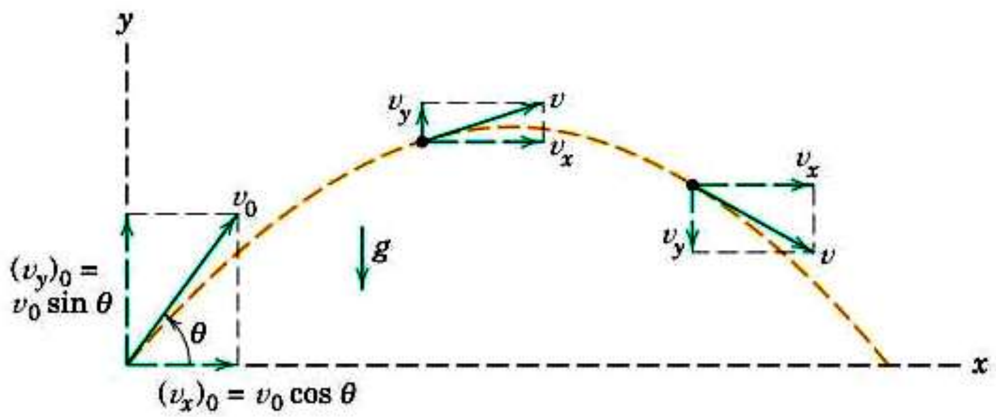
- **Projectile motion**

An important application of two-dimensional kinematic theory is the problem of projectile motion. For a first treatment of the subject, we neglect aerodynamic drag and the curvature and rotation of the earth, and we assume that the altitude change is small enough so that the acceleration due to gravity can be considered constant. With these assumptions, rectangular coordinates are useful for the trajectory analysis. For the axes shown in Fig. , the acceleration components are

$$a_x = 0 \qquad a_y = -g$$



Integration of these accelerations follows the results obtained previously for constant acceleration and yields

$$v_x = (v_x)_0 \qquad v_y = (v_y)_0 - gt$$
$$x = x_0 + (v_x)_0 t \qquad y = y_0 + (v_y)_0 t - \frac{1}{2}gt^2$$
$$v_y^2 = (v_y)_0^2 - 2g(y - y_0)$$

In all these expressions, the subscript zero denotes initial conditions.

Procedure for Analysis

Coordinate System.

- Establish the fixed x, y coordinate axes and sketch the trajectory of the particle. Between any *two points* on the path specify the given problem data and identify the *three unknowns*. In all cases the acceleration of gravity acts downward and equals 9.81 m/s^2 or 32.2 ft/s^2. The particle's initial and final velocities should be represented in terms of their x and y components.
- Remember that positive and negative position, velocity, and acceleration components always act in accordance with their associated coordinate directions.

Kinematic Equations.

- Depending upon the known data and what is to be determined, a choice should be made as to which three of the following four equations should be applied between the two points on the path to obtain the most direct solution to the problem.

Horizontal Motion.

- The *velocity* in the horizontal or x direction is *constant*, i.e., $v_x = (v_0)_x$, and

$$x = x_0 + (v_0)_x t$$

Vertical Motion.

- In the vertical or y direction *only two* of the following three equations can be used for solution.

$$v_y = (v_0)_y + a_c t$$

$$y = y_0 + (v_0)_y t + \frac{1}{2} a_c t^2$$

$$v_y^2 = (v_0)_y^2 + 2a_c(y - y_0)$$

For example, if the particle's final velocity v_y is not needed, then the first and third of these equations will not be useful.

Sample Problem 2/5

The curvilinear motion of a particle is defined by $v_x = 50 - 16t$ and $y = 100 - 4t^2$, where v_x is in meters per second, y is in meters, and t is in seconds. It is also known that $x = 0$ when $t = 0$. Plot the path of the particle and determine its velocity and acceleration when the position $y = 0$ is reached.

Solution. The x-coordinate is obtained by integrating the expression for v_x, and the x-component of the acceleration is obtained by differentiating v_x. Thus,

$$\left[\int dx = \int v_x\, dt\right] \qquad \int_0^x dx = \int_0^t (50 - 16t)\, dt \qquad x = 50t - 8t^2 \text{ m}$$

$$[a_x = \dot{v}_x] \qquad a_x = \frac{d}{dt}(50 - 16t) \qquad a_x = -16 \text{ m/s}^2$$

The y-components of velocity and acceleration are

$$[v_y = \dot{y}] \qquad v_y = \frac{d}{dt}(100 - 4t^2) \qquad v_y = -8t \text{ m/s}$$

$$[a_y = \dot{v}_y] \qquad a_y = \frac{d}{dt}(-8t) \qquad a_y = -8 \text{ m/s}^2$$

We now calculate corresponding values of x and y for various values of t and plot x against y to obtain the path as shown.

