Semester I (2019-2020)

Chapter Four
Kinematics of rigid bodies

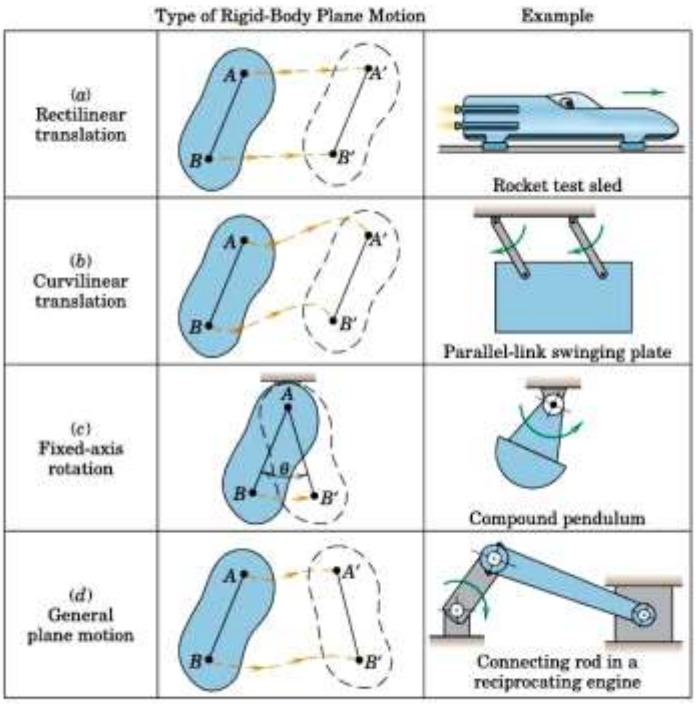
Rigid-Body Assumption:

*A **rigid body** as a system of particles for which the distances between the particles remain unchanged*

All solid materials change shape to some extent when forces are applied to them. Nevertheless, if the movements associated with the changes in shape are very small compared with the movements of the body as a whole, then the assumption of rigidity is usually acceptable. The displacements due to the flutter of an aircraft wing, for instance, do not affect the description of the flight path of the aircraft as a whole, and thus the rigid-body assumption is clearly acceptable. On the other hand, if the problem is one of describing, as a function of time, the internal wing stress due to wing flutter, then the relative motions of portions of the wing cannot be neglected, and the wing may not be considered a rigid body. In this and the next two chapters, almost all of the material is based on the assumption of rigidity.

Plane Motion

A rigid body executes plane motion when all parts of the body move in parallel planes. For convenience, we generally consider the plane of motion to be the plane which contains the mass center, and we treat the body as a thin slab whose motion is confined to the plane of the slab. This idealization adequately describes a very large category of rigid-body motions encountered in engineering. The plane motion of a rigid body may be divided into several categories, as represented in Fig.



Rotation

The rotation of a rigid body is described by its angular motion. Figure 5/2 shows a rigid body which is rotating as it undergoes plane motion in the plane of the figure. The angular positions of any two lines 1 and 2 attached to the body are specified by θ_1 and θ_2 measured from any convenient fixed reference direction. Because the angle β is invariant, the relation $\theta_2 = \theta_1 + \beta$ upon differentiation with respect to time gives $\dot{\theta}_2 = \dot{\theta}_1$ and $\ddot{\theta}_2 = \ddot{\theta}_1$ or, during a finite interval, $\Delta\theta_2 = \Delta\theta_1$. Thus, *all lines on a rigid body in its plane of motion have the same angular displacement, the same angular velocity, and the same angular acceleration*.

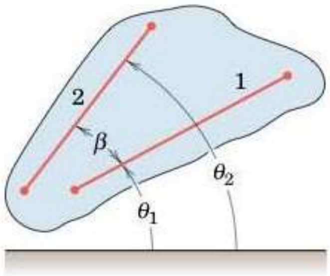


Figure 5/2

Angular-Motion Relations

The angular velocity ω and angular acceleration α of a rigid body in plane rotation are, respectively, the first and second time derivatives of the angular position coordinate θ of any line in the plane of motion of the body. These definitions give

$$\omega = \frac{d\theta}{dt} = \dot{\theta}$$

$$\alpha = \frac{d\omega}{dt} = \dot{\omega} \quad \text{or} \quad \alpha = \frac{d^2\theta}{dt^2} = \ddot{\theta} \qquad (5/1)$$

$$\omega \, d\omega = \alpha \, d\theta \quad \text{or} \quad \dot{\theta} \, d\dot{\theta} = \ddot{\theta} \, d\theta$$

