Optimization Problems

In solving such practical problems the greatest challenge is often to convert the word problem into a mathematical optimization problem by setting up the function that is to be maximized or minimized

Solving Applied Optimization Problems

1. *Read the problem.* Read the problem until you understand it. What is given? What is the unknown quantity to be optimized?
2. *Draw a picture.* Label any part that may be important to the problem.
3. *Introduce variables.* List every relation in the picture and in the problem as an equation or algebraic expression, and identify the unknown variable.
4. *Write an equation for the unknown quantity.* If you can, express the unknown as a function of a single variable or in two equations in two unknowns. This may require considerable manipulation.
5. *Test the critical points and endpoints in the domain of the unknown.* Use what you know about the shape of the function's graph. Use the first and second derivatives to identify and classify the function's critical points.

Example 8:
A cylindrical can is to be made to hold 1 L of oil. Find the dimensions that will minimize the cost of the metal to manufacture the can?.

SOLUTION Draw the diagram as in Figure 3, where r is the radius and h the height (both in centimeters). In order to minimize the cost of the metal, we minimize the total surface area of the cylinder (top, bottom, and sides). From Figure 4 we see that the sides are made from a rectangular sheet with dimensions $2\pi r$ and h. So the surface area is

$$A = 2\pi r^2 + 2\pi rh$$

We would like to express A in terms of one variable, r. To eliminate h we use the fact that the volume is given as 1 L, which is equivalent to 1000 cm^3. Thus

$$\pi r^2 h = 1000$$

which gives $h = 1000/(\pi r^2)$. Substitution of this into the expression for A gives

$$A = 2\pi r^2 + 2\pi r\left(\frac{1000}{\pi r^2}\right) = 2\pi r^2 + \frac{2000}{r}$$

We know r must be positive, and there are no limitations on how large r can be. Therefore the function that we want to minimize is

$$A(r) = 2\pi r^2 + \frac{2000}{r} \qquad r > 0$$

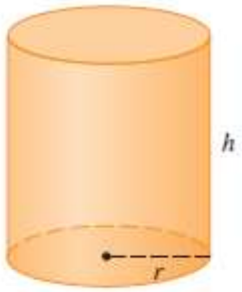


FIGURE 3

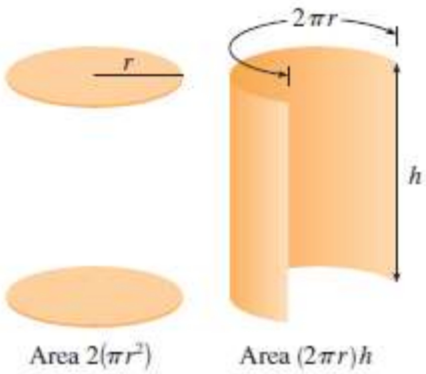


FIGURE 4

To find the critical numbers, we differentiate:

$$A'(r) = 4\pi r - \frac{2000}{r^2} = \frac{4(\pi r^3 - 500)}{r^2}$$

Then $A'(r) = 0$ when $\pi r^3 = 500$, so the only critical number is $r = \sqrt[3]{500/\pi}$.

The value of h corresponding to $r = \sqrt[3]{500/\pi}$ is

$$h = \frac{1000}{\pi r^2} = \frac{1000}{\pi(500/\pi)^{2/3}} = 2\sqrt[3]{\frac{500}{\pi}} = 2r$$

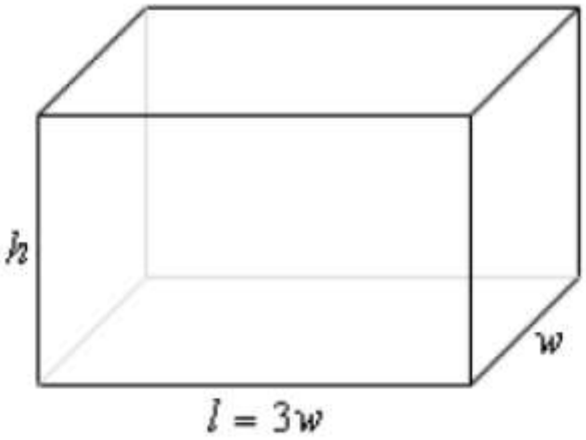
Thus, to minimize the cost of the can, the radius should be $\sqrt[3]{500/\pi}$ cm and the height should be equal to twice the radius, namely, the diameter. ■

