a) $\lim\limits_{x\to\infty} \mathbf{e}^{2-4x-8x^2}$

In this part what we need to note (using Fact 2 above) is that in the limit the exponent of the exponential does the following,

$$\lim_{x\to\infty} 2-4x-8x^2 = -\infty$$

So, the exponent goes to minus infinity in the limit and so the exponential must go to zero in the limit using the ideas from the previous set of examples. So, the answer here is,

$$\lim_{x\to\infty} \mathbf{e}^{2-4x-8x^2} = 0$$

[Return to Problems]

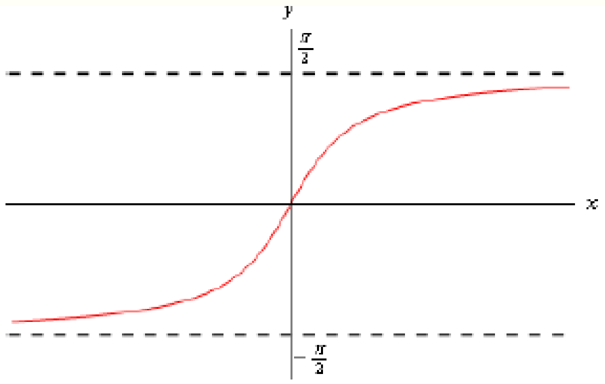
Example 7 Evaluate each of the following limits.

(a) $\lim\limits_{x\to\infty} \tan^{-1} x$ [Solution]

(b) $\lim\limits_{x\to-\infty} \tan^{-1} x$ [Solution]

(c) $\lim\limits_{x\to\infty} \tan^{-1}\left(x^3-5x+6\right)$ [Solution]

(d) $\lim\limits_{x\to0^-} \tan^{-1}\left(\frac{1}{x}\right)$ [Solution]



(a) $\lim\limits_{x\to\infty} \tan^{-1} x$

As noted above all we really need to do here is look at the graph of the inverse tangent. Doing this shows us that we have the following value of the limit.

$$\lim_{x\to\infty} \tan^{-1} x = \frac{\pi}{2}$$

[Return to Problems]

(b) $\lim_{x\to-\infty} \tan^{-1} x$

Again, not much to do here other than examine the graph of the inverse tangent.

$$\lim_{x\to-\infty} \tan^{-1} x = -\frac{\pi}{2}$$

[Return to Problems]

(c) $\lim_{x\to\infty} \tan^{-1}\left(x^3 - 5x + 6\right)$

Okay, in part (a) above we saw that if the argument of the inverse tangent function (the stuff inside the parenthesis) goes to plus infinity then we know the value of the limit. In this case (using the techniques from the previous section) we have,

$$\lim_{x\to\infty} x^3 - 5x + 6 = \infty$$

So, this limit is,

$$\lim_{x\to\infty} \tan^{-1}\left(x^3 - 5x + 6\right) = \frac{\pi}{2}$$

[Return to Problems]

(d) $\lim_{x\to 0^-} \tan^{-1}\left(\frac{1}{x}\right)$

Even though this limit is not a limit at infinity we're still looking at the same basic idea here. We'll use part (b) from above as a guide for this limit. We know from the Infinite Limits section that we have the following limit for the argument of this inverse tangent,

$$\lim_{x\to 0^-} \frac{1}{x} = -\infty$$

So, since the argument goes to minus infinity in the limit we know that this limit must be,

$$\lim_{x\to 0^-} \tan^{-1}\left(\frac{1}{x}\right) = -\frac{\pi}{2}$$

[Return to Problems]

Continuity

If the limit of a function as approaches can often be found simply by calculating the value of the function at . Functions with this property are called continuous at a.

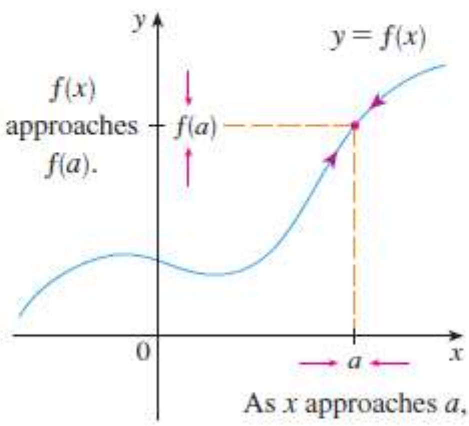
1 DEFINITION A function f is **continuous at a number** a if

$$\lim_{x \to a} f(x) = f(a)$$

Notice that Definition 1 implicitly requires three things if f is continuous at a:

1. $f(a)$ is defined (that is, a is in the domain of f)
2. $\lim_{x \to a} f(x)$ exists
3. $\lim_{x \to a} f(x) = f(a)$

As illustrated in Figure 1, if f is continuous, then the points $(x, f(x))$ on the graph of f approach the point $(a, f(a))$ on the graph. So there is no gap in the curve.

