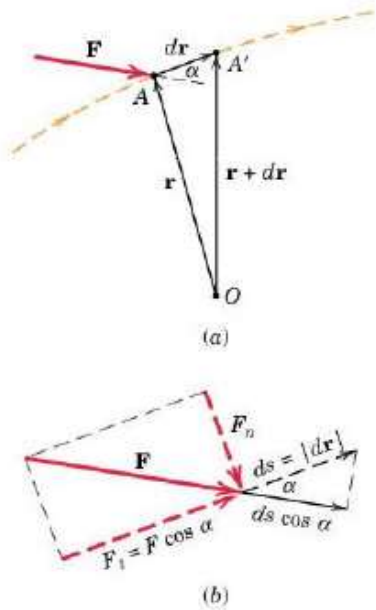
- **Section B: Work and Energy**

1. Definition of Work

Figure shows a force F acting on a particle at A which moves along the path shown. The position vector r measured from some convenient origin O locates the particle as it passes point A, and dr is the differential displacement associated with an infinitesimal movement from A to A'. The work done by the force F during the displacement dr is defined as



$$dU = \mathbf{F} \cdot d\mathbf{r}$$

The magnitude of this dot product is dU = F ds cos a, where a is the angle between F and dr and where dr is the magnitude of dr. This expression may be interpreted as the displacement multiplied by the force component Fe = F cos a in the direction of the displacement, as represented by the dashed lines in Fig.

The work dU may be written as

$$dU = F_t\, ds$$

Work is positive if the working component Ft is in the direction of the displacement and negative if is in the opposite direction. Forces which do work are termed *active forces*. Constraint forces which do no work are termed reactive forces.

2. Units of Work

The SI units of work are those of force (N) times displacement (m) or N•m. This unit is given a secial name joule (J), which is defined as the work done by a force of 1 N acting through a distance of 1 m in the direction of the force.

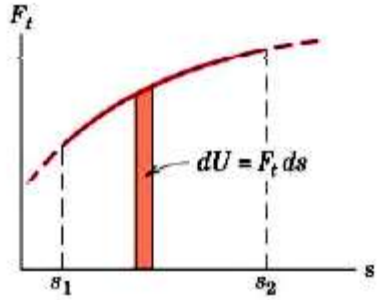
In the U.S. customary system, work has the units of ft-lb. Dimensionally, work and moment are the same. In order to distinguish between the two quantities, it is recommended that work be expressed as foot pounds (ft-lb) and moment as pound feet (lb-ft). It should be noted that work is a scalar as given by the dot product and involves the product of a force and a distance, both measured along the same line. Moment, on the other hand, is a vector as given by the cross product and involves the product of force and distance measured at right angles to the force.

3. Calculation of Work

During a finite movement of the point of application of a force, the force does an amount of work equal to

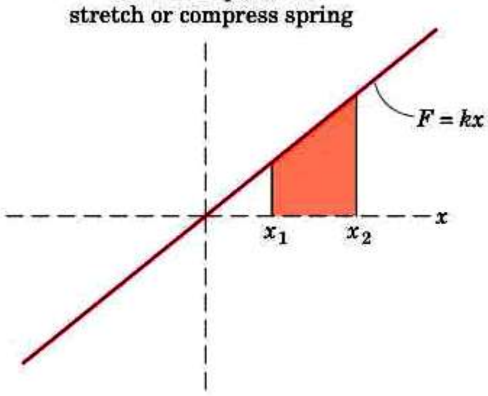
$$U = \int_1^2 \mathbf{F}\cdot d\mathbf{r} = \int_1^2 (F_x\,dx + F_y\,dy + F_z\,dz)$$

$$U = \int_{s_1}^{s_2} F_t\,ds$$

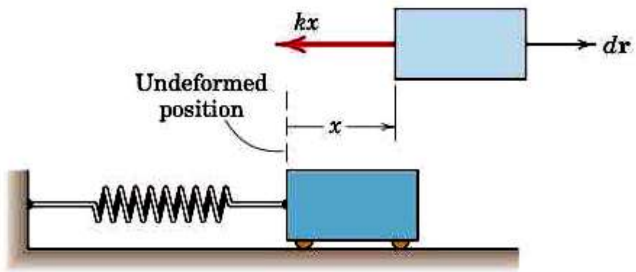


(a) Work Associated with a Spring Force. We consider here the common linear spring of stiffness k where the force required to stretch or compress the spring is proportional to the deformation x, as shown in Fig. We wish to determine the work done on the body by the spring force as the body undergoes an arbitrary displacement from an initial position x1 to a final position x2. The force exerted by the spring on the body is F = -kxi, as shown in Fig. From the definition of work, we have

