Semester I (2019-2020)

Chapter Two
Kinematics of Particles

1. Introduction

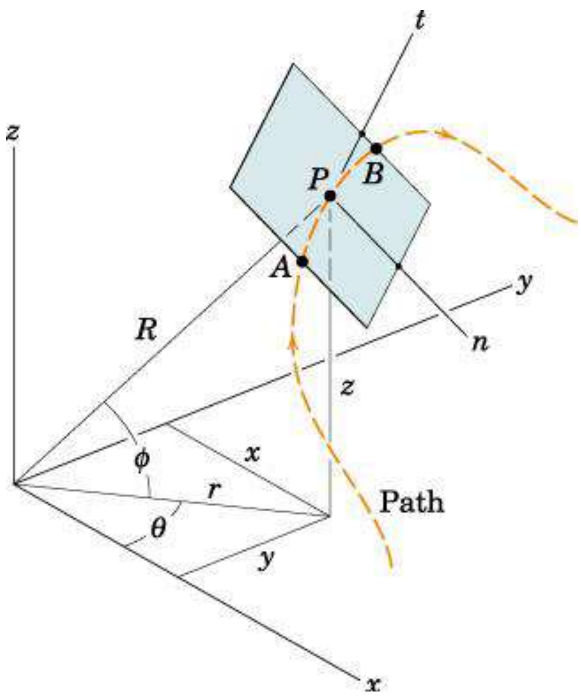
Kinematics is the branch of dynamics which describes the motion of bodies without reference to the forces which either cause the motion or are generated as a result of the motion. Kinematics is often described as the "geometry of motion". Some engineering applications of kinematics include the design of cams, gears, linkages, and other machine elements to control or produce certain desired motions, and the calculation of flight trajectories for aircraft, rockets, and spacecraft. A thorough working knowledge of kinematics is a prerequisite to kinetics, which is the study of the relationships between motion and the corresponding forces which cause or accompany the motion.

2. Choice of Coordinates

Rectangular Coordinates $\mathbf{r}(x, y, z)$

Cylindrical Coordinates $\mathbf{r}(r, \theta, z)$

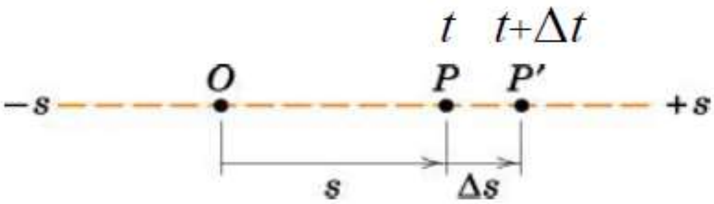
Spherical Coordinates $\mathbf{r}(R, \theta, \phi)$

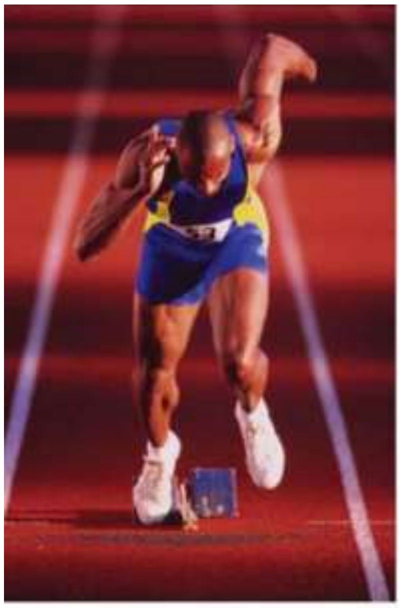


The position of particle P is shown in fig and can be specified at any time t. Motion of P can also be de-scribed by measurements along the tangent t and normal n to the curve. The direction of n lies in the local plane of the curve. These last two measurements are called *path variables*.