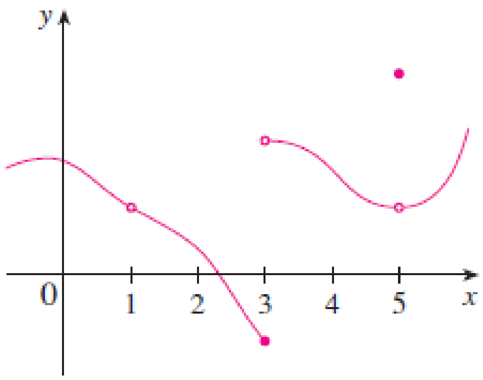


Physical phenomena are usually continuous. For instance, the displacement or velocity of a vehicle varies continuously with time, as does a person's height. But discontinuities do occur in such situations as electric currents.

Example 10: Where are each of the following functions discontinuous?

(a) $f(x) = \dfrac{x^2 - x - 2}{x - 2}$

(c) $f(x) = \begin{cases} \dfrac{x^2 - x - 2}{x - 2} & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$



Solution:

(a) Notice that $f(2)$ is not defined, so f
continuous at all other numbers.

(b) Here $f(0) = 1$ is defined but

$$\lim_{x \to 0}$$

does not exist. (See Example 8 in Secti

(c) Here $f(2) = 1$ is defined and

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{x^2 - x - 2}{x - 2} =$$

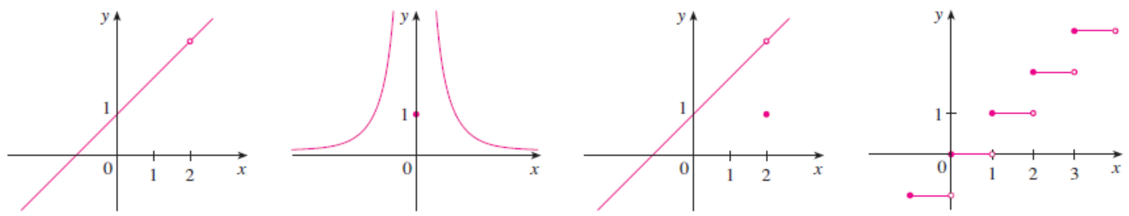
exists. But

$$\lim_{x \to}$$

so f is not continuous at 2.

(d) The greatest integer function $f(x) = [\![x]\!]$ has discontinuities at all of the integers because $\lim_{x \to n} [\![x]\!]$ does not exist if n is an integer.

Figure shows the graphs of the functions in Example 23. In each case the graph can't be drawn without lifting the pen from the paper because a hole or break or jump occurs in the graph. The kind of discontinuity illustrated in parts (a) and (c) is called **removable** because we could remove the discontinuity by redefining f at just the single number 2. [The function $g(x) = x + 1$ is continuous.] The discontinuity in part (b) is called an **infinite discontinuity**. The discontinuities in part (d) are called **jump discontinuities** because the function "jumps" from one value to another.



(a) $f(x)=\dfrac{x^2-x-2}{x-2}$ (b) $f(x)=\begin{cases} \dfrac{1}{x^2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$ (c) $f(x)=\begin{cases} \dfrac{x^2-x-2}{x-2} & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$ (d) $f(x)=[\![x]\!]$

5 THEOREM

(a) Any polynomial is continuous everywhere; that is, it is continuous on $\mathbb{R} = (-\infty, \infty)$.

(b) Any rational function is continuous wherever it is defined; that is, it is continuous on its domain.

7 THEOREM The following types of functions are continuous at every number in their domains:

polynomials	rational functions	root functions
trigonometric functions	inverse trigonometric functions	
exponential functions	logarithmic functions	

Example 11: $\lim_{x \to -2} \dfrac{x^3+2x^2-1}{5-3x}$

Solution:

The function

$$f(x) = \frac{x^3 + 2x^2 - 1}{5 - 3x}$$

is rational, so by Theorem 5 it is continuous on its domain, which is $\{x \mid x \neq \frac{5}{3}\}$. Therefore

$$\lim_{x \to -2} \frac{x^3 + 2x^2 - 1}{5 - 3x} = \lim_{x \to -2} f(x) = f(-2)$$

$$= \frac{(-2)^3 + 2(-2)^2 - 1}{5 - 3(-2)} = -\frac{1}{11}$$ □

EXAMPLE Show that the function $f(x) = 1 - \sqrt{1 - x^2}$ is continuous on the interval $[-1, 1]$.