For rotation with *constant* angular acceleration, the integrals of Eqs. 5/1 becomes

$$\omega = \omega_0 + \alpha t$$

$$\omega^2 = {\omega_0}^2 + 2\alpha(\theta - \theta_0)$$

$$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$$

Here θ_0 and ω_0 are the values of the angular position coordinate and angular velocity, respectively, at $t = 0$, and t is the duration of the motion considered. You should be able to carry out these integrations easily, as they are completely analogous to the corresponding equations for rectilinear motion with constant acceleration covered in Art. 2/2.

Rotation about fixed axis:

When a rigid body rotates about a fixed axis, all points other than those on the axis move in concentric circles about the fixed axis. Thus, for tho rigid body in Fig. 5/3 rotating about a fixed axis normal to tho plane of the figure through O, any point such as A moves in a circle of radius r.

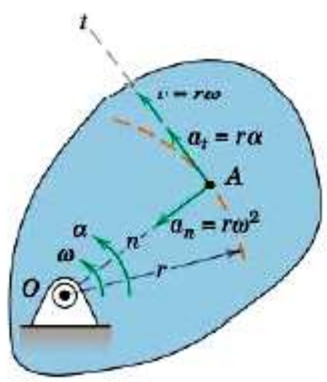


Figure 5/3

$$v = r\omega$$
$$a_n = r\omega^2 = v^2/r = v\omega \qquad (5/2)$$
$$a_t = r\alpha$$

The acceleration of point A is obtained by differentiating the cross-product expression for $\mathbf{v}$, which gives

$$\begin{aligned} \mathbf{a} = \dot{\mathbf{v}} &= \boldsymbol{\omega} \times \dot{\mathbf{r}} + \dot{\boldsymbol{\omega}} \times \mathbf{r} \\ &= \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + \dot{\boldsymbol{\omega}} \times \mathbf{r} \\ &= \boldsymbol{\omega} \times \mathbf{v} + \boldsymbol{\alpha} \times \mathbf{r} \end{aligned}$$

Here $\boldsymbol{\alpha} = \dot{\boldsymbol{\omega}}$ stands for the angular acceleration of the body. Thus, the vector equivalents to Eqs. 5/2 are

$$\begin{aligned} \mathbf{v} &= \boldsymbol{\omega} \times \mathbf{r} \\ \mathbf{a}_n &= \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \\ \mathbf{a}_t &= \boldsymbol{\alpha} \times \mathbf{r} \end{aligned} \qquad \textbf{(5/3)}$$

and are shown in Fig. 5/4*b*.

For three-dimensional motion of a rigid body, the angular-velocity vector $\boldsymbol{\omega}$ may change direction as well as magnitude, and in this case, the angular acceleration, which is the time derivative of angular velocity, $\boldsymbol{\alpha} = \dot{\boldsymbol{\omega}}$, will no longer be in the same direction as $\boldsymbol{\omega}$.