3. Rectilinear Motion (Motion along a straight line)

Consider a particle P moving along a straight line, Fig. below. The position of P at any instant of time t can be specified by its distance s measured from some convenient reference point O fixed on the line. At time t + Δt the particle has moved to P' and its coordinate becomes s + Δs. The change in the position coordinate during the interval Δt is called the displacement Δs of the particle, The displacement would be negative if the particle moved in the negative s-direction.





This sprinter will undergo rectilinear acceleration until he reaches his terminal speed.

4. Velocity and acceleration

Velocity of the particle:

$$v = \frac{ds}{dt} = \dot{s}$$

Both are vector quantities

+ve or –ve depending on +ve or –ve displacement

Acceleration of the particle:

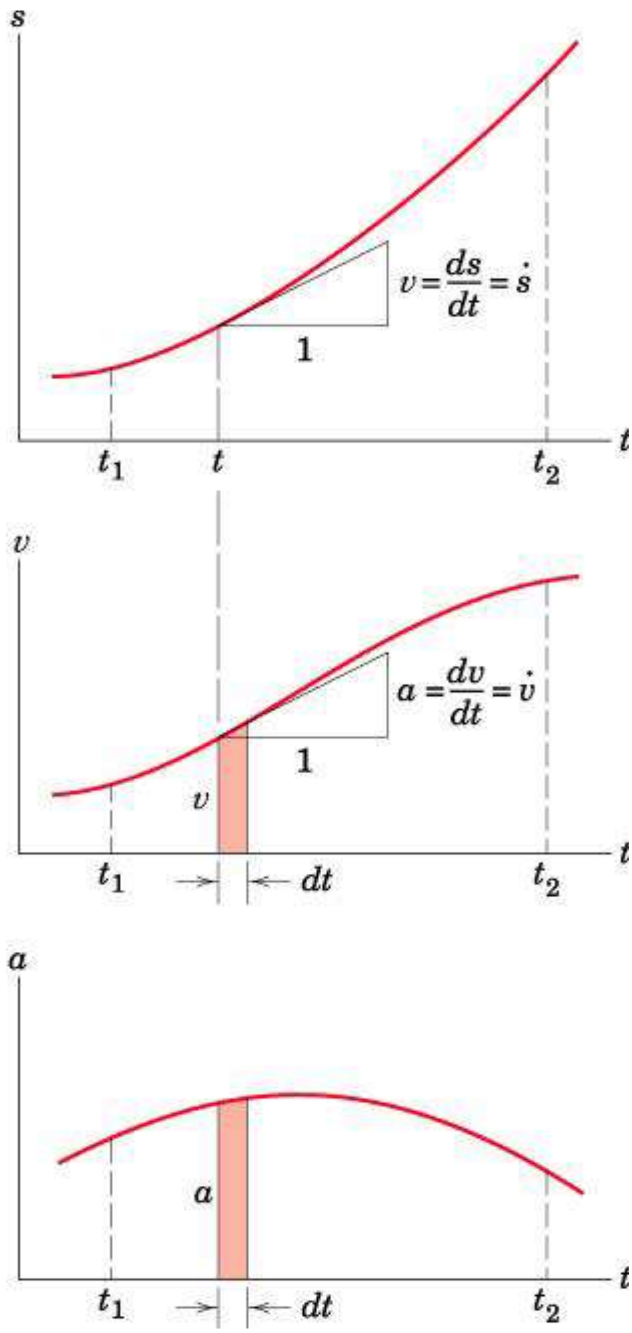
$$a = \frac{dv}{dt} = \dot{v} \quad \text{or} \quad a = \frac{d^2s}{dt^2} = \ddot{s}$$