When $y = 0$, $0 = 100 - 4t^2$, so $t = 5$ s. For this value of the time, we have

$$v_x = 50 - 16(5) = -30 \text{ m/s}$$

$$v_y = -8(5) = -40 \text{ m/s}$$

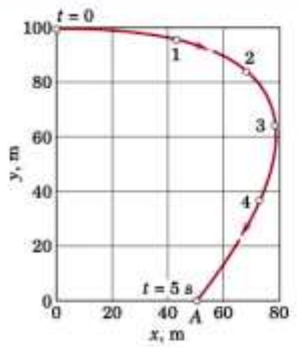
$$v = \sqrt{(-30)^2 + (-40)^2} = 50 \text{ m/s}$$

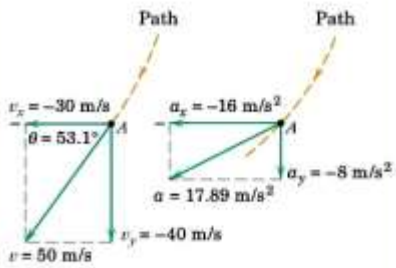
$$a = \sqrt{(-16)^2 + (-8)^2} = 17.89 \text{ m/s}^2$$

The velocity and acceleration components and their resultants are shown on the separate diagrams for point A, where $y = 0$. Thus, for this condition we may write

$$\mathbf{v} = -30\mathbf{i} - 40\mathbf{j} \text{ m/s} \qquad \textit{Ans.}$$

$$\mathbf{a} = -16\mathbf{i} - 8\mathbf{j} \text{ m/s}^2 \qquad \textit{Ans.}$$



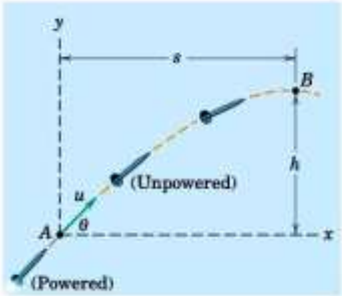


Helpful Hint

We observe that the velocity vector lies along the tangent to the path as it should, but that the acceleration vector is not tangent to the path. Note especially that the acceleration vector has a component that points toward the inside of the curved path. We concluded from our diagram in Fig. 2/5 that it is impossible for the acceleration to have a component that points toward the outside of the curve.

Sample Problem 2/6

A rocket has expended all its fuel when it reaches position A, where it has a velocity of $\mathbf{u}$ at an angle θ with respect to the horizontal. It then begins unpowered flight and attains a maximum added height h at position B after traveling a horizontal distance s from A. Determine the expressions for h and s, the time t of flight from A to B, and the equation of the path. For the interval concerned, assume a flat earth with a constant gravitational acceleration g and neglect any atmospheric resistance.



Solution. Since all motion components are directly expressible in terms of horizontal and vertical coordinates, a rectangular set of axes x-y will be employed. ① With the neglect of atmospheric resistance, $a_x = 0$ and $a_y = -g$, and the resulting motion is a direct superposition of two rectilinear motions with constant acceleration. Thus,

$$[dx = v_x\,dt] \qquad x = \int_0^t u\cos\theta\,dt \qquad x = ut\cos\theta$$

$$[dv_y = a_y\,dt] \qquad \int_{u\sin\theta}^{v_y} dv_y = \int_0^t (-g)\,dt \qquad v_y = u\sin\theta - gt$$

$$[dy = v_y\,dt] \qquad y = \int_0^t (u\sin\theta - gt)\,dt \qquad y = ut\sin\theta - \frac{1}{2}gt^2$$

Position B is reached when $v_y = 0$, which occurs for $0 = u\sin\theta - gt$ or

$$t = (u\sin\theta)/g \qquad \textit{Ans.}$$

Substitution of this value for the time into the expression for y gives the maximum added altitude

$$h = u\left(\frac{u\sin\theta}{g}\right)\sin\theta - \frac{1}{2}g\left(\frac{u\sin\theta}{g}\right)^2 \qquad h = \frac{u^2\sin^2\theta}{2g} \qquad \textit{Ans.}$$

