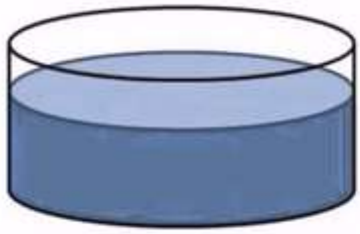
4 Applications of Differentiation

Introduction

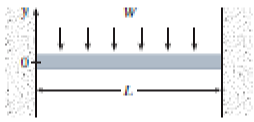
We use the derivative to determine the maximum and minimum values of particular functions (e.g. cost, strength, amount of material used in a building, profit, loss, etc.).

Change of velocity with time

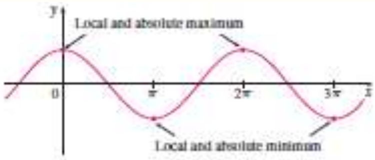
flow of tank



Displacement

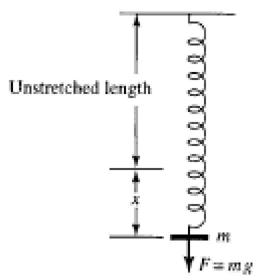


Maximum and Minimum Values



Simple circuit with light

Engineering mechanics



Summary

Mechanics

$v = \frac{dx}{dt}$, where v = velocity, x = distance, t = time.

$a = \frac{dv}{dt}$, where a = acceleration, v = velocity, t = time.

$F = \frac{dW}{dx}$, where F = force, W = work done (or energy used), x = distance moved in the direction of the force.

$F = \frac{dp}{dt}$, where F = force, p = momentum, t = time.

$P = \frac{dW}{dt}$, where P = power, W = work done (or energy used), t = time.

$\frac{dE}{dv} = p$, where E = kinetic energy, v = velocity, p = momentum.

Gases

$\frac{dW}{dV} = p$, where p = pressure, W = work done under isothermal expansion, V = volume.

Circuits

$I = \frac{dQ}{dt}$, where I = current, Q = charge, t = time.

$V = \left(L\frac{dI}{dt}\right)$, where V is the voltage drop across an inductor, L = inductance, I = current, t = time.

Electrostatics

$E = -\frac{dV}{dx}$, where V = potential, E = electric field, x = distance.

Maximum and Minimum Values

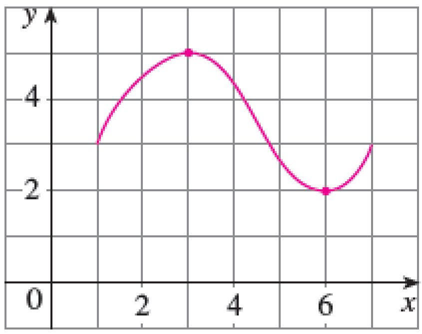
Some of the most important applications of differential calculus are optimization problems, in which we are required to find the optimal (best) way of doing something.

These problems can be reduced to finding the maximum or minimum values of a function.

Let’s first explain exactly what we mean by maximum and minimum values.

We see that the highest point on the graph of the function f shown in Figure is the point (3,5). In other words, the largest value of f is f(3)= 5. Likewise, the smallest value is f(6)= 2. We say that f (3)= 5 is the ***absolute maximum*** of f and f (6)= 2 is

the ***absolute minimum***.



In general, we use the following definition

1 Definition Let c be a number in the domain D of a function f. Then $f(c)$ is the

- **absolute maximum** value of f on D if $f(c) \geq f(x)$ for all x in D.
- **absolute minimum** value of f on D if $f(c) \leq f(x)$ for all x in D.