+ve or –ve depending on whether velocity is increasing or decreasing

$$v\,dv = a\,ds \qquad or \qquad \dot{s}\,d\dot{s} = \ddot{s}\,ds$$

5. Graphical Interpretations

• Interpretation of the differential equations governing rectilinear motion is considerably clarified by representing the relationships among s, v, a, and t graphically. Fig. a is a schematic plot of the variation of s with t from time t_1 to time t_2 for some given rectilinear motion. By constructing the tangent to the curve at any time t, we obtain the slope, which is the velocity v = ds/dt. Thus, the velocity can be determined at all points on the curve and plotted against the corresponding time as shown in Fig. b. Similarly, the slope dv/dt of the u-t curve at any instant gives the acceleration at that instant, and the a-t curve can there-fore be plotted as in Fig. c. We now see from Fig. b that the area under the v-t curve during time dt is v*dt, which is the displacement ds. Consequently, the net displacement of the particle during the interval from t_1 to t_y is the corresponding area under the curve,



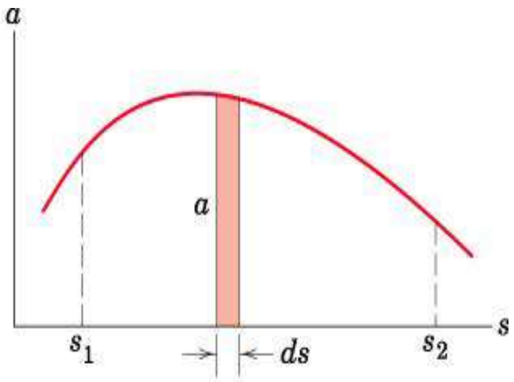
which is

$$\int_{s_1}^{s_2} ds = \int_{t_1}^{t_2} v dt$$

Similarly, from Fig. c we see that the area under the a-t curve during time dt is a* dt, which is dv. Thus, the net change in velocity between t_l and t_2 is the corresponding area under the curve, which is

$$\int_{v_1}^{v_2} dv = \int_{t_1}^{t_2} a dt$$

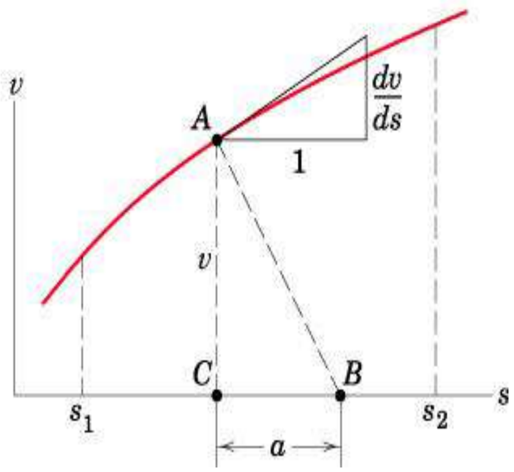
Two additional graphical relations. When the acceleration a is plotted as a function of the position coordinate s, Fig. d, the area under the curve during a displacement ds is a *ds, which is dv = $d(v^2/2)$. Thus, the net area under the curve between position coordinates s_i and s_2 is



$$\int_{v_1}^{v_2} v dv = \int_{s_1}^{s_2} a ds$$

or

½ $(v_2^2 - v_1^2)$ = (area under *a-s* curve)



When the velocity v is plotted as a function of the position coordinate s, Fig. e, the slope of the curve at any point A is dv/ds. By constructing the normal AB to the curve at this point, we see from the similar triangles that

$$\frac{\overline{CB}}{v} = \frac{dv}{ds} \quad \rightarrow \quad \overline{CB} = v\frac{dv}{ds} = a \quad (acceleration)$$

6. **Analytical integration**

(a) Constant acceleration: when a is constant, the first of previous equations can be integrated directly. For simplicity with $s = s_o$, $v = v_o$, and $t = 0$ designated at the beginning of the interval, then for a time interval t the integrated equations become

$$\int_{v_0}^{v} dv = a\int_{0}^{t} dt \qquad \text{or} \qquad v = v_0 + at$$

$$\int_{v_0}^{v} v\,dv = a\int_{s_0}^{s} ds \qquad \text{or} \qquad v^2 = {v_0}^2 + 2a(s - s_0)$$

Substitution of the integrated expression for v into $v = ds/dt$ and integration with respect to t give:

$$\int_{s_0}^{s} ds = \int_{0}^{t} (v_0 + at)\,dt \qquad \text{or} \qquad s = s_0 + v_0 t + \frac{1}{2}at^2$$

These relations are necessarily restricted to the special case where the acceleration is constant. The integration limits depend on the initial and final conditions, which for a given problem may be different from those used here. It may be more convenient, for instance, to begin the integration at some specified time t_1 rather than at time $t = 0$.

