EX-4- Find all derivatives of the following function :

$$y = 3x^3 - 4x^2 + 7x + 10$$

Sol.-

$$\frac{dy}{dx} = 9x^2 - 8x + 7 \quad , \quad \frac{d^2y}{dx^2} = 18x - 8$$

$$\frac{d^3y}{dx^3} = 18 \quad , \quad \frac{d^4y}{dx^4} = 0 = \frac{d^5y}{dx^5} = ...$$

Ex-5 – Find the third derivative of the following function :

$$y = \frac{1}{x} + \sqrt{x^3}$$

Sol.-

$$\frac{dy}{dx} = -\frac{1}{x^2} + \frac{3}{2}x^{\frac{1}{2}}$$

$$\frac{d^2y}{dx^2} = \frac{2}{x^3} + \frac{3}{4}x^{-\frac{1}{2}}$$

$$\frac{d^3y}{dx^3} = -\frac{6}{x^4} - \frac{3}{8}x^{-\frac{3}{2}} \quad \Rightarrow \frac{d^3y}{dx^3} = -\frac{6}{x^4} - \frac{3}{8\sqrt{x^3}}$$

The Chain Rule

The Chain Rule If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $F = f \circ g$ defined by $F(x) = f(g(x))$ is differentiable at x and F' is given by the product

$$F'(x) = f'(g(x)) \cdot g'(x)$$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$ are both differentiable functions, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

COMMENTS ON THE PROOF OF THE CHAIN RULE Let Δu be the change in u corresponding to a change of Δx in x, that is,

$$\Delta u = g(x + \Delta x) - g(x)$$

Then the corresponding change in y is

$$\Delta y = f(u + \Delta u) - f(u)$$

It is tempting to write

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

1
$$= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u} \cdot \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x}$$

$$= \lim_{\Delta u \to 0} \frac{\Delta y}{\Delta u} \cdot \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x}$$

(Note that $\Delta u \to 0$ as $\Delta x \to 0$ since g is continuous.)

$$= \frac{dy}{du} \frac{du}{dx}$$

The Chain Rule can be written either in the prime notation

2
$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$

or, if $y = f(u)$ and $u = g(x)$, in Leibniz notation:

3
$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$$

Example 15:

Find $F'(x)$ if $F(x) = \sqrt{x^2 + 1}$.

SOLUTION 1 (using Equation 2): At the beginning of this section we expressed F as $F(x) = (f \circ g)(x) = f(g(x))$ where $f(u) = \sqrt{u}$ and $g(x) = x^2 + 1$. Since

$$f'(u) = \tfrac{1}{2}u^{-1/2} = \frac{1}{2\sqrt{u}} \qquad \text{and} \qquad g'(x) = 2x$$

we have

$$F'(x) = f'(g(x)) \cdot g'(x)$$

$$= \frac{1}{2\sqrt{x^2+1}} \cdot 2x = \frac{x}{\sqrt{x^2+1}}$$

SOLUTION 2 (using Equation 3): If we let $u = x^2 + 1$ and $y = \sqrt{u}$, then

$$F'(x) = \frac{dy}{du}\frac{du}{dx} = \frac{1}{2\sqrt{u}}(2x) = \frac{1}{2\sqrt{x^2+1}}(2x) = \frac{x}{\sqrt{x^2+1}}$$ ■

NOTE In using the Chain Rule we work from the outside to the inside. Formula 2 says that *we differentiate the outer function f [at the inner function $g(x)$] and then we multiply by the derivative of the inner function.*

$$\frac{d}{dx}\underbrace{f}_{\text{outer function}}\underbrace{(g(x))}_{\text{evaluated at inner function}} = \underbrace{f'}_{\text{derivative of outer function}}\underbrace{(g(x))}_{\text{evaluated at inner function}} \cdot \underbrace{g'(x)}_{\text{derivative of inner function}}$$

EXAMPLE 2 Differentiate (a) $y = \sin(x^2)$ and (b) $y = \sin^2 x$.

