Furtehr examples:

EXAMPLE 1 Find the general indefinite integral

$$\int (10x^4 - 2\sec^2 x)\,dx$$

SOLUTION Using our convention and Table 1, we have

$$\int (10x^4 - 2\sec^2 x)\,dx = 10\int x^4\,dx - 2\int \sec^2 x\,dx$$

$$= 10\frac{x^5}{5} - 2\tan x + C$$

$$= 2x^5 - 2\tan x + C$$

You should check this answer by differentiating it. ■

EXAMPLE 2 Evaluate $\int \frac{\cos\theta}{\sin^2\theta}\,d\theta$.

SOLUTION This indefinite integral isn't immediately apparent in Table 1, so we use trigonometric identities to rewrite the function before integrating:

$$\int \frac{\cos\theta}{\sin^2\theta}\,d\theta = \int \left(\frac{1}{\sin\theta}\right)\left(\frac{\cos\theta}{\sin\theta}\right)d\theta$$

$$= \int \csc\theta\cot\theta\,d\theta = -\csc\theta + C$$ ■

Integrals of inverse trigonometric and hyperbolic functions

$$\int \frac{1}{x^2+1}\,dx = \tan^{-1}x + c \qquad \int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x + c$$

$$\int \sinh x\,dx = \cosh x + c \qquad \int \cosh x\,dx = \sinh x + c$$

$$\int \operatorname{sech}^2 x\,dx = \tanh x + c \qquad \int \operatorname{sech} x\tanh x\,dx = -\operatorname{sech} x + c$$

$$\int \operatorname{csch}^2 x\,dx = -\coth x + c \qquad \int \operatorname{csch} x\coth x\,dx = -\operatorname{csch} x + c$$

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} \qquad \therefore \int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C$$

$$\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}} \qquad \therefore \int \frac{-1}{\sqrt{1-x^2}}\,dx = \cos^{-1} x + C$$

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2} \qquad \therefore \int \frac{1}{1+x^2}\,dx = \tan^{-1} x + C$$

$$\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{x^2+1}} \qquad \therefore \int \frac{1}{\sqrt{x^2+1}}\,dx = \sinh^{-1} x + C$$

$$\frac{d}{dx}(\cosh^{-1}) = \frac{1}{\sqrt{x^2-1}} \qquad \therefore \int \frac{1}{\sqrt{x^2-1}}\,dx = \cosh^{-1} x + C$$

$$\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1-x^2} \qquad \therefore \int \frac{1}{1-x^2}\,dx = \tanh^{-1} x + C$$

Substitution Rule for Indefinite Integrals

Because of the Fundamental Theorem, it's important to be able to find antiderivatives. But our antidifferentiation formulas don't tell us how to evaluate integrals such as

1 $$\int 2x\sqrt{1+x^2}\,dx$$

To find this integral we use the problem-solving strategy of *introducing something extra.*

4 The Substitution Rule If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))\,g'(x)\,dx = \int f(u)\,du$$

EXAMPLE 1 Find $\int x^3 \cos(x^4 + 2)\, dx$.

SOLUTION We make the substitution $u = x^4 + 2$ because its differential is $du = 4x^3\, dx$, which, apart from the constant factor 4, occurs in the integral. Thus, using

$x^3\, dx = \frac{1}{4}\, du$ and the Substitution Rule, we have

$$\int x^3 \cos(x^4 + 2)\, dx = \int \cos u \cdot \tfrac{1}{4}\, du = \tfrac{1}{4} \int \cos u\, du$$

$$= \tfrac{1}{4} \sin u + C$$

$$= \tfrac{1}{4} \sin(x^4 + 2) + C$$

Notice that at the final stage we had to return to the original variable x.

EXAMPLE 2 Evaluate $\int \sqrt{2x + 1}\, dx$.