(b) Acceleration Given as a Function of Time, $a = f(t)$.

Substitution of the function into the first of eq. gives $f(t) = dv/dt$. Multiplying by dt separates the variables and permits integration. Thus,

$$\int_{v_0}^{v} dv = \int_{0}^{t} f(t)\,dt \qquad \text{or} \qquad v = v_0 + \int_{0}^{t} f(t)\,dt$$

From this integrated expression for v as a function of t, the position co-ordinate s is obtained by integrating v = ds/dt , which, in form, would be

$$\int_{s_0}^{s} ds = \int_0^t v\,dt \qquad \text{or} \qquad s = s_0 + \int_0^t v\,dt$$

If the indefinite integral is employed, the end conditions are used to establish the constants of integration. The results are identical with those obtained by using the definite integral. If desired, the displacement s can be obtained by a direct solution of the second-order differential equation s = f(t) obtained by substitution of f(t) into the second of $a = \frac{d^2s}{dt^2} = \ddot{s}$

(c) Acceleration Given as a Function of Velocity, a = f(v).

Substitution of the function into the first of Eqs. 2/2 gives f(v) = dv/dt, which permits separating the variables and integrating. Thus,

$$t = \int_0^t dt = \int_{v_0}^{v} \frac{dv}{f(v)}$$

This result gives t as a function of v. Then it would be necessary to solve for u as a function of t so that Eq. 2/1 can be integrated to obtain the position coordinate s as a function of t. Another approach is to substitute the function a = f(v) into the first of Eqs. 2/3, giving v du = f(v) ds. The variables can now be separated and the equation integrated in the form

$$\int_{v_0}^{v} \frac{v\,dv}{f(v)} = \int_{s_0}^{s} ds \qquad \text{or} \qquad s = s_0 + \int_{v_0}^{v} \frac{v\,dv}{f(v)}$$

Note that this equation gives s in terms of v without explicit reference to t.

(d) Acceleration Given as a Function of Displacement a = f(s).

Substituting the function into Eq. 2/3 and integrating give the form

$$\int_{v_0}^{v} v\,dv = \int_{s_0}^{s} f(s)\,ds \qquad \text{or} \qquad v^2 = {v_0}^2 + 2\int_{s_0}^{s} f(s)\,ds$$

Next we solve for v to give v = g(s), a function of s. Now we can substitute ds/dt for v, separate variables, and integrate in the form

$$\int_{s_0}^{s} \frac{ds}{g(s)} = \int_{0}^{t} dt \qquad \text{or} \qquad t = \int_{s_0}^{s} \frac{ds}{g(s)}$$

which gives t as a function of s. Finally, we can rearrange to obtain s as a function of t.

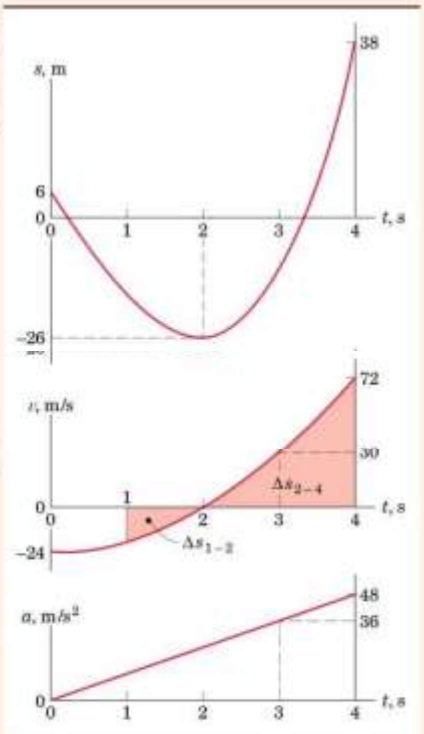
In each of the foregoing cases when the acceleration varies according to some functional relationship, the possibility of solving the equations by direct mathematical integration will depend on the form of the function. In cases where the integration is excessively awkward or difficult, integration by graphical, numerical, or computer methods can be utilized.