SOLUTION

(a) If $y = \sin(x^2)$, then the outer function is the sine function and the inner function is the squaring function, so the Chain Rule gives

$$\frac{dy}{dx} = \frac{d}{dx}\underbrace{\sin}_{\text{outer function}}\underbrace{(x^2)}_{\text{evaluated at inner function}} = \underbrace{\cos}_{\text{derivative of outer function}}\underbrace{(x^2)}_{\text{evaluated at inner function}} \cdot \underbrace{2x}_{\text{derivative of inner function}}$$

$$= 2x\cos(x^2)$$

(b) Note that $\sin^2 x = (\sin x)^2$. Here the outer function is the squaring function and the inner function is the sine function. So

$$\frac{dy}{dx} = \frac{d}{dx}\underbrace{(\sin x)^2}_{\text{inner function}} = \underbrace{2}_{\text{derivative of outer function}} \cdot \underbrace{(\sin x)}_{\text{evaluated at inner function}} \cdot \underbrace{\cos x}_{\text{derivative of inner function}}$$

4 The Power Rule Combined with the Chain Rule If n is any real number and $u = g(x)$ is differentiable, then

$$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$$

Alternatively,

$$\frac{d}{dx}[g(x)]^n = n[g(x)]^{n-1} \cdot g'(x)$$

EXAMPLE 3 Differentiate $y = (x^3 - 1)^{100}$.

SOLUTION Taking $u = g(x) = x^3 - 1$ and $n = 100$ in (4), we have

$$\frac{dy}{dx} = \frac{d}{dx}(x^3 - 1)^{100} = 100(x^3 - 1)^{99}\frac{d}{dx}(x^3 - 1)$$

$$= 100(x^3 - 1)^{99} \cdot 3x^2 = 300x^2(x^3 - 1)^{99}$$

■

EXAMPLE 4 Find $f'(x)$ if $f(x) = \dfrac{1}{\sqrt[3]{x^2 + x + 1}}$.

SOLUTION First rewrite f: $\quad f(x) = (x^2 + x + 1)^{-1/3}$

Thus

$$f'(x) = -\tfrac{1}{3}(x^2 + x + 1)^{-4/3}\frac{d}{dx}(x^2 + x + 1)$$

$$= -\tfrac{1}{3}(x^2 + x + 1)^{-4/3}(2x + 1)$$

EXAMPLE 5 Find the derivative of the function

$$g(t) = \left(\frac{t - 2}{2t + 1}\right)^9$$

SOLUTION Combining the Power Rule, Chain Rule, and Quotient Rule, we get

$$g'(t) = 9\left(\frac{t - 2}{2t + 1}\right)^8 \frac{d}{dt}\left(\frac{t - 2}{2t + 1}\right)$$

$$= 9\left(\frac{t - 2}{2t + 1}\right)^8 \frac{(2t + 1) \cdot 1 - 2(t - 2)}{(2t + 1)^2} = \frac{45(t - 2)^8}{(2t + 1)^{10}}$$

EX-3 – Use the chain rule to express dy/dx in terms of x and y :

a) $y = \dfrac{t^2}{t^2+1}$ and $t = \sqrt{2x+1}$

b) $y = \dfrac{1}{t^2+1}$ and $x = \sqrt{4t+1}$

c) $y = \left(\dfrac{t-1}{t+1}\right)^2$ and $x = \dfrac{1}{t^2} - 1$ at $t = 2$

d) $y = 1 - \dfrac{1}{t}$ and $t = \dfrac{1}{1-x}$ at $x = 2$

Sol.-

a) $y = \dfrac{t^2}{t^2+1} \Rightarrow \dfrac{dy}{dt} = \dfrac{2t(t^2+1) - 2t.t^2}{(t^2+1)^2} = \dfrac{2t}{(t^2+1)^2}$

$t = (2x+1)^{\frac{1}{2}} \Rightarrow \dfrac{dt}{dx} = \dfrac{1}{2}.(2x+1)^{-\frac{1}{2}}.2 = \dfrac{1}{\sqrt{2x+1}}$

