Differentiation

Introduction

Derivative: it's a function we use to measure the rates at which things change, like slope and velocity and accelerations.

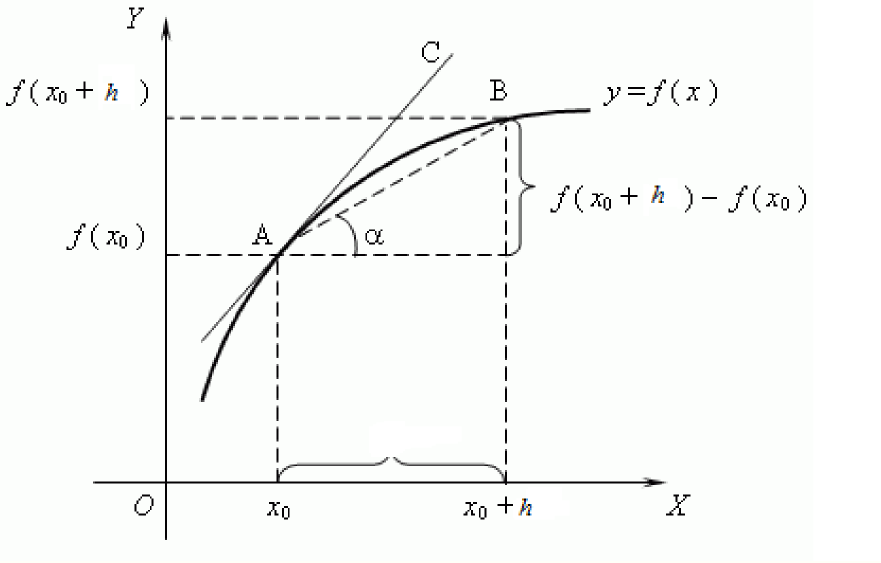
The derivative of a function is a function f′ where value at x is defined in the equation:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

The function $\frac{f(x+h) - f(x)}{h}$ is the difference quotient for f at x.

h is the difference increment.

f′(x) is the first derivate of the function f at x. See figure below.



The most common notation for the differentiation of a function y = f(x) besides f′(x) or dy/dx and df/dx Dx(f) (Dx of f) .. etc..

Application of differentiation:

- The velocity and acceleration at time t and
- Problems of cost , maxima and minima
- Electrical circuits' problem
- Flow of water (change in volume with time)
- Any other problems related to rate of change.

EX-1 – **Find the derivative of the function :** $f(x) = \frac{1}{\sqrt{2x+3}}$

Sol.:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\frac{1}{\sqrt{2(x+\Delta x)+3}} - \frac{1}{\sqrt{2x+3}}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\sqrt{2x+3} - \sqrt{2(x+\Delta x)+3}}{\Delta x . \sqrt{2(x+\Delta x)+3}\sqrt{2x+3}} \cdot \frac{\sqrt{2x+3} + \sqrt{2(x+\Delta x)+3}}{\sqrt{2x+3} + \sqrt{2(x+\Delta x)+3}}$$

$$= \lim_{\Delta x \to 0} \frac{(2x+3) - (2(x+\Delta x)+3)}{\Delta x . \sqrt{2(x+\Delta x)+3}\sqrt{2x+3}(\sqrt{2x+3} + \sqrt{2(x+\Delta x)+3})}$$

$$= \frac{-2}{(2x+3)(\sqrt{2x+3} + \sqrt{2x+3})} = -\frac{1}{\sqrt{(2x+3)^3}}$$

Example 2: Find the derivative of f(x) = $x^2 -$ 2x using the definition.

Solution

f(x) = $x^2 -$ 2x

$$f'(x) = \lim_{h \to -0} \frac{f(x+h) - f(x)}{h}$$

$$f(x+h) = (x+h)^2 - 2(x+h) = x^2 + 2xh + h^2 - 2x - 2h$$

$$\frac{f(x+h)-f(x)}{h}=\frac{x^2+2xh+h^2-2x-2h-(x^2-2x)}{h}$$

$$=\frac{x^2+2xh+h^2-2x-2h-x^2+2x}{h}=\frac{h^2+2hx-2h}{h}=h+2x-2$$

We can take the limit as h ⟶ 0: -

$$f'(x)=\lim_{h\to 0}(h+2x-2)=2x-2$$

Example 3: Show that the derivative of $y=\sqrt{x}$ is $\frac{dy}{dx}=\frac{1}{2\sqrt{x}}$

Solution:

$f(x+h)=\sqrt{x+h}$ and $f(x)=\sqrt{x}$

$$\frac{f(x+h)-f(x)}{h}=\frac{\sqrt{x+h}-\sqrt{x}}{h}$$ ÷ 0 (Not Ok)

$$\frac{f(x+h)-f(x)}{h}=\frac{\sqrt{x+h}-\sqrt{x}}{h}\times\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$$

$$= \;=\frac{(x+h)-x}{h\sqrt{x+h}+\sqrt{x}}=\frac{1}{h\sqrt{x+h}+\sqrt{x}}$$

$$\frac{dy}{dx}=\lim_{h\to 0}\frac{1}{h\sqrt{x+h}+\sqrt{x}}=\frac{1}{2\sqrt{x}}$$ Ok

The Power Rule If n is a positive integer, then

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

Example 4: Differentiate (a) $f(x) = \frac{1}{x^2}$ (b) $y = \sqrt[3]{x^2}$

Solution:

In each case we rewrite the function as a power of x.

(a) Since $f(x) = x^{-2}$, we use the Power Rule with $n = -2$:

$$f'(x) = \frac{d}{dx}(x^{-2}) = -2x^{-2-1} = -2x^{-3} = -\frac{2}{x^3}$$

(b)
$$\frac{dy}{dx} = \frac{d}{dx}\left(\sqrt[3]{x^2}\right) = \frac{d}{dx}(x^{2/3}) = \tfrac{2}{3}x^{(2/3)-1} = \tfrac{2}{3}x^{-1/3}$$

Slope and tangent lines:

Example 5: Find an eq. for the tangent to the curve y = 2/x at x = 3

Solution:

$$m = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f(x+h) = \frac{2}{x+h}$$

$$\frac{f(x+h) - f(x)}{h} = \frac{\frac{2}{x+h} - \frac{2}{x}}{h} = \frac{\frac{2x - 2x - 2h}{(x+h)x}}{h} = \frac{-2h}{(x+h)x} = \frac{-2}{x^2}$$

$m = f'(x) = -2/x^2$ at x = 3 m = $-2/(3)^2$

Then y = -2/9

y + 2/3 = -2/9 (x-3)

Rules for differentiation

If f and g are differentiable functions, the following differentiation rules are valid

1. $\frac{d}{dx}\{f(x)+g(x)\} = \frac{d}{dx}f(x) + \frac{d}{dx}g(x) = f'(x) + g'(x)$ **(Addition Rule)**

2. $\frac{d}{dx}\{f(x)-g(x)\} = \frac{d}{dx}f(x) - \frac{d}{dx}g(x) = f'(x) - g'(x)$

3. $\frac{d}{dx}\{Cf(x)\} = C\frac{d}{dx}f(x) = Cf'(x)$ where C is any constant

4. $\frac{d}{dx}\{f(x)g(x)\} = f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x) = f(x)g'(x) + g(x)f'(x)$ **(Product Rule)**

5. $\frac{d}{dx}\left\{\frac{f(x)}{g(x)}\right\} = \frac{g(x)\frac{d}{dx}f(x) - f(x)\frac{d}{dx}g(x)}{[g(x)]^2} = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$ if $g(x) \neq 0$ **(Quotient Rule)**

6. $\frac{d}{dx}(C) = 0$

7. $\frac{d}{dx}(x^n) = nx^{n-1}$

8. $\frac{d}{dx}(\ln x) = \frac{dx}{x}$ or $\frac{1}{x}dx$

9. $\frac{d}{dx}(e^x) = e^x \ dx$

10. $\frac{d}{dx}(a^x) = a^x \ \ln a \ dx$

11. $\frac{d}{dx}(\log_a x) = \frac{1}{x}\log_a .e = \frac{1}{x}.\frac{1}{\ln a} dx$

Example 6: If $f(x) = e^x - x$, find f' and f''. Compare the graphs of f and f'.