Important Points

- Dynamics is concerned with bodies that have accelerated motion.
- Kinematics is a study of the geometry of the motion.
- Kinetics is a study of the forces that cause the motion.
- Rectilinear kinematics refers to straight-line motion.
- Speed refers to the magnitude of velocity.
- Average speed is the total distance traveled divided by the total time. This is different from the average velocity, which is the displacement divided by the time.
- A particle that is slowing down is decelerating.
- A particle can have an acceleration and yet have zero velocity.
- The relationship $a\,ds = v\,dv$ is derived from $a = dv/dt$ and $v = ds/dt$, by eliminating dt.

Sample Problem 2/1

The position coordinate of a particle which is confined to move along a straight line is given by $s = 2t^3 - 24t + 6$, where s is measured in meters from a convenient origin and t is in seconds. Determine (*a*) the time required for the particle to reach a velocity of 72 m/s from its initial condition at $t = 0$, (*b*) the acceleration of the particle when $v = 30$ m/s, and (*c*) the net displacement of the particle during the interval from $t = 1$ s to $t = 4$ s.



Solution. The velocity and acceleration are obtained by successive differentiation of s with respect to the time. Thus,

$[v = \dot{s}]$ $\qquad v = 6t^2 - 24$ m/s

$[a = \dot{v}]$ $\qquad a = 12t$ m/s²

(a) Substituting $v = 72$ m/s into the expression for v gives us $72 = 6t^2 - 24$, from which $t = \pm 4$ s. The negative root describes a mathematical solution for t before ① the initiation of motion, so this root is of no physical interest. Thus, the desired result is

$t = 4$ s *Ans.*

(b) Substituting $v = 30$ m/s into the expression for v gives $30 = 6t^2 - 24$, from which the positive root is $t = 3$ s, and the corresponding acceleration is

$a = 12(3) = 36$ m/s² *Ans.*

(c) The net displacement during the specified interval is

$$\Delta s = s_4 - s_1 \qquad \text{or}$$

$$\Delta s = [2(4^3) - 24(4) + 6] - [2(1^3) - 24(1) + 6]$$

$$= 54 \text{ m}$$ *Ans.*

② which represents the net advancement of the particle along the s-axis from the position it occupied at $t = 1$ s to its position at $t = 4$ s.

To help visualize the motion, the values of s, v, and a are plotted against the time t as shown. Because the area under the v-t curve represents displacement, ③ we see that the net displacement from $t = 1$ s to $t = 4$ s is the positive area Δs_{2-4} less the negative area Δs_{1-2}.

Helpful Hints

① Be alert to the proper choice of sign when taking a square root. When the situation calls for only one answer, the positive root is not always the one you may need.

② Note carefully the distinction between italic s for the position coordinate and the vertical s for seconds.

③ Note from the graphs that the values for v are the slopes ($\dot{s}$) of the s-t curve and that the values for a are the slopes ($\dot{v}$) of the v-t curve. *Suggestion:* Integrate $v\,dt$ for each of the two intervals and check the answer for Δs. Show that the total distance traveled during the interval $t = 1$ s to $t = 4$ s is 74 m.