The horizontal distance is seen to be

② $$s = u\left(\frac{u\sin\theta}{g}\right)\cos\theta \qquad s = \frac{u^2\sin 2\theta}{2g} \qquad \textit{Ans.}$$

which is clearly a maximum when $\theta = 45°$. The equation of the path is obtained by eliminating t from the expressions for x and y, which gives

$$y = x\tan\theta - \frac{gx^2}{2u^2}\sec^2\theta \qquad \textit{Ans.}$$

③ This equation describes a vertical parabola as indicated in the figure.

Helpful Hints

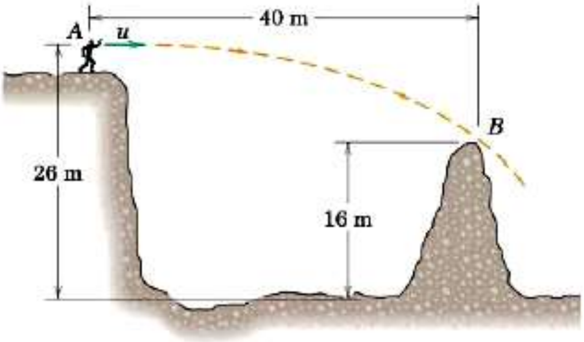
① Note that this problem is simply the description of projectile motion neglecting atmospheric resistance.

② We see that the total range and time of flight for a projectile fired above a horizontal plane would be twice the respective values of s and t given here.

③ If atmospheric resistance were to be accounted for, the dependency of the acceleration components on the velocity would have to be established before an integration of the equations could be carried out. This becomes a much more difficult problem.

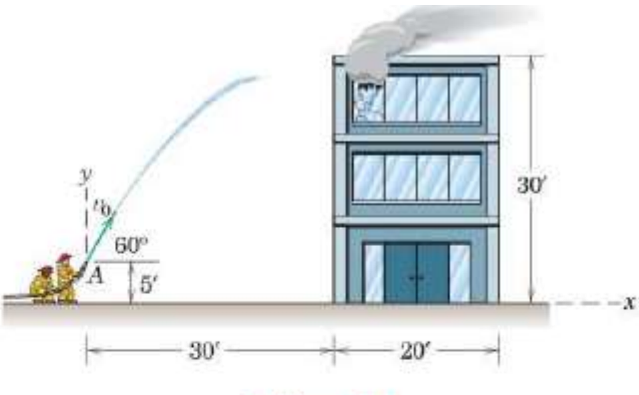
PROBLEMS

2/72: With what minimum horizontal velocity ***u*** can a boy throw a rock at A and have it just clear the obstruction at B?



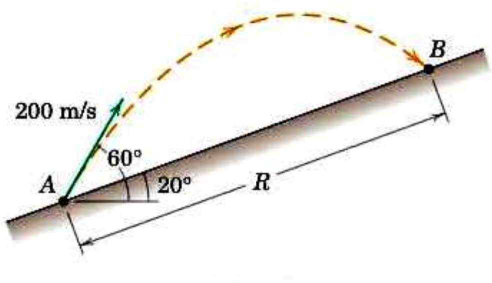
Problem 2/72

2/74: Water issues from the nozzle at A, which is 5 ft above the ground. Determine the coordinates of the point of impact of the stream if the initial water speed is (a) v_o = 45 ft/sec and (b) v_o= 60 ft/see.



Problem 2/74

2/85: A projectile is launched with an initial speed of 200 m/s at an angle of 60° with respect to the horizontal. Compute the range R as measured up the incline.



Problem 2/85

1. **Normal and Tangential coordinates**

The n- and t-coordinates are considered to move along the path with the particle, as seen in next Fig. where the particle advances from A to B to C. The positive direction for n at any position is always taken toward the center of curvature of the path. As seen from Fig., the positive n-direction will shift from one side of the curve to the other side if the curvature changes direction.