SOLUTION If $-1 < a < 1$, then using the Limit Laws, we have

$$\lim_{x \to a} f(x) = \lim_{x \to a} \left(1 - \sqrt{1 - x^2}\right)$$

$$= 1 - \lim_{x \to a} \sqrt{1 - x^2} \qquad \text{(by Laws 2 and 7)}$$

$$= 1 - \sqrt{\lim_{x \to a} (1 - x^2)} \qquad \text{(by 11)}$$

$$= 1 - \sqrt{1 - a^2} \qquad \text{(by 2, 7, and 9)}$$

$$= f(a)$$

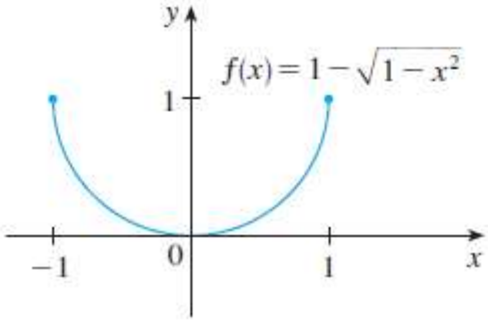
Thus, by Definition l, f is continuous at a if $-1 < a < 1$. Similar calculations show that

$$\lim_{x \to -1^+} f(x) = 1 = f(-1) \qquad \text{and} \qquad \lim_{x \to 1^-} f(x) = 1 = f(1)$$

so f is continuous from the right at -1 and continuous from the left at 1. Therefore, according to Definition 3, f is continuous on $[-1, 1]$.

The graph of f is sketched in Figure 4. It is the lower half of the circle

$$x^2 + (y - 1)^2 = 1$$ ■



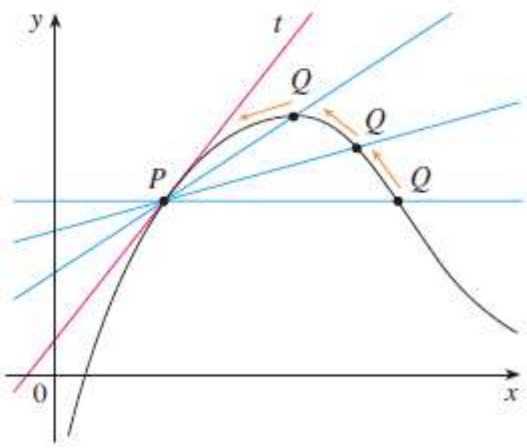
Tangent line, Derivatives and Rates of Change

The problem of finding the tangent line to a curve and the problem of finding the velocity of an object both involve finding the same type of limit, as we saw in previous section. This special type of limit is called a derivative and we will see that it can be interpreted as a rate of change in any of the sciences or engineering.

1 DEFINITION The **tangent line** to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with slope

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

provided that this limit exists.



Example 12: Find an equation of the tangent line to the parabola $y = x^2$ at point P(1,1).

Solution:

Here we have $a = 1$ and $f(x) = x^2$, so the slope is

$$m = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1} \frac{x^2 - 1}{x - 1}$$

$$= \lim_{x \to 1} \frac{(x - 1)(x + 1)}{x - 1}$$

$$= \lim_{x \to 1} (x + 1) = 1 + 1 = 2$$

Using the point-slope form of the equation of a line, we find that an equation of the tangent line at (1,1) is

$$y - 1 = 2(x - 1) \qquad \text{or} \qquad y = 2x - 1$$

Note:

There is another expression for the slope of a tangent line that is sometimes easier to use. If $h = x - a$, then $x = a + h$ and so the slope of the secant line PQ is

$$m_{PQ} = \frac{f(a + h) - f(a)}{h}$$

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Example 13: Find an equation of the tangent line to the hyperbola y = 3/x at point (3,1).