SOLUTION 1 Let $u = 2x + 1$. Then $du = 2\, dx$, so $dx = \frac{1}{2}\, du$. Thus the Substitution Rule gives

$$\int \sqrt{2x + 1}\, dx = \int \sqrt{u} \cdot \tfrac{1}{2}\, du = \tfrac{1}{2} \int u^{1/2}\, du$$

$$= \frac{1}{2} \cdot \frac{u^{3/2}}{3/2} + C = \tfrac{1}{3} u^{3/2} + C$$

$$= \tfrac{1}{3}(2x + 1)^{3/2} + C$$

SOLUTION 2 Another possible substitution is $u = \sqrt{2x + 1}$. Then

$$du = \frac{dx}{\sqrt{2x + 1}} \qquad \text{so} \qquad dx = \sqrt{2x + 1}\, du = u\, du$$

(Or observe that $u^2 = 2x + 1$, so $2u\, du = 2\, dx$.) Therefore

$$\int \sqrt{2x + 1}\, dx = \int u \cdot u\, du = \int u^2\, du$$

$$= \frac{u^3}{3} + C = \tfrac{1}{3}(2x + 1)^{3/2} + C$$

EXAMPLE 3 Find $\int \frac{x}{\sqrt{1-4x^2}}\,dx$.

SOLUTION Let $u = 1 - 4x^2$. Then $du = -8x\,dx$, so $x\,dx = -\frac{1}{8}\,du$ and

$$\int \frac{x}{\sqrt{1-4x^2}}\,dx = -\frac{1}{8}\int \frac{1}{\sqrt{u}}\,du = -\frac{1}{8}\int u^{-1/2}\,du$$

$$= -\frac{1}{8}(2\sqrt{u}) + C = -\frac{1}{4}\sqrt{1-4x^2} + C$$

EXAMPLE 4 Calculate $\int e^{5x}\,dx$.

SOLUTION If we let $u = 5x$, then $du = 5\,dx$, so $dx = \frac{1}{5}\,du$. Therefore

$$\int e^{5x}\,dx = \frac{1}{5}\int e^u\,du = \frac{1}{5}e^u + C = \frac{1}{5}e^{5x} + C$$

NOTE With some experience, you might be able to evaluate integrals like those in Examples 1–4 without going to the trouble of making an explicit substitution. By recognizing the pattern in Equation 3, where the integrand on the left side is the product of the derivative of an outer function and the derivative of the inner function, we could work Example 1 as follows:

$$\int x^3\cos(x^4+2)\,dx = \int \cos(x^4+2)\cdot x^3\,dx = \frac{1}{4}\int \cos(x^4+2)\cdot(4x^3)\,dx$$

$$= \frac{1}{4}\int \cos(x^4+2)\cdot\frac{d}{dx}(x^4+2)\,dx = \frac{1}{4}\sin(x^4+2) + C$$

Similarly, the solution to Example 4 could be written like this:

$$\int e^{5x}\,dx = \frac{1}{5}\int 5e^{5x}\,dx = \frac{1}{5}\int \frac{d}{dx}(e^{5x})\,dx = \frac{1}{5}e^{5x} + C$$

The following example, however, is more complicated and so an explicit substitution is advisable.

EXAMPLE 5 Find $\int \sqrt{1+x^2}\, x^5\, dx$.

SOLUTION An appropriate substitution becomes more obvious if we factor x^5 as $x^4 \cdot x$. Let $u = 1 + x^2$. Then $du = 2x\,dx$, so $x\,dx = \frac{1}{2}\,du$. Also $x^2 = u - 1$, so $x^4 = (u-1)^2$:

$$\begin{aligned}
\int \sqrt{1+x^2}\, x^5\, dx &= \int \sqrt{1+x^2}\, x^4 \cdot x\, dx \\
&= \int \sqrt{u}\,(u-1)^2 \cdot \tfrac{1}{2}\,du = \tfrac{1}{2}\int \sqrt{u}\,(u^2 - 2u + 1)\,du \\
&= \tfrac{1}{2}\int (u^{5/2} - 2u^{3/2} + u^{1/2})\,du \\
&= \tfrac{1}{2}\left(\tfrac{2}{7}u^{7/2} - 2\cdot\tfrac{2}{5}u^{5/2} + \tfrac{2}{3}u^{3/2}\right) + C \\
&= \tfrac{1}{7}(1+x^2)^{7/2} - \tfrac{2}{5}(1+x^2)^{5/2} + \tfrac{1}{3}(1+x^2)^{3/2} + C
\end{aligned}$$

■

EXAMPLE 6 Calculate $\int \tan x\, dx$.