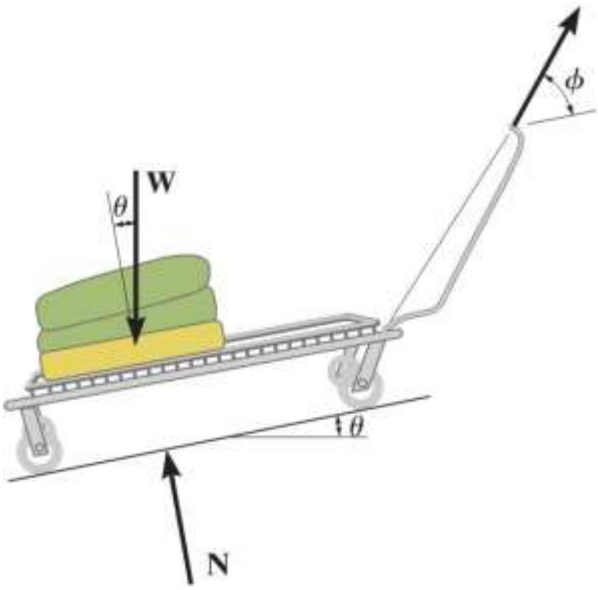
$$U_{1\text{-}2} = \int_1^2 \mathbf{F}\cdot d\mathbf{r} = \int_1^2 (-kx\mathbf{i})\cdot dx\,\mathbf{i} = -\int_{x_1}^{x_2} kx\,dx = \frac{1}{2}k({x_1}^2 - {x_2}^2) \qquad \textbf{(3/10)}$$



(a)



(b)



(b) Work associated with a constant external force

$$U_{1\text{-}2} = \int_1^2 \mathbf{F}\cdot d\mathbf{r} = \int_1^2 [(P\cos\alpha)\mathbf{i} + (P\sin\alpha)\mathbf{j}]\cdot dx\,\mathbf{i}$$

$$= \int_{x_1}^{x_2} P\cos\alpha\, dx = P\cos\alpha(x_2 - x_1) = PL\cos\alpha \qquad \textbf{(3/9)}$$

(c)Work Associated with Weight

If g is constant

$$U_{1\text{-}2} = \int_1^2 \mathbf{F}\cdot d\mathbf{r} = \int_1^2 (-mg\mathbf{j})\cdot(dx\mathbf{i} + dy\mathbf{j})$$

$$= -mg\int_{y_1}^{y_2} dy = -mg(y_2 - y_1) \qquad \textbf{(3/11)}$$

If g is not constant

$$U_{1\text{-}2} = \int_1^2 \mathbf{F}\cdot d\mathbf{r} = \int_1^2 \frac{-Gm_e m}{r^2}\mathbf{e}_r\cdot dr\,\mathbf{e}_r = -Gm_e m\int_{r_1}^{r_2}\frac{dr}{r^2}$$

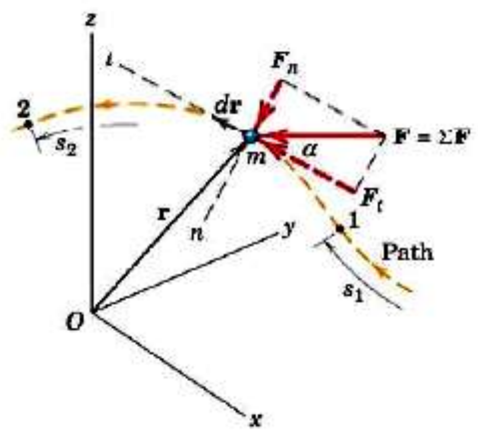
$$= Gm_e m\left(\frac{1}{r_2} - \frac{1}{r_1}\right) = mgR^2\left(\frac{1}{r_2} - \frac{1}{r_1}\right) \qquad \textbf{(3/12)}$$

4. Work and Curvilinear Motion

The work done by F during a finite movement of the particle from point 1 to point 2 is

$$U_{1\text{-}2} = \int_1^2 \mathbf{F}\cdot d\mathbf{r} = \int_{s_1}^{s_2} F_t\, ds$$

$$U_{1\text{-}2} = \int_1^2 \mathbf{F}\cdot d\mathbf{r} = \int_1^2 m\mathbf{a}\cdot d\mathbf{r}$$



But $\mathbf{a}\cdot d\mathbf{r} = a_t ds$, where a_t is the tangential component of the acceleration of m. In terms of the velocity v of the particle, Eq. 2/3 gives $a_t\, ds = v\, dv$. Thus, the expression for the work of **F** becomes

$$U_{1\text{-}2} = \int_1^2 \mathbf{F}\cdot d\mathbf{r} = \int_{v_1}^{v_2} mv\, dv = \frac{1}{2}m(v_2^2 - v_1^2) \qquad \textbf{(3/13)}$$

5. Principle of Work and Kinetic Energy

The kinetic energy T of the particle is defined as

$$T = \frac{1}{2}mv^2 \qquad \textbf{(3/14)}$$

and is the total work which must be done on the particle to bring it from a state of rest to a velocity u. Kinetic energy T is a scalar quantity with the units of N•m or joules (J) in SI units and ft-lb in U.S. customary units. Kinetic energy is always positive, regardless of the direction of the velocity.

$$U_{1\text{-}2} = T_2 - T_1 = \Delta T \qquad \textbf{(3/15)}$$

which is the **work-energy equation** for a particle. The equation states that the total work done by all forces acting on a particle as it moves from point 1 to point 2 equals the corresponding change in kinetic energy of the particle. Although T is always positive, the change ΔT may be positive, negative, or zero.

$$T_1 + U_{1\text{-}2} = T_2 \qquad \textbf{(3/15a)}$$