Solution:

Let $f(x) = 3/x$. Then the slope of the tangent at (3, 1) is

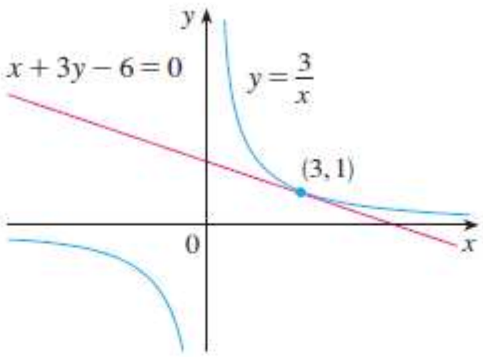
$$m = \lim_{h \to 0} \frac{f(3+h) - f(3)}{h} = \lim_{h \to 0} \frac{\dfrac{3}{3+h} - 1}{h} = \lim_{h \to 0} \frac{\dfrac{3-(3+h)}{3+h}}{h}$$

$$= \lim_{h \to 0} \frac{-h}{h(3+h)} = \lim_{h \to 0} -\frac{1}{3+h} = -\frac{1}{3}$$

Therefore an equation of the tangent at the point (3, 1) is

$$y - 1 = -\tfrac{1}{3}(x - 3)$$

which simplifies to $\quad x + 3y - 6 = 0$



4 DEFINITION The **derivative of a function** f **at a number** a, denoted by $f'(a)$, is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

if this limit exists.

Example 14: Find the derivative of the function $f(x) = x^2 - 8x + 9$ at the number a.

Solution: From Definition 4 we have

$$\begin{aligned} f'(a) &= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \\ &= \lim_{h \to 0} \frac{[(a+h)^2 - 8(a+h) + 9] - [a^2 - 8a + 9]}{h} \\ &= \lim_{h \to 0} \frac{a^2 + 2ah + h^2 - 8a - 8h + 9 - a^2 + 8a - 9}{h} \\ &= \lim_{h \to 0} \frac{2ah + h^2 - 8h}{h} = \lim_{h \to 0} (2a + h - 8) \\ &= 2a - 8 \end{aligned}$$

RATES OF CHANGE

Suppose y is a quantity that depends on another quantity x. Thus y is a function of x and we write $y = f(x)$. If x changes from x_1 to x_2, then the change in x (also called the **increment** of x) is

$$\Delta x = x_2 - x_1$$

and the corresponding change in y is

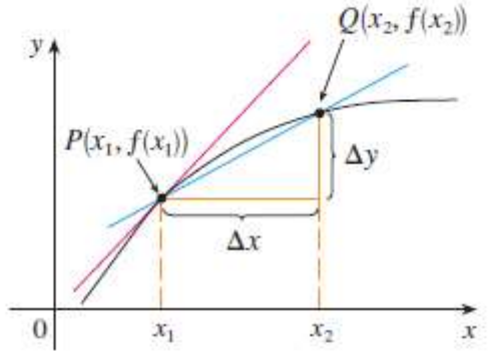
$$\Delta y = f(x_2) - f(x_1)$$

The difference quotient

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

is called the **average rate of change of y with respect to x** over an interval

$$\boxed{6} \qquad \text{instantaneous rate of change} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$



average rate of change = m_{PQ}
instantaneous rate of change = slope of tangent at P

Example 15: A manufacturer produces bolts of a fabric with a fixed width. The cost of
producing x yards of this fabric is C= f(x) dollars,

(a) What is the meaning of the derivative f'(x), what are its units?

(b) In practical terms, what does it mean to say that f'(1000) = 9?

(c) Which do you think is greater f'(50) or f'(500), what about f'(5000)?

Solution:

(a) The derivative $f'(x)$ is the instantaneous rate of change of C with respect to x; that is, $f'(x)$ means the rate of change of the production cost with respect to the number of yards produced. (Economists call this rate of change the *marginal cost*.

Because

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta C}{\Delta x}$$

the units for $f'(x)$ are the same as the units for the difference quotient $\Delta C/\Delta x$. Since ΔC is measured in dollars and Δx in yards, it follows that the units for $f'(x)$ are dollars per yard.

(b) The statement that $f'(1000) = 9$ means that, after 1000 yards of fabric have been manufactured, the rate at which the production cost is increasing is \$9/yard. (When $x = 1000$, C is increasing 9 times as fast as x.)

Since $\Delta x = 1$ is small compared with $x = 1000$, we could use the approximation

$$f'(1000) \approx \frac{\Delta C}{\Delta x} = \frac{\Delta C}{1} = \Delta C$$

and say that the cost of manufacturing the 1000th yard (or the 1001st) is about \$9.

(c) The rate at which the production cost is increasing (per yard) is probably lower when $x = 500$ than when $x = 50$ (the cost of making the 500th yard is less than the cost of the 50th yard) because of economies of scale. (The manufacturer makes more efficient use of the fixed costs of production.) So

$$f'(50) > f'(500)$$

But, as production expands, the resulting large-scale operation might become inefficient and there might be overtime costs. Thus it is possible that the rate of increase of costs will eventually start to rise. So it may happen that

$$f'(5000) > f'(500)$$

□