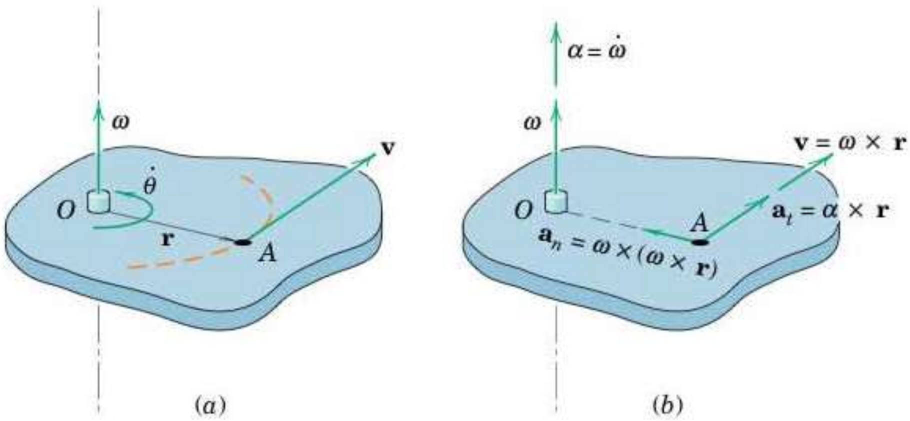


Figure 5/4

Sample Problem 5/1

A flywheel rotating freely at 1800 rev/min clockwise is subjected to a variable counterclockwise torque which is first applied at time $t = 0$. The torque produces a counterclockwise angular acceleration $\alpha = 4t$ rad/s^2, where t is the time in seconds during which the torque is applied. Determine (*a*) the time required for the flywheel to reduce its clockwise angular speed to 900 rev/min, (*b*) the time required for the flywheel to reverse its direction of rotation, and (*c*) the total number of revolutions, clockwise plus counterclockwise, turned by the flywheel during the first 14 seconds of torque application.

Solution. The counterclockwise direction will be taken arbitrarily as positive.

(a) Since α is a known function of the time, we may integrate it to obtain angular ① velocity. With the initial angular velocity of $-1800(2\pi)/60 = -60\pi$ rad/s, we have

$$[d\omega = \alpha\, dt] \qquad \int_{-60\pi}^{\omega} d\omega = \int_0^t 4t\, dt \qquad \omega = -60\pi + 2t^2$$

Substituting the clockwise angular speed of 900 rev/min or $\omega = -900(2\pi)/60 = -30\pi$ rad/s gives

$$-30\pi = -60\pi + 2t^2 \qquad t^2 = 15\pi \qquad t = 6.86 \text{ s} \qquad \textit{Ans.}$$

(b) The flywheel changes direction when its angular velocity is momentarily zero. Thus,

$$0 = -60\pi + 2t^2 \qquad t^2 = 30\pi \qquad t = 9.71 \text{ s} \qquad \textit{Ans.}$$

(c) The total number of revolutions through which the flywheel turns during 14 seconds is the number of clockwise turns N_1 during the first 9.71 seconds, plus the number of counterclockwise turns N_2 during the remainder of the interval. Integrating the expression for ω in terms of t gives us the angular displacement in radians. Thus, for the first interval

$$[d\theta = \omega\, dt] \qquad \int_0^{\theta_1} d\theta = \int_0^{9.71} (-60\pi + 2t^2)\, dt$$

② $$\theta_1 = [-60\pi t + \tfrac{2}{3}t^3]_0^{9.71} = -1220 \text{ rad}$$

or $N_1 = 1220/2\pi = 194.2$ revolutions clockwise.

For the second interval

$$\int_0^{\theta_2} d\theta = \int_{9.71}^{14} (-60\pi + 2t^2)\, dt$$

③ $$\theta_2 = [-60\pi t + \tfrac{2}{3}t^3]_{9.71}^{14} = 410 \text{ rad}$$

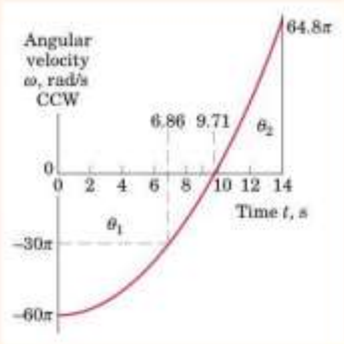
or $N_2 = 410/2\pi = 65.3$ revolutions counterclockwise. Thus, the total number of revolutions turned during the 14 seconds is

$$N = N_1 + N_2 = 194.2 + 65.3 = 259 \text{ rev} \qquad \textit{Ans.}$$

We have plotted ω versus t and we see that θ_1 is represented by the negative area and θ_2 by the positive area. If we had integrated over the entire interval in one step, we would have obtained $|\theta_2| - |\theta_1|$.

Helpful Hints

① We must be very careful to be consistent with our algebraic signs. The lower limit is the negative (clockwise) value of the initial angular velocity. Also we must convert revolutions to radians since α is in radian units.