Example 9:

We want to construct a box whose base length is 3 times the base width. The material used to build the top and bottom cost $\$10/\text{ft}^2$ and the material used to build the sides cost $\$6/\text{ft}^2$. If the box must have a volume of 50ft^3 determine the dimensions that will minimize the cost to build the box.

Solution:

First, we sketch a figure as below:



We want to minimize the cost of the materials subject to the constraint that the volume must be 50ft^3. Note as well that the cost for each side is just the area of that side times the appropriate cost.

The two functions we'll be working with here this time are,

$$\text{Minimize}: C = 10(2lw) + 6(2wh + 2lh) = 60w^2 + 48wh$$
$$\text{Constraint}: 50 = lwh = 3w^2h$$

As with the first example, we will solve the constraint for one of the variables and plug this into the cost. It will definitely be easier to solve the constraint for h so let's do that.

$$h = \frac{50}{3w^2}$$

Plugging this into the cost gives,

$$C(w) = 60w^2 + 48w\left(\frac{50}{3w^2}\right) = 60w^2 + \frac{800}{w}$$

Now, let's get the first and second (we'll be needing this later…) derivatives,

$$C'(w) = 120w - 800w^{-2} = \frac{120w^3 - 800}{w^2} \qquad C''(w) = 120 + 1600w^{-3}$$

The next critical point will come from determining where the numerator is zero.

$$120w^3 - 800 = 0 \qquad \Rightarrow \qquad w = \sqrt[3]{\frac{800}{120}} = \sqrt[3]{\frac{20}{3}} = 1.8821$$

First, we know that whatever the value of w that we get it will have to be positive and we can see second derivative above that provided $w > 0$ we will have $C''(w) > 0$ and so in the interval of possible optimal values the cost function will always be concave up and so $w = 1.8821$ must give the absolute minimum cost.

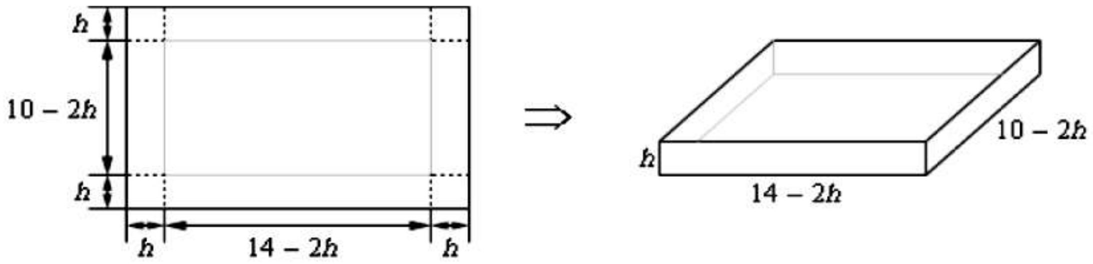
All we need to do now is to find the remaining dimensions.

$$w = 1.8821$$
$$l = 3w = 3(1.8821) = 5.6463$$
$$h = \frac{50}{3w^2} = \frac{50}{3(1.8821)^2} = 4.7050$$

Also, even though it was not asked for, the minimum cost is : $C(1.8821) = \$637.60$.

Example 10:

We have a piece of cardboard that is 14 inches by 10 inches and we're going to cut out the corners as shown below and fold up the sides to form a box, also shown below. Determine the height of the box that will give a maximum volume.