$\dfrac{dy}{dx} = \dfrac{dy}{dt}.\dfrac{dt}{dx} = \dfrac{2t}{(t^2+1)^2}.\dfrac{1}{\sqrt{2x+1}} = \dfrac{2\sqrt{2x+1}}{((2x+1)+1)^2}.\dfrac{1}{\sqrt{2x+1}} = \dfrac{1}{2(x+1)^2}$

b) $y = (t^2+1)^{-1} \Rightarrow \dfrac{dy}{dx} = -2t(t^2+1)^{-2} = -\dfrac{2t}{(t^2+1)^2}$

$x = (4t+1)^{\frac{1}{2}} \Rightarrow \dfrac{dx}{dt} = \dfrac{1}{2}(4t+1)^{-\frac{1}{2}}.4 = \dfrac{2}{\sqrt{4t+1}}$

$\dfrac{dy}{dx} = \dfrac{dy}{dt} \div \dfrac{dx}{dt} = -\dfrac{2t}{(t^2+1)^2} \div \dfrac{2}{\sqrt{4t+1}} = -\dfrac{t\sqrt{4t+1}}{(t^2+1)^2}$

$= -\dfrac{x^2-1}{4}.x \div \dfrac{1}{y^2} = -\dfrac{xy^2(x^2-1)}{4}$

where $x = \sqrt{4t+1} \Rightarrow t = \dfrac{x^2-1}{4}$

where $y = \dfrac{1}{t^2+1} \Rightarrow t^2+1 = \dfrac{1}{y}$

Implicit Differentiation

To find dy/dx for any equation involving x and y differentiation each of term in the equation with respect to x instead of finding y in terms of x.

EXAMPLE 4 Find y'' if $x^4 + y^4 = 16$.

SOLUTION Differentiating the equation implicitly with respect to x, we get

$$4x^3 + 4y^3y' = 0$$

Solving for y' gives

$$y' = -\frac{x^3}{y^3} \qquad (3)$$

To find y'' we differentiate this expression for y' using the Quotient Rule and remembering that y is a function of x:

$$y'' = \frac{d}{dx}\left(-\frac{x^3}{y^3}\right) = -\frac{y^3\,(d/dx)(x^3) - x^3\,(d/dx)(y^3)}{(y^3)^2}$$

$$= -\frac{y^3 \cdot 3x^2 - x^3(3y^2y')}{y^6}$$

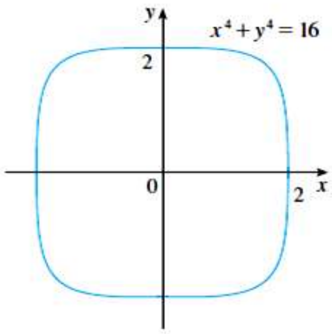
If we now substitute Equation 3 into this expression, we get

$$y'' = -\frac{3x^2y^3 - 3x^3y^2\left(-\dfrac{x^3}{y^3}\right)}{y^6}$$

$$= -\frac{3(x^2y^4 + x^6)}{y^7} = -\frac{3x^2(y^4 + x^4)}{y^7}$$

But the values of x and y must satisfy the original equation $x^4 + y^4 = 16$. So the answer simplifies to

$$y'' = -\frac{3x^2(16)}{y^7} = -48\,\frac{x^2}{y^7}$$



EXAMPLE 3 Find y' if $\sin(x + y) = y^2 \cos x$.