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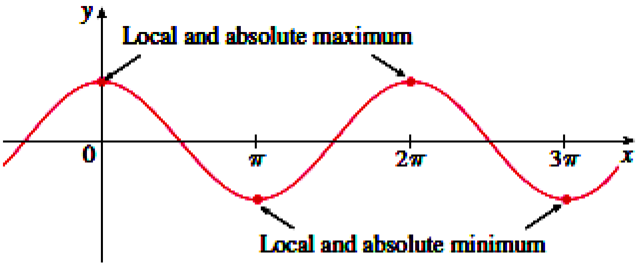
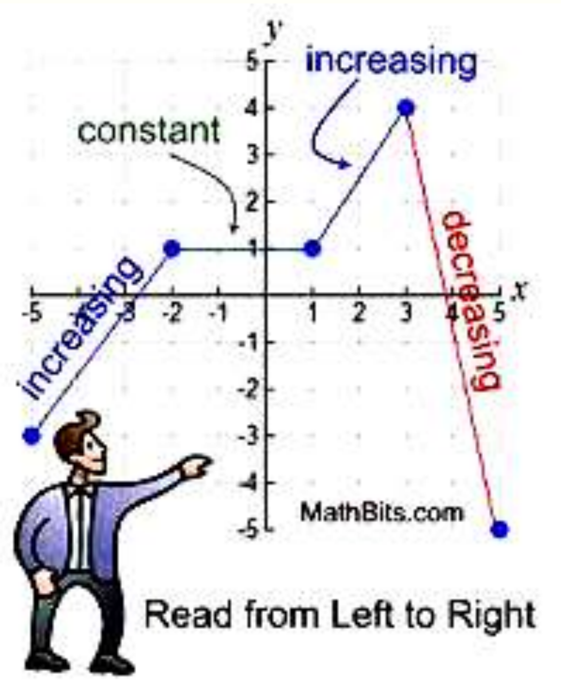
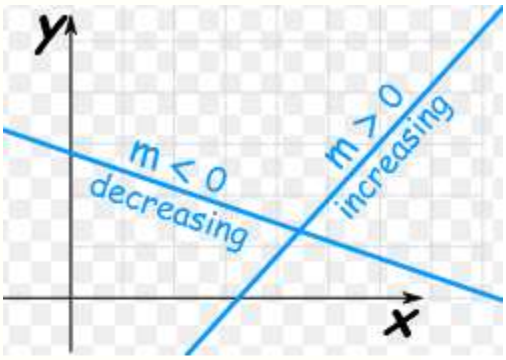
Example 48 The function $f(x) = \cos x$ takes on its (local and absolute) maximum value of 1 infinitely many times, since $\cos 2n\pi = 1$ for any integer n and $-1 \leq \cos x \leq 1$ for all x. (See Figure) Likewise, $\cos(2n + 1)\pi = -1$ is its minimum value, where n is any integer.



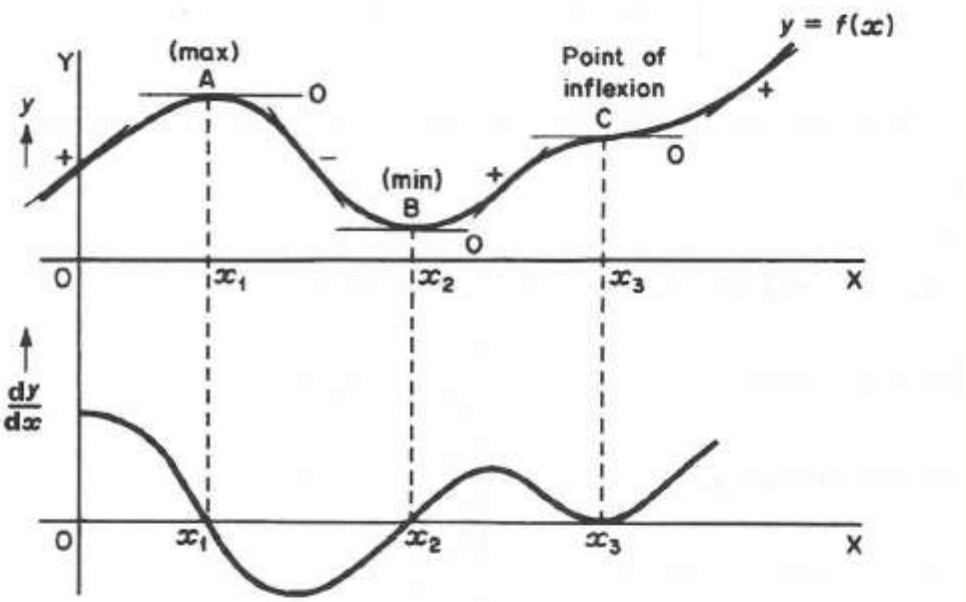




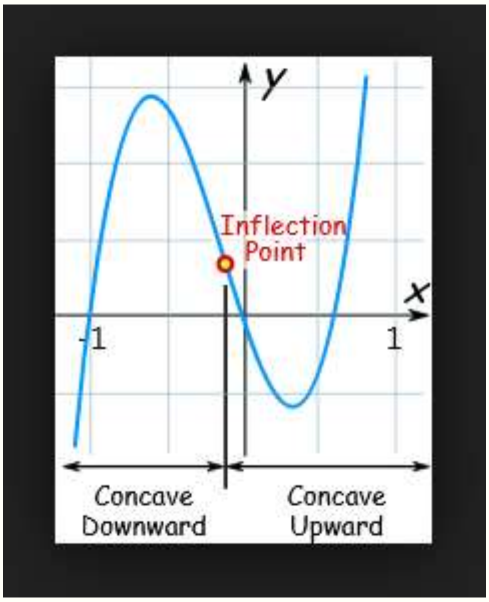
If f(x2) > f(x1) then the function is called **increasing** on its interval