- **Velocity and Acceleration**:

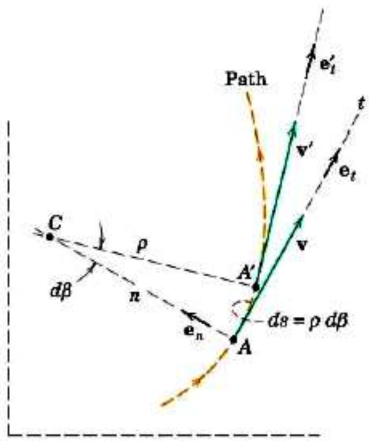
We now use the coordinates n and t to describe the velocity v and acceleration a which were introduced previously for the curvilinear motion of a particle. For this purpose, we introduce unit vectors e_n in the n-direction and e_t in the t-direction, as shown in Fig. below for the position of the particle at point A on its path.

$ds = \rho\, d\beta$, where β is in radians.

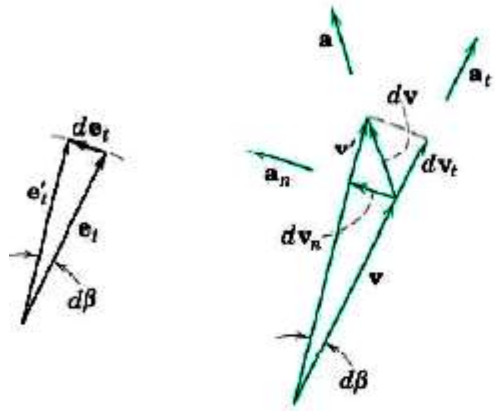
$v = ds/dt$

$v = ds/dt = \rho\, d\beta/dt$

$$\mathbf{v} = v\mathbf{e}_t = \rho\dot{\beta}\mathbf{e}_t$$



$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d(v\mathbf{e}_t)}{dt} = v\dot{\mathbf{e}}_t + \dot{v}\mathbf{e}_t$$



$$|\mathbf{e}_t|\ d\beta\ =\ d\beta$$

$$d\mathbf{e}_t = \mathbf{e}_n\, d\beta$$

$$\frac{d\mathbf{e}_t}{d\beta} = \mathbf{e}_n$$

Dividing by dt gives $d\mathbf{e}_t/dt = (d\beta/dt)\mathbf{e}_n$, which can be written

$$\dot{\mathbf{e}}_t = \dot{\beta}\mathbf{e}_n$$

$\dot{\beta}$ from the relation $v = \rho\dot{\beta}$

$$\mathbf{a} = \frac{v^2}{\rho}\mathbf{e}_n + \dot{v}\mathbf{e}_t$$

where

$$a_n = \frac{v^2}{\rho} = \rho\dot{\beta}^2 = v\dot{\beta}$$

$$a_t = \dot{v} = \ddot{s}$$

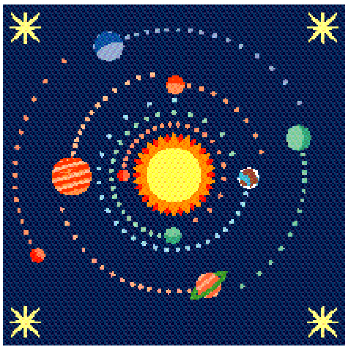
$$a = \sqrt{{a_n}^2 + {a_t}^2}$$

- **Circular Motion:**

Circular motion is movement of an object along the circumference of a circle , or rotation along a circular path. It is a motion with constant acceleration in angular direction.

Semester I (2019-2020)

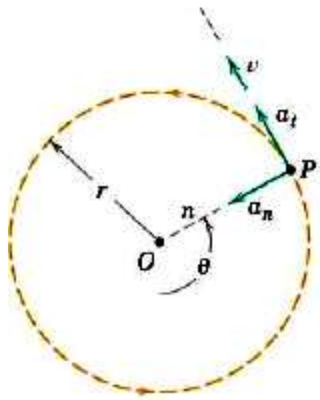
Examples of circular motion:



an artificial satellite orbiting the Earth at a constant height, a fan's blades rotating around a hub, a stone which is tied to a rope and is being swung in circles, a car turning through a curve in a race track, an electron moving perpendicular to a uniform magnetic field, and a gear turning inside a mechanism.

An example of circular motion is this car moving with constant speed around a skid pad, which is a circular roadway with a diameter of about 60 m.