In this case we want to maximize the volume. Here is the volume, in terms of h and its first derivative.

$$V(h) = h(14-2h)(10-2h) = 140h - 48h^2 + 4h^3 \qquad V'(h) = 140 - 96h + 12h^2$$

Setting the first derivative equal to zero and solving gives the following two critical points,

$$h = \frac{12 \pm \sqrt{39}}{3} = 1.9183, \quad 6.0817$$

So, knowing that whatever h is it must be in the range $0 \le h \le 5$ we can see that the second critical point is outside this range and so the only critical point that we need to worry about is 1.9183.

Finally, since the volume is defined and continuous on $0 \le h \le 5$ all we need to do is plug in the critical points and endpoints into the volume to determine which gives the largest volume. Here are those function evaluations.

$$V(0) = 0 \qquad V(1.9183) = 120.1644 \qquad V(5) = 0$$

So, if we take $h = 1.9183$ we get a maximum volume.

Example 11: a rectangle is to be inscribed in a semicircle of radius 2. What is the largest area the rectangle can have, and what are its dimensions?

Solution Let $\left(x, \sqrt{4-x^2}\right)$ be the coordinates of the corner of the rectangle obtained by placing the circle and rectangle in the coordinate plane (Figure 4.40). The length, height, and area of the rectangle can then be expressed in terms of the position x of the lower right-hand corner:

$$\text{Length: } 2x, \qquad \text{Height: } \sqrt{4-x^2}, \qquad \text{Area: } 2x\sqrt{4-x^2}.$$

Notice that the values of x are to be found in the interval $0 \le x \le 2$, where the selected corner of the rectangle lies.

Our goal is to find the absolute maximum value of the function

$$A(x) = 2x\sqrt{4-x^2}$$

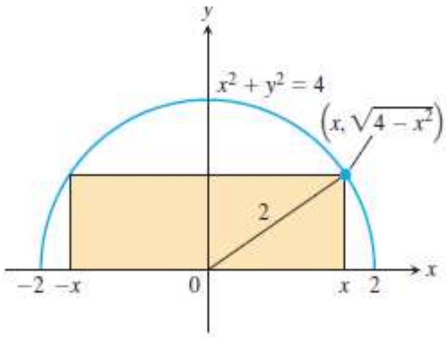
on the domain $[0, 2]$.

The derivative

$$\frac{dA}{dx} = \frac{-2x^2}{\sqrt{4-x^2}} + 2\sqrt{4-x^2}$$

is not defined when $x = 2$ and is equal to zero when

$$\begin{aligned} \frac{-2x^2}{\sqrt{4-x^2}} + 2\sqrt{4-x^2} &= 0 \\ -2x^2 + 2(4-x^2) &= 0 \\ 8 - 4x^2 &= 0 \\ x^2 &= 2 \\ x &= \pm\sqrt{2}. \end{aligned}$$

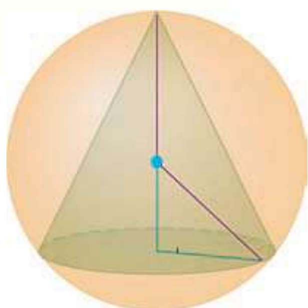


Of the two zeros, $x = \sqrt{2}$ and $x = -\sqrt{2}$, only $x = \sqrt{2}$ lies in the interior of A's domain and makes the critical-point list. The values of A at the endpoints and at this one critical point are

$$\text{Critical point value: } \quad A\left(\sqrt{2}\right) = 2\sqrt{2}\sqrt{4-2} = 4$$
$$\text{Endpoint values: } \quad A(0) = 0, \quad A(2) = 0.$$

The area has a maximum value of 4 when the rectangle is $\sqrt{4-x^2} = \sqrt{2}$ units high and $2x = 2\sqrt{2}$ units long.

Example 12**(Homework)** Find the volume of the largest right circular cone that can be inscribed in a sphere of radius 3?



V = 32π / 3