If f(x2) < f(x1) then the function is called **decreasing** on its interval

If f(x2) = f(x1) then the function is called **constant** on its interval



Concavity



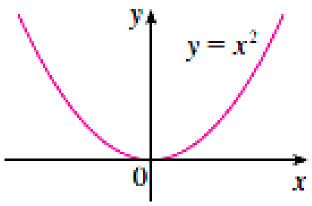
Remember:

The graph of y= f(x) is

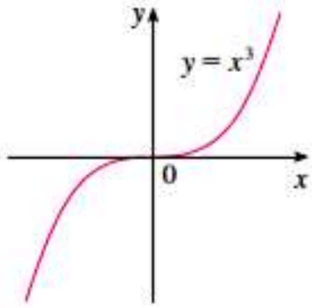
Concve up when y" > 0

Concave down when y" < 0

Example 1:



Mimimum value 0, no maximum

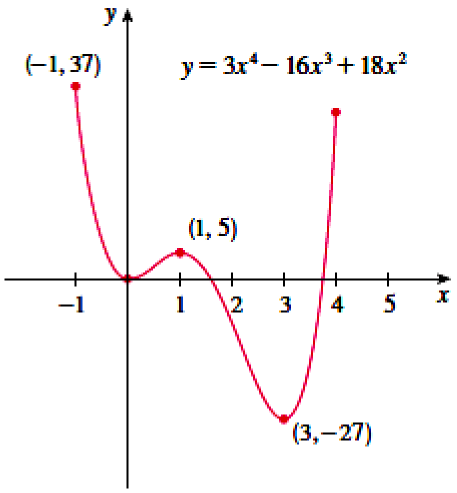


No mimimum, no maximum

Example 2: The graph of the function

$$f(x) = 3x^4 - 16x^3 + 18x^2 \qquad -1 \leqslant x \leqslant 4$$

is shown in Figure . You can see that $f(1) = 5$ is a local maximum, whereas the absolute maximum is $f(-1) = 37$. (This absolute maximum is not a local maximum because it occurs at an endpoint.) Also, $f(0) = 0$ is a local minimum and $f(3) = -27$ is both a local and an absolute minimum. Note that f has neither a local nor an absolute maximum at $x = 4$.



We have seen that some functions have extreme values, whereas others do not. The following theorem gives conditions under which a function is guaranteed to possess extreme values.

Extrema of a function (maxima and minima)

3 The Extreme Value Theorem If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$.

The Second Derivative Test Suppose f'' is continuous near c.

(a) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.

(b) If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.

Example 3: Discuss the curve $y = x^4 - 4x^3$ with respect to concavity, points of inflection, and local maxima and minima. Use this information to sketch the curve

SOLUTION If $f(x) = x^4 - 4x^3$, then

$$f'(x) = 4x^3 - 12x^2 = 4x^2(x - 3)$$

$$f''(x) = 12x^2 - 24x = 12x(x - 2)$$

To find the critical numbers we set $f'(x) = 0$ and obtain $x = 0$ and $x = 3$. (Note that f' is a polynomial and hence defined everywhere.) To use the Second Derivative Test we evaluate f'' at these critical numbers:

$$f''(0) = 0 \qquad f''(3) = 36 > 0$$

Since $f'(3) = 0$ and $f''(3) > 0$, $f(3) = -27$ is a local minimum. [In fact, the expression for $f'(x)$ shows that f decreases to the left of 3 and increases to the right of 3.] Since $f''(0) = 0$, the Second Derivative Test gives no information about the critical number 0. But since $f'(x) < 0$ for $x < 0$ and also for $0 < x < 3$, the First Derivative Test tells us that f does not have a local maximum or minimum at 0.

Since $f''(x) = 0$ when $x = 0$ or 2, we divide the real line into intervals with these numbers as endpoints and complete the following chart.

Interval	$f''(x) = 12x(x-2)$	Concavity
$(-\infty, 0)$	+	upward
$(0, 2)$	−	downward
$(2, \infty)$	+	upward

The point (0, 0) is an inflection point since the curve changes from concave up to concave downward there. Also (2, −16) is an inflection point since the curve changes from concave downward to concave upward there.

Using the local minimum, the intervals of concavity, and the inflection points,