The velocity and acceleration components for the circular motion of the particle P becomes:



$$v = r\dot{\theta}$$

$$a_n = v^2/r = r\dot{\theta}^2 = v\dot{\theta}$$

$$a_t = \dot{v} = r\ddot{\theta}$$

Note:

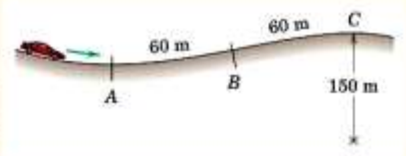
- If the path is expressed as $y = f(x)$, the radius of curvature ρ at any point on the path is determined from the equation

$$\rho = \frac{[1 + (dy/dx)^2]^{3/2}}{|d^2y/dx^2|}$$

The derivation of this result is given in any standard calculus text.

Sample Problem 2/7

To anticipate the dip and hump in the road, the driver of a car applies her brakes to produce a uniform deceleration. Her speed is 100 km/h at the bottom A of the dip and 50 km/h at the top C of the hump, which is 120 m along the road from A. If the passengers experience a total acceleration of 3 m/s² at A and if the radius of curvature of the hump at C is 150 m, calculate (*a*) the radius of curvature ρ at A, (*b*) the acceleration at the inflection point B, and (*c*) the total acceleration at C.



Solution. The dimensions of the car are small compared with those of the ① path, so we will treat the car as a particle. The velocities are

$$v_A = \left(100\,\frac{\text{km}}{\text{h}}\right)\left(\frac{1\text{ h}}{3600\text{ s}}\right)\left(1000\,\frac{\text{m}}{\text{km}}\right) = 27.8\text{ m/s}$$

$$v_C = 50\,\frac{1000}{3600} = 13.89\text{ m/s}$$

We find the constant deceleration along the path from

$$\left[\int v\,dv = \int a_t\,ds\right] \qquad \int_{v_A}^{v_C} v\,dv = a_t\int_0^s ds$$

$$a_t = \frac{1}{2s}(v_C^2 - v_A^2) = \frac{(13.89)^2 - (27.8)^2}{2(120)} = -2.41\text{ m/s}^2$$

(a) Condition at A. With the total acceleration given and a_t determined, we can easily compute a_n and hence ρ from

$[a^2 = a_n^2 + a_t^2]$ $\quad a_n^2 = 3^2 - (2.41)^2 = 3.19 \quad a_n = 1.785\text{ m/s}^2$

$[a_n = v^2/\rho]$ $\quad \rho = v^2/a_n = (27.8)^2/1.785 = 432\text{ m}$ *Ans.*

(b) Condition at B. Since the radius of curvature is infinite at the inflection point, $a_n = 0$ and

$$a = a_t = -2.41\text{ m/s}^2 \qquad \textit{Ans.}$$

(c) Condition at C. The normal acceleration becomes

$[a_n = v^2/\rho]$ $\quad a_n = (13.89)^2/150 = 1.286\text{ m/s}^2$

With unit vectors $\mathbf{e}_n$ and $\mathbf{e}_t$ in the n- and t-directions, the acceleration may be written

$$\mathbf{a} = 1.286\mathbf{e}_n - 2.41\mathbf{e}_t\text{ m/s}^2$$

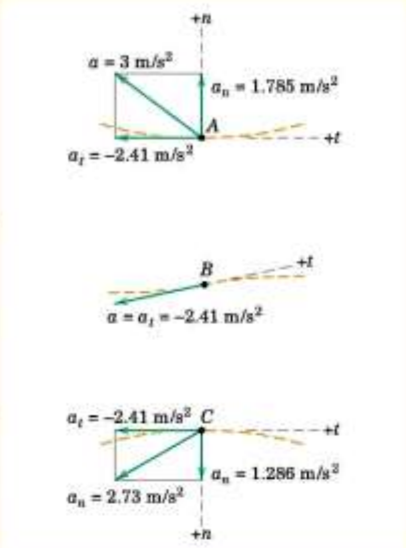
where the magnitude of $\mathbf{a}$ is

$[a = \sqrt{a_n^2 + a_t^2}]$ $\quad a = \sqrt{(1.286)^2 + (-2.41)^2} = 2.73\text{ m/s}^2$ *Ans.*

The acceleration vectors representing the conditions at each of the three points are shown for clarification.

