Indeterminate Forms and L'Hospital's Rule

John Bernoulli discovered a rule using derivatives to calculate limits of fractions whose numerators and denominators both approach zero or +∞. The rule is known today as l'Hôpital's Rule, after Guillaume de l'Hôpital.

Indeterminate Form 0/0

If we want to know how the function

$$F(x) = \frac{x - \sin x}{x^3}$$

behaves *near* $x = 0$ (where it is undefined), we can examine the limit of $F(x)$ as $x \to 0$. We cannot apply the Quotient Rule for limits (Theorem 1 of Chapter 2) because the limit of the denominator is 0. Moreover, in this case, *both* the numerator and denominator approach 0, and $0/0$ is undefined. Such limits may or may not exist in general, but the limit does exist for the function $F(x)$ under discussion by applying l'Hôpital's Rule, as we will see in Example 1d.

If the continuous functions $f(x)$ and $g(x)$ are both zero at $x = a$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

cannot be found by substituting $x = a$. The substitution produces $0/0$, a meaningless expression, which we cannot evaluate. We use $0/0$ as a notation for an expression known as an **indeterminate form**. Other meaningless expressions often occur, such as ∞/∞, $\infty \cdot 0$, $\infty - \infty$, 0^0, and 1^∞, which cannot be evaluated in a consistent way; these are called indeterminate forms as well. Sometimes, but not always, limits that lead to indeterminate forms may be found by cancelation, rearrangement of terms, or other algebraic

THEOREM 6—L'Hôpital's Rule Suppose that $f(a) = g(a) = 0$, that f and g are differentiable on an open interval I containing a, and that $g'(x) \neq 0$ on I if $x \neq a$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

assuming that the limit on the right side of this equation exists.

Example 13:

The following limits involve 0/0 indeterminate forms, so we apply l'Hôpital's Rule. In some cases, it must be applied repeatedly.

(a) $\lim_{x\to 0}\frac{3x-\sin x}{x} = \lim_{x\to 0}\frac{3-\cos x}{1} = \left.\frac{3-\cos x}{1}\right|_{x=0} = 2$

(b) $\lim_{x\to 0}\frac{\sqrt{1+x}-1}{x} = \lim_{x\to 0}\frac{\frac{1}{2\sqrt{1+x}}}{1} = \frac{1}{2}$

(c) $\lim_{x\to 0}\frac{\sqrt{1+x}-1-x/2}{x^2}$ $\quad\frac{0}{0}$; apply l'Hôpital's Rule.

$= \lim_{x\to 0}\frac{(1/2)(1+x)^{-1/2}-1/2}{2x}$ $\quad$ Still $\frac{0}{0}$; apply l'Hôpital's Rule again.

$= \lim_{x\to 0}\frac{-(1/4)(1+x)^{-3/2}}{2} = -\frac{1}{8}$ $\quad$ Not $\frac{0}{0}$; limit is found.

(d) $\lim_{x\to 0}\frac{x-\sin x}{x^3}$ $\quad\frac{0}{0}$; apply l'Hôpital's Rule.

$= \lim_{x\to 0}\frac{1-\cos x}{3x^2}$ $\quad$ Still $\frac{0}{0}$; apply l'Hôpital's Rule again.

$= \lim_{x\to 0}\frac{\sin x}{6x}$ $\quad$ Still $\frac{0}{0}$; apply l'Hôpital's Rule again.

$= \lim_{x\to 0}\frac{\cos x}{6} = \frac{1}{6}$ $\quad$ Not $\frac{0}{0}$; limit is found.

Example 14: Calculate $\lim_{x\to\infty}\frac{\ln x}{\sqrt{x}}$.

SOLUTION Since $\ln x \to \infty$ and $\sqrt{x} \to \infty$ as $x \to \infty$, l'Hospital's Rule applies:

$$\lim_{x\to\infty}\frac{\ln x}{\sqrt{x}} = \lim_{x\to\infty}\frac{1/x}{\frac{1}{2}x^{-1/2}} = \lim_{x\to\infty}\frac{1/x}{1/(2\sqrt{x})}$$

Notice that the limit on the right side is now indeterminate of type $\frac{0}{0}$. But instead of applying l'Hospital's Rule a second time as we did in Example 2, we simplify the expression and see that a second application is unnecessary:

Example 15: Find the limits of these ∞/∞ forms:

(a) $\lim_{x\to\pi/2} \frac{\sec x}{1+\tan x}$ (b) $\lim_{x\to\infty} \frac{\ln x}{2\sqrt{x}}$ (c) $\lim_{x\to\infty} \frac{e^x}{x^2}$.

Solution

(a) The numerator and denominator are discontinuous at $x = \pi/2$, so we investigate the one-sided limits there. To apply l'Hôpital's Rule, we can choose I to be any open interval with $x = \pi/2$ as an endpoint.

$\lim_{x\to(\pi/2)^-} \frac{\sec x}{1+\tan x}$ $\frac{\infty}{\infty}$ from the left so we apply l'Hôpital's Rule.

$$= \lim_{x\to(\pi/2)^-} \frac{\sec x \tan x}{\sec^2 x} = \lim_{x\to(\pi/2)^-} \sin x = 1$$

The right-hand limit is 1 also, with $(-\infty)/(-\infty)$ as the indeterminate form. Therefore, the two-sided limit is equal to 1.

(b) $\lim_{x\to\infty} \frac{\ln x}{2\sqrt{x}} = \lim_{x\to\infty} \frac{1/x}{1/\sqrt{x}} = \lim_{x\to\infty} \frac{1}{\sqrt{x}} = 0$ $\quad \frac{1/x}{1/\sqrt{x}} = \frac{\sqrt{x}}{x} = \frac{1}{\sqrt{x}}$

(c) $\lim_{x\to\infty} \frac{e^x}{x^2} = \lim_{x\to\infty} \frac{e^x}{2x} = \lim_{x\to\infty} \frac{e^x}{2} = \infty$ ■

Example 16:

Evaluate $\lim_{x\to 0^+} x \ln x$.

SOLUTION The given limit is indeterminate because, as $x \to 0^+$, the first factor (x) approaches 0 while the second factor ($\ln x$) approaches $-\infty$. Writing $x = 1/(1/x)$, we have $1/x \to \infty$ as $x \to 0^+$, so l'Hospital's Rule gives

$$\lim_{x\to 0^+} x \ln x = \lim_{x\to 0^+} \frac{\ln x}{1/x} = \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+} (-x) = 0$$ ■