SOLUTION Differentiating implicitly with respect to x and remembering that y is a function of x, we get

$$\cos(x + y) \cdot (1 + y') = y^2(-\sin x) + (\cos x)(2yy')$$

(Note that we have used the Chain Rule on the left side and the Product Rule and Chain Rule on the right side.) If we collect the terms that involve y', we get

$$\cos(x + y) + y^2 \sin x = (2y \cos x)y' - \cos(x + y) \cdot y'$$

So

$$y' = \frac{y^2 \sin x + \cos(x + y)}{2y \cos x - \cos(x + y)}$$

EX-6- Find $\frac{dy}{dx}$ for the following functions:

a) $x^2.y^2 = x^2 + y^2$

b) $(x+y)^3 + (x-y)^3 = x^4 + y^4$

c) $\frac{x-y}{x-2y} = 2$ at $P(3,1)$

d) $xy + 2x - 5y = 2$ at $P(3,2)$

Sol.

a) $x^2(2y\frac{dy}{dx}) + y^2(2x) = 2x + 2y\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{x - xy^2}{x^2y - y}$

b) $3(x+y)^2(1+\frac{dy}{dx}) + 3(x-y)^2(1-\frac{dy}{dx}) = 4x^3 + 4y^3\frac{dy}{dx}$

$$\Rightarrow \frac{dy}{dx} = \frac{4x^3 - 3(x+y)^2 - 3(x-y)^2}{3(x+y)^2 - 3(x-y)^2 - 4y^3} \Rightarrow \frac{dy}{dx} = \frac{2x^3 - 3x^2 - 3y^2}{6xy - 2y^3}$$

c) $\frac{(x-2y)(1-\frac{dy}{dx}) - (x-y)(1-2\frac{dy}{dx})}{(x-2y)^2} = 0 \Rightarrow \frac{dy}{dx} = \frac{y}{x} \Rightarrow \left[\frac{dy}{dx}\right]_{(3,1)} = \frac{1}{3}$

d) $x\frac{dy}{dx} + y + 2 - 5\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{y+2}{5-x} \Rightarrow \left[\frac{dy}{dx}\right]_{(3,2)} = \frac{2+2}{5-3} = 2$

Derivatives of Inverse Trigonometric Functions

Derivatives of Inverse Trigonometric Functions

$$\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\csc^{-1}x) = -\frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2} \qquad \frac{d}{dx}(\cot^{-1}x) = -\frac{1}{1+x^2}$$

EXAMPLE 5 Differentiate (a) $y = \dfrac{1}{\sin^{-1}x}$ and (b) $f(x) = x \arctan\sqrt{x}$.

SOLUTION

(a)
$$\frac{dy}{dx} = \frac{d}{dx}(\sin^{-1}x)^{-1} = -(\sin^{-1}x)^{-2}\frac{d}{dx}(\sin^{-1}x)$$
$$= -\frac{1}{(\sin^{-1}x)^2\sqrt{1-x^2}}$$

(b)
$$f'(x) = x\frac{1}{1+(\sqrt{x})^2}\left(\tfrac{1}{2}x^{-1/2}\right) + \arctan\sqrt{x}$$
$$= \frac{\sqrt{x}}{2(1+x)} + \arctan\sqrt{x}$$ ■

Further examples:

Example 1

Find $\dfrac{dy}{dx}$, given that $y = (1-x^2)\sin^{-1}x$

Here we have a product

$$\therefore\ \frac{dy}{dx} = (1-x^2)\frac{1}{\sqrt{1-x^2}} + \sin^{-1}x.(-2x)$$
$$= \sqrt{1-x^2} - 2x.\sin^{-1}x$$

Example 2

If $y = \tan^{-1}(2x-1)$, find $\dfrac{dy}{dx}$

This time, it is a function of a function

$$\frac{dy}{dx} = \frac{1}{1+(2x-1)^2}.2 = \frac{2}{1+4x^2-4x+1}$$
$$= \frac{2}{2+4x^2-4x} = \frac{1}{2x^2-2x+1}$$

and so on.

Additional Exercise

For each of the following problems differentiate the given function.

1. $T(z) = 2\cos(z) + 6\cos^{-1}(z)$
2. $g(t) = \csc^{-1}(t) - 4\cot^{-1}(t)$
3. $y = 5x^6 - \sec^{-1}(x)$
4. $f(w) = \sin(w) + w^2 \tan^{-1}(w)$
5. $h(x) = \dfrac{\sin^{-1}(x)}{1+x}$