

Related Rates

In a related rates problem the idea is to compute the rate of change of one quantity in terms of the rate of change of another quantity (which may be more easily measured). The procedure is to find an equation that relates the two quantities and then use the Chain Rule to differentiate both sides with respect to time.

Problem Solving Strategy

1. Read the problem carefully.
2. Draw a diagram if possible.
3. Introduce notation. Assign symbols to all quantities that are functions of time.
4. Express the given information and the required rate in terms of derivatives.
5. Write an equation that relates the various quantities of the problem. If necessary, use the geometry of the situation to eliminate one of the variables by substitution.
6. Use the Chain Rule to differentiate both sides of the equation with respect to t .
7. Substitute the given information into the resulting equation and solve for the unknown rate.

Example 1:

Air is being pumped into a spherical balloon so that its volume increases at a rate of $100 \text{ cm}^3/\text{s}$. How fast is the radius of the balloon increasing when the diameter is 50 cm?

SOLUTION We start by identifying two things:

the *given information*:

the rate of increase of the volume of air is $100 \text{ cm}^3/\text{s}$

and the *unknown*:

the rate of increase of the radius when the diameter is 50 cm

In order to express these quantities mathematically, we introduce some suggestive *notation*:

Let V be the volume of the balloon and let r be its radius.

The key thing to remember is that rates of change are derivatives. In this problem, the volume and the radius are both functions of the time t . The rate of increase of the volume with respect to time is the derivative dV/dt , and the rate of increase of the radius is dr/dt . We can therefore restate the given and the unknown as follows:

$$\text{Given:} \quad \frac{dV}{dt} = 100 \text{ cm}^3/\text{s}$$

$$\text{Unknown:} \quad \frac{dr}{dt} \quad \text{when } r = 25 \text{ cm}$$

In order to connect dV/dt and dr/dt , we first relate V and r by the formula for the volume of a sphere:

$$V = \frac{4}{3}\pi r^3$$

In order to use the given information, we differentiate each side of this equation with respect to t . To differentiate the right side, we need to use the Chain Rule:

$$\frac{dV}{dt} = \frac{dV}{dr} \frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}$$

Now we solve for the unknown quantity:

$$\frac{dr}{dt} = \frac{1}{4\pi r^2} \frac{dV}{dt}$$

If we put $r = 25$ and $dV/dt = 100$ in this equation, we obtain

$$\frac{dr}{dt} = \frac{1}{4\pi(25)^2} 100 = \frac{1}{25\pi}$$

The radius of the balloon is increasing at the rate of $1/(25\pi) \approx 0.0127$ cm/s.

Example 2:

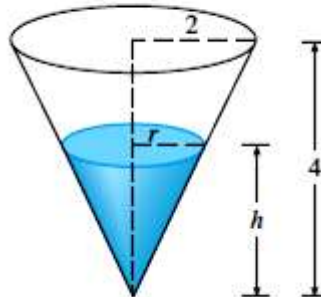
A water tank has the shape of an inverted circular cone with base radius 2 m and height 4 m. If water is being pumped into the tank at a rate of $2 \text{ m}^3/\text{min}$, find the rate at which the water level is rising when the water is 3 m deep.

SOLUTION We first sketch the cone and label it as in Figure 3. Let V , r , and h be the volume of the water, the radius of the surface, and the height of the water at time t , where t is measured in minutes.

We are given that $dV/dt = 2 \text{ m}^3/\text{min}$ and we are asked to find dh/dt when h is 3 m. The quantities V and h are related by the equation

$$V = \frac{1}{3}\pi r^2 h$$

but it is very useful to express V as a function of h alone. In order to eliminate r , we use



the similar triangles in Figure 3 to write

$$\frac{r}{h} = \frac{2}{4} \quad r = \frac{h}{2}$$

and the expression for V becomes

$$V = \frac{1}{3} \pi \left(\frac{h}{2} \right)^2 h = \frac{\pi}{12} h^3$$

Now we can differentiate each side with respect to t :

$$\frac{dV}{dt} = \frac{\pi}{4} h^2 \frac{dh}{dt}$$

so

$$\frac{dh}{dt} = \frac{4}{\pi h^2} \frac{dV}{dt}$$

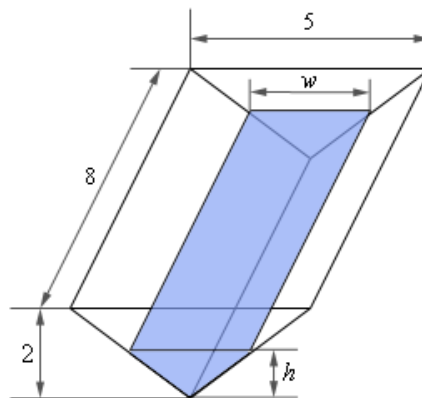
Substituting $h = 3$ m and $dV/dt = 2$ m³/min, we have

$$\frac{dh}{dt} = \frac{4}{\pi(3)^2} \cdot 2 = \frac{8}{9\pi}$$

The water level is rising at a rate of $8/(9\pi) \approx 0.28$ m/min.