SOLUTION First we write tangent in terms of sine and cosine:

$$\int \tan x\, dx = \int \frac{\sin x}{\cos x}\, dx$$

This suggests that we should substitute $u = \cos x$, since then $du = -\sin x\, dx$ and so $\sin x\, dx = -du$:

$$\begin{aligned}
\int \tan x\, dx &= \int \frac{\sin x}{\cos x}\, dx = -\int \frac{1}{u}\, du \\
&= -\ln|u| + C = -\ln|\cos x| + C
\end{aligned}$$

■

Since $-\ln|\cos x| = \ln(|\cos x|^{-1}) = \ln(1/|\cos x|) = \ln|\sec x|$, the result of Example 6 can also be written as

5 $$\int \tan x\, dx = \ln|\sec x| + C$$

Definite Integral

Given a function $f(x)$ that is continuous on the interval $[a,b]$ we divide the interval into n subintervals of equal width, Δx, and from each interval choose a point, x_i^*. Then the **definite integral of** ***f(x)*** **from** ***a*** **to** ***b*** is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x$$

EXAMPLE 1 Evaluate the integral

$$\int_3^5 \frac{2x-3}{\sqrt{x^2-3x+1}}\,dx.$$

Solution We rewrite the integral and apply the Substitution Rule for Definite Integrals presented in Section 5.6, to find

$$\int_3^5 \frac{2x-3}{\sqrt{x^2-3x+1}}\,dx = \int_1^{11}\frac{du}{\sqrt{u}}$$

$u = x^2 - 3x + 1,\ du = (2x-3)\,dx;$
$u = 1$ when $x = 3$, $u = 11$ when $x = 5$

$$= \int_1^{11} u^{-1/2}\,du$$

$$= 2\sqrt{u}\Big]_1^{11} = 2(\sqrt{11}-1) \approx 4.63.$$

Table 8.1, Formula 2 ■

EXAMPLE 4 Find $\displaystyle\int_0^{\pi/4}\frac{dx}{1-\sin x}$.

Solution We multiply the numerator and denominator of the integrand by $1+\sin x$, which is simply a multiplication by a form of the number one. This procedure transforms the integral into one we can evaluate:

$$\int_0^{\pi/4}\frac{dx}{1-\sin x} = \int_0^{\pi/4}\frac{1}{1-\sin x}\cdot\frac{1+\sin x}{1+\sin x}\,dx$$

$$= \int_0^{\pi/4}\frac{1+\sin x}{1-\sin^2 x}\,dx$$

$$= \int_0^{\pi/4}\frac{1+\sin x}{\cos^2 x}\,dx$$

$$= \int_0^{\pi/4}(\sec^2 x + \sec x\tan x)\,dx$$

Use Table 8.1, Formulas 8 and 10

$$= \Big[\tan x + \sec x\Big]_0^{\pi/4} = \left(1+\sqrt{2}-(0+1)\right) = \sqrt{2}.$$ ■

EXAMPLE 2 Evaluate $\int \frac{1}{\sqrt{25+9x^2}}\,dx$.

SOLUTION We may express the integral as in Theorem (8.18)(i), by using the substitution

$$u = 3x, \qquad du = 3\,dx.$$

Since du contains the factor 3, we adjust the integrand by multiplying by 3 and then compensate by multiplying the integral by $\frac{1}{3}$ before substituting:

$$\begin{aligned}\int \frac{1}{\sqrt{25+9x^2}}\,dx &= \frac{1}{3}\int \frac{1}{\sqrt{5^2+(3x)^2}}\,3\,dx\\ &= \frac{1}{3}\int \frac{1}{\sqrt{5^2+u^2}}\,du\\ &= \frac{1}{3}\sinh^{-1}\frac{u}{5}+C\\ &= \frac{1}{3}\sinh^{-1}\frac{3x}{5}+C\end{aligned}$$

EXAMPLE 3 Evaluate $\int \frac{e^x}{16-e^{2x}}\,dx$.

SOLUTION Substituting $u = e^x$, $du = e^x\,dx$ and applying Theorem (8.18)(iii) with $a = 4$, we have

$$\begin{aligned}\int \frac{e^x}{16-e^{2x}}\,dx &= \int \frac{1}{4^2-(e^x)^2}\,e^x\,dx\\ &= \int \frac{1}{4^2-u^2}\,du\\ &= \frac{1}{4}\tanh^{-1}\frac{u}{4}+C\\ &= \frac{1}{4}\tanh^{-1}\frac{e^x}{4}+C\end{aligned}$$