Sample Problem 2/2

A particle moves along the x-axis with an initial velocity $v_x = 50$ ft/sec at the origin when $t = 0$. For the first 4 seconds it has no acceleration, and thereafter it is acted on by a retarding force which gives it a constant acceleration $a_x = -10$ ft/sec^2. Calculate the velocity and the x-coordinate of the particle for the conditions of $t = 8$ sec and $t = 12$ sec and find the maximum positive x-coordinate reached by the particle. ①

Helpful Hints

① Learn to be flexible with symbols. The position coordinate x is just as valid as s.

Solution. The velocity of the particle after $t = 4$ sec is computed from

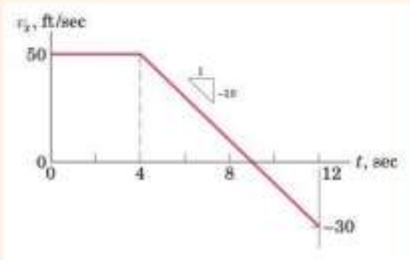
② $$\left[\int dv = \int a\,dt\right] \qquad \int_{50}^{v_x} dv_x = -10\int_4^t dt \qquad v_x = 90 - 10t \text{ ft/sec}$$

② Note that we integrate to a general time t and then substitute specific values.

and is plotted as shown. At the specified times, the velocities are

$$t = 8 \text{ sec}, \qquad v_x = 90 - 10(8) = 10 \text{ ft/sec}$$

$$t = 12 \text{ sec}, \qquad v_x = 90 - 10(12) = -30 \text{ ft/sec} \qquad \textit{Ans.}$$



The x-coordinate of the particle at any time greater than 4 seconds is the distance traveled during the first 4 seconds plus the distance traveled after the discontinuity in acceleration occurred. Thus,

$$\left[\int ds = \int v\,dt\right] \qquad x = 50(4) + \int_4^t (90 - 10t)\,dt = -5t^2 + 90t - 80 \text{ ft}$$

For the two specified times,

$$t = 8 \text{ sec}, \qquad x = -5(8^2) + 90(8) - 80 = 320 \text{ ft}$$

$$t = 12 \text{ sec}, \qquad x = -5(12^2) + 90(12) - 80 = 280 \text{ ft} \qquad \textit{Ans.}$$

The x-coordinate for $t = 12$ sec is less than that for $t = 8$ sec since the motion is in the negative x-direction after $t = 9$ sec. The maximum positive x-coordinate is, then, the value of x for $t = 9$ sec which is

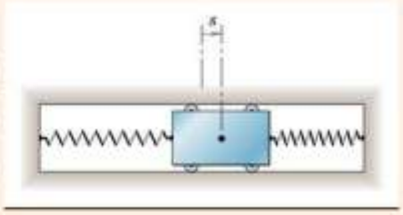
$$x_{max} = -5(9^2) + 90(9) - 80 = 325 \text{ ft} \qquad \textit{Ans.}$$

③ These displacements are seen to be the net positive areas under the v-t graph up to the values of t in question.

③ Show that the total distance traveled by the particle in the 12 sec is 370 ft.

Sample Problem 2/3

The spring-mounted slider moves in the horizontal guide with negligible friction and has a velocity v_0 in the s-direction as it crosses the mid-position where $s = 0$ and $t = 0$. The two springs together exert a retarding force to the motion of the slider, which gives it an acceleration proportional to the displacement but oppositely directed and equal to $a = -k^2s$, where k is constant. (The constant is arbitrarily squared for later convenience in the form of the expressions.) Determine the expressions for the displacement s and velocity v as functions of the time t.