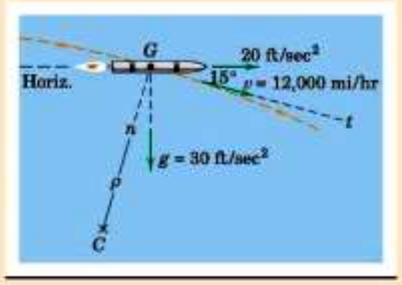
Helpful Hint

① Actually, the radius of curvature to the road differs by about 1 m from that to the path followed by the center of mass of the passengers, but we have neglected this relatively small difference.



Sample Problem 2/8

A certain rocket maintains a horizontal attitude of its axis during the powered phase of its flight at high altitude. The thrust imparts a horizontal component of acceleration of 20 ft/sec^2, and the downward acceleration component is the acceleration due to gravity at that altitude, which is g = 30 ft/sec^2. At the instant represented, the velocity of the mass center G of the rocket along the 15° direction of its trajectory is 12,000 mi/hr. For this position determine (a) the radius of curvature of the flight trajectory, (b) the rate at which the speed v is increasing, (c) the angular rate $\dot{\beta}$ of the radial line from G to the center of curvature C, and (d) the vector expression for the total acceleration $\mathbf{a}$ of the rocket.



Solution. We observe that the radius of curvature appears in the expression for the normal component of acceleration, so we use n- and t-coordinates to describe the motion of G. The n- and t-components of the total acceleration are obtained by resolving the given horizontal and vertical accelerations into their n- and t-components and then combining. From the figure we get ①

$$a_n = 30 \cos 15° - 20 \sin 15° = 23.8 \text{ ft/sec}^2$$

$$a_t = 30 \sin 15° + 20 \cos 15° = 27.1 \text{ ft/sec}^2$$

(a) We may now compute the radius of curvature from

② $[a_n = v^2/\rho]$ $\qquad \rho = \dfrac{v^2}{a_n} = \dfrac{[(12{,}000)(44/30)]^2}{23.8} = 13.01(10^6) \text{ ft}$ *Ans.*

(b) The rate at which v is increasing is simply the t-component of acceleration.

$[\dot{v} = a_t]$ $\qquad \dot{v} = 27.1 \text{ ft/sec}^2$ *Ans.*

(c) The angular rate $\dot{\beta}$ of line GC depends on v and ρ and is given by

$[v = \rho\dot{\beta}]$ $\qquad \dot{\beta} = v/\rho = \dfrac{12{,}000(44/30)}{13.01(10^6)} = 13.53(10^{-4}) \text{ rad/sec}$ *Ans.*

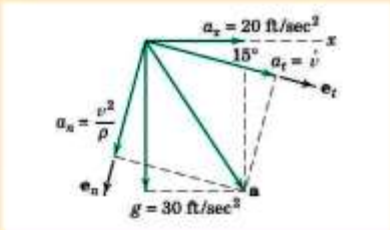
(d) With unit vectors $\mathbf{e}_n$ and $\mathbf{e}_t$ for the n- and t-directions, respectively, the total acceleration becomes

$$\mathbf{a} = 23.8\mathbf{e}_n + 27.1\mathbf{e}_t \text{ ft/sec}^2$$ *Ans.*

Helpful Hints

① Alternatively, we could find the resultant acceleration and then resolve it into n- and t-components.

② To convert from mi/hr to ft/sec, multiply by $\dfrac{5280 \text{ ft/mi}}{3600 \text{ sec/hr}} = \dfrac{44 \text{ ft/sec}}{30 \text{ mi/hr}}$ which is easily remembered, as 30 mi/hr is the same as 44 ft/sec.



PROBLEMS

2/104: The car passes through a dip in the road at A with a constant speed which gives its mass center G an acceleration equal to 0.5g. If the radius of curvature of the road at A is 100 m, and if the distance from the road to the mass center G of the car is 0.6 m, determine the speed v of the car.

Problem 2/104

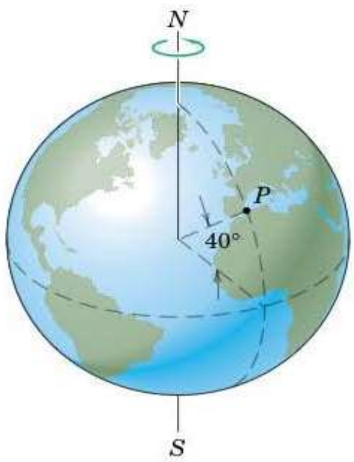
solution

$a = a_n = v^2/\rho$

$v = \sqrt{\rho a_n} = \sqrt{(100 - 0.6)\ 0.5 * (9.81)}$

$v = 22.08$ m/sec $= 22.08 \times 3.6 = 79.5$ km/h

2/110: Consider the polar axis of the earth to be fixed in space and compute the magnitude of the acceleration a of a point P on the earth's surface at latitude 40° north. The mean diameter of the earth is 12 742 km and its angular velocity is $0.729(10^{-4})$ rad/s.