Example 3: A trough of water is 8 meters deep and its ends are in the shape of isosceles triangles whose width is 5 meters and height is 2 meters. If water is being

pumped in at a constant rate of $6 \text{ m}^3/\text{sec}$. At what rate is the height of the water changing when the water has a height of 120 cm?



Now, in this problem we know that $V' = 6 \text{ m}^3/\text{sec}$ and we want to determine h' when $h = 1.2 \text{ m}$. Note that because V' is in terms of meters we need to convert h into meters as well. So, we need an equation that will relate these two quantities and the volume of the tank will do it.

The volume of this kind of tank is simple to compute. The volume is the area of the end times the depth. For our case the volume of the water in the tank is,

$$\begin{aligned} V &= (\text{Area of End})(\text{depth}) \\ &= \left(\frac{1}{2} \text{base} \times \text{height}\right)(\text{depth}) \\ &= \frac{1}{2}hw(8) \\ &= 4hw \end{aligned}$$

As with the previous example we've got an extra quantity here, w , that is also changing with time and so we need to get eliminate it from the problem. To do this we'll again make use of the idea of similar triangles. If we look at the end of the tank we'll see that we again have two similar triangles. One for the tank itself and one formed by the water in the tank. Again, remember that with similar triangles that ratios of sides must be equal. In our case we'll use,

$$\frac{w}{5} = \frac{h}{2} \quad \Rightarrow \quad w = \frac{5}{2}h$$

Plugging this into the volume gives a formula for the volume (and only for this tank) that only involved the height of the water.

$$V = 4hw = 4h\left(\frac{5}{2}h\right) = 10h^2$$

We can now differentiate this to get,

$$V' = 20hh'$$

Finally, all we need to do is plug in and solve for h' .

$$6 = 20(1.2)h' \quad \Rightarrow \quad h' = 0.25 \text{ m/sec}$$

So, the height of the water is raising at a rate of 0.25 m/sec.

Derivatives of Hyperbolic Functions

Certain even and odd combinations of the exponential functions e^x and e^{-x} arise so frequently in mathematics and its applications that they deserve to be given special names. In many ways they are analogous to the trigonometric functions, and they have the same relationship to the hyperbola that the trigonometric functions have to the circle. For this reason they are collectively called **hyperbolic functions** and individually called **hyperbolic sine**, **hyperbolic cosine**, and so on.

Definition of the Hyperbolic Functions

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\operatorname{csch} x = \frac{1}{\sinh x}$$

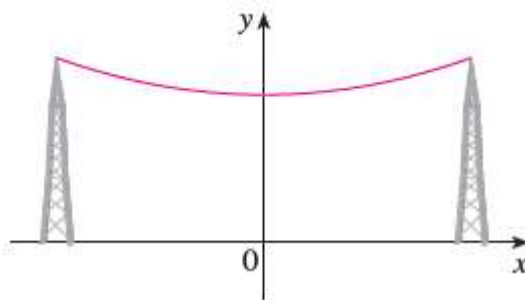
$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$\operatorname{sech} x = \frac{1}{\cosh x}$$

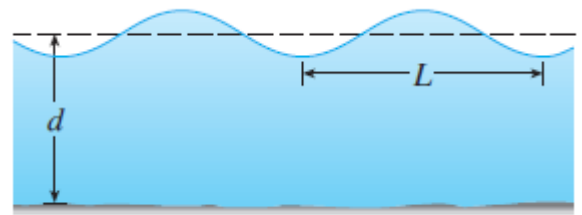
$$\tanh x = \frac{\sinh x}{\cosh x}$$

$$\operatorname{coth} x = \frac{\cosh x}{\sinh x}$$

Applications to science and engineering occur whenever an entity such as light, velocity, electricity, or radioactivity is gradually absorbed or extinguished, for the decay can be represented by hyperbolic functions. The most famous application is the use of hyperbolic cosine to describe the shape of a hanging wire.