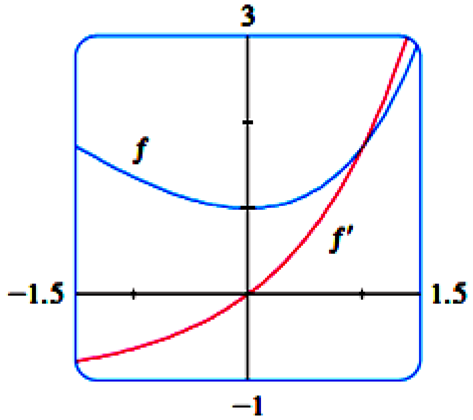
Solution: Using the Difference Rule, we have

$$f'(x) = \frac{d}{dx}(e^x - x) = \frac{d}{dx}(e^x) - \frac{d}{dx}(x) = e^x - 1$$

we defined the second derivative as the derivative of f', so

$$f''(x) = \frac{d}{dx}(e^x - 1) = \frac{d}{dx}(e^x) - \frac{d}{dx}(1) = e^x$$

The function f and its derivative f' are graphed in Figure . Notice that f has a horizontal tangent when $x = 0$; this corresponds to the fact that $f'(0) = 0$. Notice also that, for $x > 0$, $f'(x)$ is positive and f is increasing. When $x < 0$, $f'(x)$ is negative and f is decreasing. ■



EX-8 – Find $\frac{dy}{dx}$ for the following functions :

a) $y = \log_{10} e^x$

b) $y = \log_5 (x+1)^2$

c) $y = \log_2 (3x^2+1)^3$

d) $y = \left[\ln(x^2+2)^2\right]^3$

e) $y + \ln(xy) = 1$

f) $y = \dfrac{(2x^3-4)^{\frac{2}{3}} . (2x^2+3)^{\frac{5}{2}}}{(7x^3+4x-3)^2}$

a) $y = \log_{10} e^x \Rightarrow y = x \log_{10} e \Rightarrow \dfrac{dy}{dx} = \log_{10} e = \dfrac{\ln e}{\ln 10} = \dfrac{1}{\ln 10}$

b) $y = \log_5 (x+1)^2 = 2\log_5 (x+1) \Rightarrow \dfrac{dy}{dx} = \dfrac{2}{(x+1)\ln 5}$

c) $y = 3\log_2 (3x^2+1) \Rightarrow \dfrac{dy}{dx} = \dfrac{3}{3x^2+1} . \dfrac{6x}{\ln 2} = \dfrac{18x}{(3x^2+1)\ln 2}$

d) $\dfrac{dy}{dx} = 3\left[2\ln(x^2+2)\right]^2 \dfrac{2}{x^2+2} .2x = \dfrac{48x\left[\ln(x^2+2)\right]^2}{x^2+2}$

e) $y + \ln x + \ln y = 1 \Rightarrow \dfrac{dy}{dx} + \dfrac{1}{x} + \dfrac{1}{y} . \dfrac{dy}{dx} = 0 \Rightarrow \dfrac{dy}{dx} = -\dfrac{y}{x(y+1)}$

f) $\ln y = \dfrac{2}{3}\ln(2x^3-4) + \dfrac{5}{2}\ln(2x^2+3) - 2\ln(7x^3+4x-3)$

$$\Rightarrow \frac{1}{y} . \frac{dy}{dx} = \frac{2}{3} . \frac{6x^2}{2x^3-4} + \frac{5}{2} . \frac{4x}{2x^2+3} - 2 . \frac{21x^2+4}{7x^3+4x-3}$$

$$\Rightarrow \frac{dy}{dx} = 2y\left[\frac{2x^2}{2x^3-4} + \frac{5x}{2x^2+3} - \frac{21x^2+4}{7x^3+4x-3}\right]$$

EXAMPLE 3 If $f(x) = \sqrt{x}\, g(x)$, where $g(4) = 2$ and $g'(4) = 3$, find $f'(4)$.

SOLUTION Applying the Product Rule, we get

$$f'(x) = \frac{d}{dx}\left[\sqrt{x}\, g(x)\right] = \sqrt{x}\,\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}\left[\sqrt{x}\right]$$

$$= \sqrt{x}\, g'(x) + g(x)\cdot \tfrac{1}{2}x^{-1/2}$$

$$= \sqrt{x}\, g'(x) + \frac{g(x)}{2\sqrt{x}}$$

So $$f'(4) = \sqrt{4}\, g'(4) + \frac{g(4)}{2\sqrt{4}} = 2\cdot 3 + \frac{2}{2\cdot 2} = 6.5$$ ■

Example 7:

Let $y = \dfrac{x^2 + x - 2}{x^3 + 6}$. Then

$$y' = \frac{(x^3+6)\dfrac{d}{dx}(x^2+x-2) - (x^2+x-2)\dfrac{d}{dx}(x^3+6)}{(x^3+6)^2}$$

$$= \frac{(x^3+6)(2x+1) - (x^2+x-2)(3x^2)}{(x^3+6)^2}$$

Reminder:

$$= \frac{(2x^4 + x^3 + 12x + 6) - (3x^4 + 3x^3 - 6x^2)}{(x^3+6)^2}$$

The (

$$= \frac{-x^4 - 2x^3 + 6x^2 + 12x + 6}{(x^3+6)^2}$$

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{\cdots}{[g(x)]^2}$$

Example 8:

