5 Integrals

The Indefinite Integral

An integral can be considered to be an **antiderivative**. Thus, if we know that the derivative of $F(x)$ is $f(x)$ $[=F'(x)]$, an integral of $f(x)$ is $F(x)$. For example, the derivative of $\frac{1}{3}x^3$ is x^2, and an integral of x^2 is $\frac{1}{3}x^3$. Note that we have used the article an. Since the derivative of a constant is zero, $F(x)$ is arbitrary to the extent of an arbitrary constant. The integral we have defined is known as an **indefinite integral** which is usually denoted by the symbol $\int$. Thus, we write

$$\int f(x)\,dx = F(x) + C,$$

where C is any arbitrary constant.

Example 79: Evaluate the following indifinite integral $\int x^4 + 3x - 9\, dx$

Solution

The indefinite integral is

$$\int x^4 + 3x - 9\, dx = \frac{1}{x}x^5 + \frac{3}{2}x^2 - 9x + c$$

PROPERTIES OF INDEFINITE INTEGRALS

1. *A constant factor can be taken outside the integral sign:*

$$\int af(x)\,dx = a\int f(x)\,dx \qquad (a = \text{const}).$$

2. *Integral of the sum or difference of functions (additivity):*

$$\int [f(x) \pm g(x)]\, dx = \int f(x)\, dx \pm \int g(x)\, dx.$$

Computing Indefinite Integrals

$$\int cf(x)\, dx = c\int f(x)\, dx \qquad \int [f(x) + g(x)]\, dx = \int f(x)\, dx + \int g(x)\, dx$$

$$\int k\, dx = kx + C$$

$$\int x^n\, dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1) \qquad \int \frac{1}{x}\, dx = \ln|x| + C$$

$$\int e^x\, dx = e^x + C \qquad \int b^x\, dx = \frac{b^x}{\ln b} + C$$

$$\int \sin x\, dx = -\cos x + C \qquad \int \cos x\, dx = \sin x + C$$

$$\int \sec^2 x\, dx = \tan x + C \qquad \int \csc^2 x\, dx = -\cot x + C$$

$$\int \sec x \tan x\, dx = \sec x + C \qquad \int \csc x \cot x\, dx = -\csc x + C$$

Example 1: Evaluate each of the following integrals

(a) $\int 5t^3 - 10t^{-6} + 4\,dt$

(b) $\int x^8 + x^{-8}\,dx$

(c) $\int 3\sqrt[4]{x^3} + \frac{7}{x^5} + \frac{1}{6\sqrt{x}}\,dx$

(d) $\int dy$

(e) $\int \left(w + \sqrt[3]{w}\right)\left(4 - w^2\right)dw$

(f) $\int \frac{4x^{10} - 2x^4 + 15x^2}{x^3}\,dx$

Solution:

(a) $\int 5t^3 - 10t^{-6} + 4\,dt$

$$\int 5t^3 - 10t^{-6} + 4\,dt = 5\left(\frac{1}{4}\right)t^4 - 10\left(\frac{1}{-5}\right)t^{-5} + 4t + c$$

$$= \frac{5}{4}t^4 + 2t^{-5} + 4t + c$$

(b) $\int x^8 + x^{-8}\,dx$

$$\int x^8 + x^{-8}\,dx = \frac{1}{9}x^9 - \frac{1}{7}x^{-7} + c$$

(c) $\int 3\sqrt[4]{x^3} + \frac{7}{x^5} + \frac{1}{6\sqrt{x}}\,dx$

$$\int 3\sqrt[4]{x^3}+\frac{7}{x^5}+\frac{1}{6\sqrt{x}}\,dx=\int 3x^{\frac{3}{4}}+7x^{-5}+\frac{1}{6}x^{-\frac{1}{2}}\,dx$$

$$=3\frac{1}{7/4}x^{\frac{7}{4}}-\frac{7}{4}x^{-4}+\frac{1}{6}\left(\frac{1}{1/2}\right)x^{\frac{1}{2}}+c$$

$$=\frac{12}{7}x^{\frac{7}{4}}-\frac{7}{4}x^{-4}+\frac{1}{3}x^{\frac{1}{2}}+c$$

(d) $\int dy$

$$\int dy=\int 1\,dy=y+c$$

(e) $\int\left(w+\sqrt[3]{w}\right)\left(4-w^2\right)dw$

$$\int\left(w+\sqrt[3]{w}\right)\left(4-w^2\right)dw=\int 4w-w^3+4w^{\frac{1}{3}}-w^{\frac{7}{3}}dw$$

$$=2w^2-\frac{1}{4}w^4+3w^{\frac{4}{3}}-\frac{3}{10}w^{\frac{10}{3}}+c$$

(f) $\int \frac{4x^{10}-2x^4+15x^2}{x^3}dx$

$$\int \frac{4x^{10}-2x^4+15x^2}{x^3}dx=\int \frac{4x^{10}}{x^3}-\frac{2x^4}{x^3}+\frac{15x^2}{x^3}dx$$

$$=\int 4x^7-2x+\frac{15}{x}dx$$

$$=\frac{1}{2}x^8-x^2+15\ln|x|+c$$