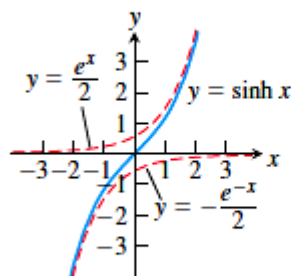


A catenary $y = c + a \cosh(x/a)$

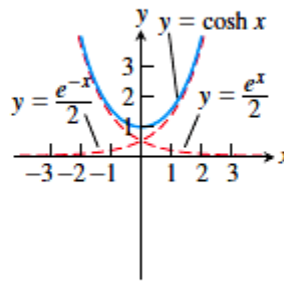


Idealized ocean wave

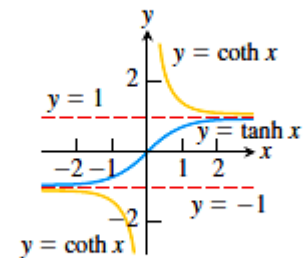
The six basic hyperbolic functions



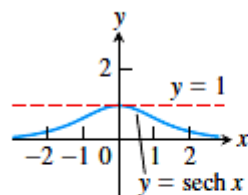
(a)
Hyperbolic sine:
 $\sinh x = \frac{e^x - e^{-x}}{2}$



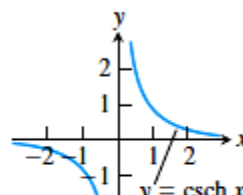
(b)
Hyperbolic cosine:
 $\cosh x = \frac{e^x + e^{-x}}{2}$



(c)
Hyperbolic tangent:
 $\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$



(d)
Hyperbolic secant:
 $\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$



(e)
Hyperbolic cosecant:
 $\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$

Hyperbolic cotangent:
 $\operatorname{coth} x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$

Hyperbolic Identities

$$\sinh(-x) = -\sinh x$$

$$\cosh(-x) = \cosh x$$

$$\cosh^2 x - \sinh^2 x = 1$$

$$1 - \tanh^2 x = \operatorname{sech}^2 x$$

$$\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$$

$$\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$$

Evaluation of hyperbolic functions

To evaluate $\sinh 1.275$

$$\text{Now } \sinh x = \frac{1}{2}(e^x - e^{-x}) \quad \therefore \sinh 1.275 = \frac{1}{2}(e^{1.275} - e^{-1.275}).$$

We now have to evaluate $e^{1.275}$ and $e^{-1.275}$.

Using your calculator, you will find that:

$$e^{1.275} = 3.579 \text{ and } e^{-1.275} = \frac{1}{3.579} = 0.2794$$

$$\begin{aligned} \therefore \sinh 1.275 &= \frac{1}{2}(3.579 - 0.279) \\ &= \frac{1}{2}(3.300) = 1.65 \end{aligned}$$

$$\therefore \sinh 1.275 = 1.65$$

Example 70:

Prove (a) $\cosh^2 x - \sinh^2 x = 1$ and (b) $1 - \tanh^2 x = \operatorname{sech}^2 x$.

SOLUTION

$$\begin{aligned} \text{(a)} \quad \cosh^2 x - \sinh^2 x &= \left(\frac{e^x + e^{-x}}{2} \right)^2 - \left(\frac{e^x - e^{-x}}{2} \right)^2 \\ &= \frac{e^{2x} + 2 + e^{-2x}}{4} - \frac{e^{2x} - 2 + e^{-2x}}{4} \\ &= \frac{4}{4} = 1 \end{aligned}$$

(b) We start with the identity proved in part (a):

$$\cosh^2 x - \sinh^2 x = 1$$

If we divide both sides by $\cosh^2 x$, we get

$$1 - \frac{\sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x}$$

or

$$1 - \tanh^2 x = \operatorname{sech}^2 x$$

1 Derivatives of Hyperbolic Functions

$$\frac{d}{dx} (\sinh x) = \cosh x$$

$$\frac{d}{dx} (\operatorname{csch} x) = -\operatorname{csch} x \coth x$$

$$\frac{d}{dx} (\cosh x) = \sinh x$$

$$\frac{d}{dx} (\operatorname{sech} x) = -\operatorname{sech} x \tanh x$$

$$\frac{d}{dx} (\tanh x) = \operatorname{sech}^2 x$$

$$\frac{d}{dx} (\operatorname{coth} x) = -\operatorname{csch}^2 x$$

Example 71: Differentiate each of the following functions

(a) $f(x) = 2x^5 \cosh x$

(b) $h(t) = \frac{\sinh t}{t+1}$

Solution

(a)

$$f'(x) = 10x^4 \cosh x + 2x^5 \sinh x$$

(b)

$$h'(t) = \frac{(t+1)\cosh t - \sinh t}{(t+1)^2}$$