Solution I. Since the acceleration is specified in terms of the displacement, the differential relation $v\,dv = a\,ds$ may be integrated. Thus,

① $$\int v\,dv = \int -k^2 s\,ds + C_1 \text{ a constant,} \quad \text{or} \quad \frac{v^2}{2} = -\frac{k^2s^2}{2} + C_1$$

When $s = 0$, $v = v_0$, so that $C_1 = v_0^2/2$, and the velocity becomes

$$v = +\sqrt{v_0^2 - k^2s^2}$$

The plus sign of the radical is taken when v is positive (in the plus s-direction). This last expression may be integrated by substituting $v = ds/dt$. Thus,

② $$\int \frac{ds}{\sqrt{v_0^2 - k^2s^2}} = \int dt + C_2 \text{ a constant,} \quad \text{or} \quad \frac{1}{k}\sin^{-1}\frac{ks}{v_0} = t + C_2$$

With the requirement of $t = 0$ when $s = 0$, the constant of integration becomes $C_2 = 0$, and we may solve the equation for s so that

$$s = \frac{v_0}{k}\sin kt \qquad \textit{Ans.}$$

The velocity is $v = \dot{s}$, which gives

$$v = v_0 \cos kt \qquad \textit{Ans.}$$

Solution II. Since $a = \ddot{s}$, the given relation may be written at once as

$$\ddot{s} + k^2s = 0$$

This is an ordinary linear differential equation of second order for which the solution is well known and is

$$s = A\sin Kt + B\cos Kt$$

where A, B, and K are constants. Substitution of this expression into the differential equation shows that it satisfies the equation, provided that $K = k$. The velocity is $v = \dot{s}$, which becomes

$$v = Ak\cos kt - Bk\sin kt$$

The initial condition $v = v_0$ when $t = 0$ requires that $A = v_0/k$, and the condition $s = 0$ when $t = 0$ gives $B = 0$. Thus, the solution is

③ $$s = \frac{v_0}{k}\sin kt \quad \text{and} \quad v = v_0\cos kt \qquad \textit{Ans.}$$

Helpful Hints

① We have used an indefinite integral here and evaluated the constant of integration. For practice, obtain the same results by using the definite integral with the appropriate limits.

② Again try the definite integral here as above.

③ This motion is called *simple harmonic motion* and is characteristic of all oscillations where the restoring force, and hence the acceleration, is proportional to the displacement but opposite in sign.

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Sample Problem 2/4

① A freighter is moving at a speed of 8 knots when its engines are suddenly stopped. If it takes 10 minutes for the freighter to reduce its speed to 4 knots, determine and plot the distance s in nautical miles moved by the ship and its speed v in knots as functions of the time t during this interval. The deceleration of the ship is proportional to the square of its speed, so that $a = -kv^2$.

Solution. The speeds and the time are given, so we may substitute the expression for acceleration directly into the basic definition $a = dv/dt$ and integrate. Thus,

$$-kv^2 = \frac{dv}{dt} \qquad \frac{dv}{v^2} = -k\,dt \qquad \int_8^v \frac{dv}{v^2} = -k\int_0^t dt$$

② $$-\frac{1}{v} + \frac{1}{8} = -kt \qquad v = \frac{8}{1 + 8kt}$$

Now we substitute the end limits of $v = 4$ knots and $t = \frac{10}{60} = \frac{1}{6}$ hour and get

$$4 = \frac{8}{1 + 8k(1/6)} \qquad k = \frac{3}{4}\text{ mi}^{-1} \qquad v = \frac{8}{1 + 6t} \qquad \textit{Ans.}$$

The speed is plotted against the time as shown.

The distance is obtained by substituting the expression for v into the definition $v = ds/dt$ and integrating. Thus,

$$\frac{8}{1 + 6t} = \frac{ds}{dt} \qquad \int_0^t \frac{8\,dt}{1 + 6t} = \int_0^s ds \qquad s = \frac{4}{3}\ln(1 + 6t) \qquad \textit{Ans.}$$

The distance s is also plotted against the time as shown, and we see that the ship has moved through a distance $s = \frac{4}{3}\ln(1 + \frac{6}{6}) = \frac{4}{3}\ln 2 = 0.924$ mi (nautical) during the 10 minutes.

Helpful Hints

① Recall that one knot is the speed of one nautical mile (6076 ft) per hour. Work directly in the units of nautical miles and hours.

② We choose to integrate to a general value of v and its corresponding time t so that we may obtain the variation of v with t.