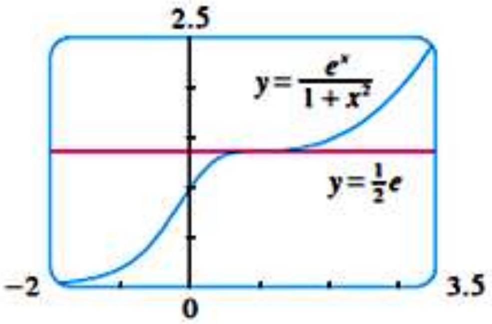
Find an equation of the tangent line to the curve $y = e^x/(1 + x^2)$ at the point $(1, \tfrac{1}{2}e)$.

According to the Quotient Rule, we have

$$\frac{dy}{dx} = \frac{(1+x^2)\dfrac{d}{dx}(e^x) - e^x\dfrac{d}{dx}(1+x^2)}{(1+x^2)^2}$$

$$= \frac{(1+x^2)e^x - e^x(2x)}{(1+x^2)^2} = \frac{e^x(1-2x+x^2)}{(1+x^2)^2}$$

$$= \frac{e^x(1-x)^2}{\cdots}$$



Additional Example

$$\frac{d}{dx}(x^8 + 12x^5 - 4x^4 + 10x^3 - 6x + 5)$$
$$= \frac{d}{dx}(x^8) + 12\frac{d}{dx}(x^5) - 4\frac{d}{dx}(x^4) + 10\frac{d}{dx}(x^3) - 6\frac{d}{dx}(x) + \frac{d}{dx}(5)$$
$$= 8x^7 + 12(5x^4) - 4(4x^3) + 10(3x^2) - 6(1) + 0$$
$$= 8x^7 + 60x^4 - 16x^3 + 30x^2 - 6$$

Derivatives of trigonometric functions

- X is measured in radians
- $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Example 9: Find d/dx {sinx} using the definition?

Solution:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sin(x+h)-\sin x}{h}$$

$$= \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$

$$= \lim_{h\to 0} \left[\frac{\sin x \cos h - \sin x}{h} + \frac{\cos x \sin h}{h}\right]$$

$$= \lim_{h\to 0} \left[\sin x \left(\frac{\cos h - 1}{h}\right) + \cos x \left(\frac{\sin h}{h}\right)\right]$$

1 $$= \lim_{h\to 0} \sin x \cdot \lim_{h\to 0} \frac{\cos h - 1}{h} + \lim_{h\to 0} \cos x \cdot \lim_{h\to 0} \frac{\sin h}{h}$$

Two of these four limits are easy to evaluate. Since we regard x as a constant when computing a limit as $h \to 0$, we have

$$\lim_{h\to 0} \sin x = \sin x \qquad \text{and} \qquad \lim_{h\to 0} \cos x = \cos x$$

But :

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$$

$$\lim_{\theta\to 0} \frac{\cos\theta - 1}{\theta} = 0$$

Then

$$f'(x) = \lim_{h\to 0} \sin x \cdot \lim_{h\to 0} \frac{\cos h - 1}{h} + \lim_{h\to 0} \cos x \cdot \lim_{h\to 0} \frac{\sin h}{h}$$

$$= (\sin x)\cdot 0 + (\cos x)\cdot 1 = \cos x$$

So we have proved the formula for the derivative of the sine function:

4 $$\frac{d}{dx}(\sin x) = \cos x$$

Derivatives of Trigonometric Functions

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

$$\frac{d}{dx}(\cos x) = -\sin x \qquad \frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

Example 10: Differentiate $y = x^2 \sin x$.

Solution: Using the Product Rule and Formula 4, we have

$$\frac{dy}{dx} = x^2 \frac{d}{dx}(\sin x) + \sin x \frac{d}{dx}(x^2)$$
$$= x^2 \cos x + 2x \sin x$$

Example 11:

Differentiate $f(x) = \dfrac{\sec x}{1 + \tan x}$. For what values of x does the graph of f have a horizontal tangent?