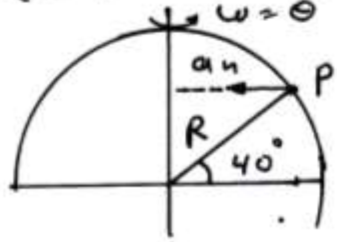
Problem 2/110

Solution.

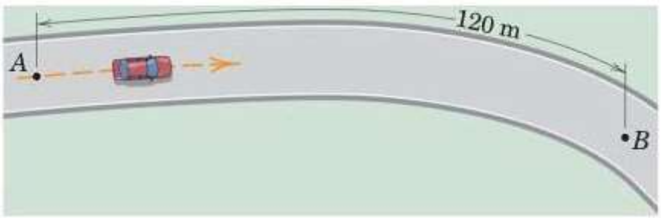
$a = a_n = r\dot{\theta}^2 = R\cos\gamma\,\dot{\theta}^2$

$= \frac{12.742\,(10^6)}{2}\cos 40^\circ \times (0.729\times10^{-4})^2$

$= 0.0259\ m/sec^2$



2/116: A car travels along the level curved road with a speed which is decreasing at the constant rate of 0.6 m/s each second. The speed of the car as it passes point A is 16 m/s. Calculate the magnitude of the total acceleration of the car as it passes point B which is 120 m along the road from A. The radius of curvature of the road at B is 60 m.



Problem 2/116

Solution

$a_t = -0.6\ \frac{m}{s^2}$ · constant acceleration

$v_B^2 = v_A^2 + 2a_t s = 16^2 - 2\times0.6\times120$

$v_B = 10.58 \text{ m/s}$

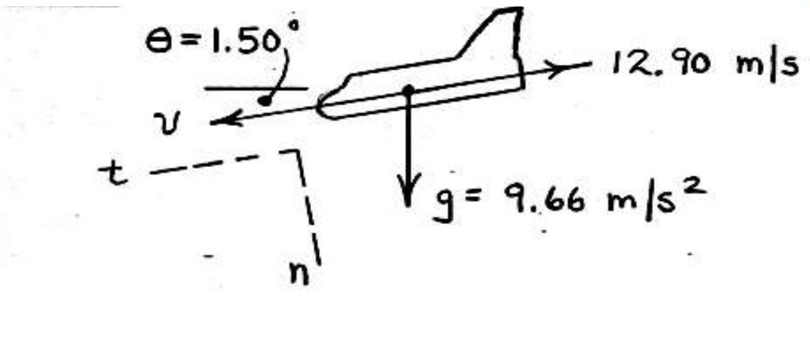
$a_n = v_B^2/\rho = \frac{10.58^2}{60} = 1.867 \text{ m/s}^2$

$a = \sqrt{a_t^2 + a_n^2} = \sqrt{0.6^2 + 1.867^2} =$

$= 1.961 \text{ m/sec}^2$

2/128: At a certain point in the reentry of the space shuttle into the earth's atmosphere, the total acceleration of the shuttle may be represented by two components. One component is the gravitational acceleration g = 9.66 m/s^2 at this altitude. The second component equals 12.90 m/s^2 due to atmospheric resistance and is directed opposite to the velocity. The shuttle is at an altitude of' 48.2 km and has reduced its orbital velocity of 28 300 km/h to 15 450 km/h in the direction θ=1.50°. For this instant, calculate the radius of curvature p of the path and the rate *v* at which the speed is changing.

Problem 2/128



$$\dot{v} = a_t = 9.66 \sin 1.50° - 12.90 = \underline{-12.65 \ m/s^2}$$

$$a_n = g \cos\theta = 9.66 \cos 1.5° = 9.657 \ m/s^2$$

$$a_n = \frac{v^2}{\rho} \Rightarrow \rho = \frac{v^2}{a_n} = \frac{(15\,450/3.6)^2}{9.657}$$

$$\underline{\rho = 1907 \ km}$$