Solution:

$$f'(x) = \frac{(1 + \tan x)\frac{d}{dx}(\sec x) - \sec x \frac{d}{dx}(1 + \tan x)}{(1 + \tan x)^2}$$

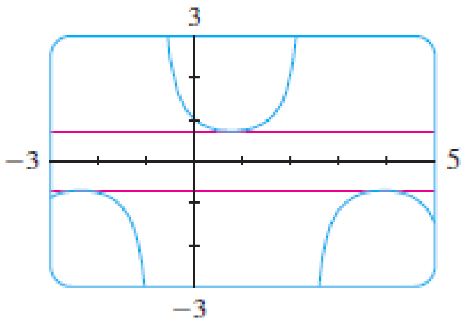
$$= \frac{(1 + \tan x)\sec x \tan x - \sec x \cdot \sec^2 x}{(1 + \tan x)^2}$$

$$= \frac{\sec x(\tan x + \tan^2 x - \sec^2 x)}{(1 + \tan x)^2}$$

$$= \frac{\sec x(\tan x - 1)}{(1 + \tan x)^2}$$

In simplifying the answer we have used the identity $\tan^2 x + 1 = \sec^2 x$.

Since $\sec x$ is never 0, we see that $f'(x) = 0$ when $\tan x = 1$, and this occurs when $x = n\pi + \pi/4$, where n is an integer (see Figure).



Example 2 Differentiate each of the following functions.

(a) $g(x) = 3\sec(x) - 10\cot(x)$ [Solution]

(b) $h(w) = 3w^{-4} - w^2\tan(w)$ [Solution]

(c) $y = 5\sin(x)\cos(x) + 4\csc(x)$ [Solution]

(d) $P(t) = \dfrac{\sin(t)}{3 - 2\cos(t)}$ [Solution]

Solution

(a) $g(x) = 3\sec(x) - 10\cot(x)$

There really isn't a whole lot to this problem. We'll just differentiate each term using the formulas from above.

$$\begin{aligned} g'(x) &= 3\sec(x)\tan(x) - 10\left(-\csc^2(x)\right) \\ &= 3\sec(x)\tan(x) + 10\csc^2(x) \end{aligned}$$

[Return to Problems]

(b) $h(w) = 3w^{-4} - w^2\tan(w)$

In this part we will need to use the product rule on the second term and note that we really will need the product rule here. There is no other way to do this derivative unlike what we saw when

$$h'(w) = -12w^{-5} - \left(2w\tan(w) + w^2\sec^2(w)\right)$$
$$= -12w^{-5} - 2w\tan(w) - w^2\sec^2(w)$$

$$h'(w) = -12w^{-5} - 2w\tan(w) - w^2\sec^2(w)$$

Motion and velocity

Example 13:

The equation of motion of a particle is $s = 2t^3 - 5t^2 + 3t + 4$, where s is measured in centimeters and t in seconds. Find the acceleration as a function of time. What is the acceleration after 2 seconds?

The velocity and acceleration are

$$v(t) = \frac{ds}{dt} = 6t^2 - 10t + 3$$

$$a(t) = \frac{dv}{dt} = 12t - 10$$

The acceleration after 2 s is $a(2) = 14\ \text{cm/s}^2$. ■

NOTE Don't use the Quotient Rule *every* time you see a quotient. Sometimes it's easier to rewrite a quotient first to put it in a form that is simpler for the purpose of differentiation. For instance, although it is possible to differentiate the function

$$F(x) = \frac{3x^2 + 2\sqrt{x}}{x}$$

using the Quotient Rule, it is much easier to perform the division first and write the function as

$$F(x) = 3x + 2x^{-1/2}$$

before differentiating.

Higher order derivatives

Example 14: Find y′, y″ , and y‴ for the following functions:

1. $y = 2x^3 + x - 5$

2. $y = \dfrac{2x}{1-2x}$

Solution

$y = 2x^3 + x - 5$

$y' = 6x^2 + 1$

$y'' = 12x$

$y''' = 12$

2. $y = \dfrac{2x}{1-2x}$

$$y' = \frac{(1-2x)*2 - 2x*(-2)}{(1-2x)^2} = \frac{2-4x+4x}{(1-2x)^2} = \frac{2}{(1-2x)^2}$$

$$y'' = \frac{-2*2(1-2x)*(-2)}{(1-2x)^4} = \frac{8}{(1-2x)^3}$$

$$y'' = \frac{-8*3(1-2x)^2*(-2)}{(1-2x)^6} = \frac{48}{(1-2x)^4}$$