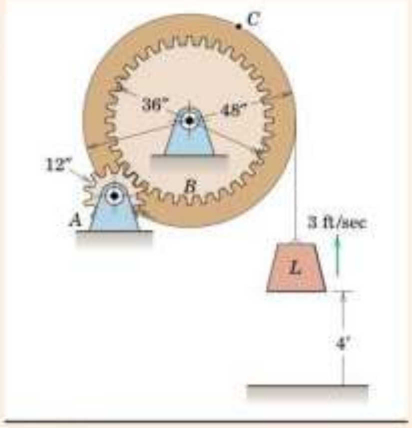


② Again note that the minus sign signifies clockwise in this problem.

③ We could have converted the original expression for α into the units of rev/s^2, in which case our integrals would have come out directly in revolutions.

Sample Problem 5/2

The pinion A of the hoist motor drives gear B, which is attached to the hoisting drum. The load L is lifted from its rest position and acquires an upward velocity of 3 ft/sec in a vertical rise of 4 ft with constant acceleration. As the load passes this position, compute (a) the acceleration of point C on the cable in contact with the drum and (b) the angular velocity and angular acceleration of the pinion A.



Solution. **(a)** If the cable does not slip on the drum, the vertical velocity and acceleration of the load L are, of necessity, the same as the tangential velocity v and tangential acceleration a_t of point C. For the rectilinear motion of L with constant acceleration, the n- and t-components of the acceleration of C become

$[v^2 = 2as]$ $\quad a = a_t = v^2/2s = 3^2/[2(4)] = 1.125 \text{ ft/sec}^2$

① $[a_n = v^2/r]$ $\quad a_n = 3^2/(24/12) = 4.5 \text{ ft/sec}^2$

$[a = \sqrt{a_n^2 + a_t^2}]$ $\quad a_C = \sqrt{(4.5)^2 + (1.125)^2} = 4.64 \text{ ft/sec}^2$ *Ans.*

(b) The angular motion of gear A is determined from the angular motion of gear B by the velocity v_1 and tangential acceleration a_1 of their common point of contact. First, the angular motion of gear B is determined from the motion of point C on the attached drum. Thus,

$[v = r\omega]$ $\quad \omega_B = v/r = 3/(24/12) = 1.5 \text{ rad/sec}$

$[a_t = r\alpha]$ $\quad \alpha_B = a_t/r = 1.125/(24/12) = 0.562 \text{ rad/sec}^2$

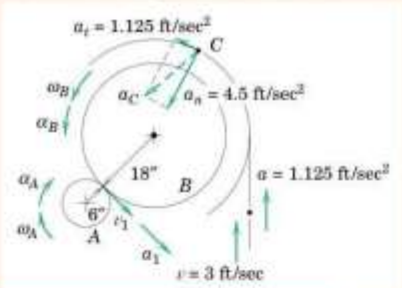
Then from $v_1 = r_A\omega_A = r_B\omega_B$ and $a_1 = r_A\alpha_A = r_B\alpha_B$, we have

$$\omega_A = \frac{r_B}{r_A}\omega_B = \frac{18/12}{6/12}1.5 = 4.5 \text{ rad/sec CW} \qquad \textit{Ans.}$$

$$\alpha_A = \frac{r_B}{r_A}\alpha_B = \frac{18/12}{6/12}0.562 = 1.688 \text{ rad/sec}^2 \text{ CW} \qquad \textit{Ans.}$$

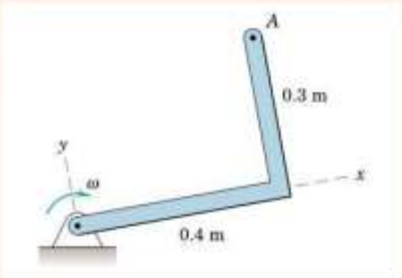
Helpful Hint

① Recognize that a point on the cable changes the direction of its velocity after it contacts the drum and acquires a normal component of acceleration.



Sample Problem 5/3

The right-angle bar rotates clockwise with an angular velocity which is decreasing at the rate of 4 rad/s². Write the vector expressions for the velocity and acceleration of point A when $\omega = 2$ rad/s.



Solution. Using the right-hand rule gives

$$\boldsymbol{\omega} = -2\mathbf{k} \text{ rad/s} \quad \text{and} \quad \boldsymbol{\alpha} = +4\mathbf{k} \text{ rad/s}^2$$

The velocity and acceleration of A become

$[\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}]$ $\quad \mathbf{v} = -2\mathbf{k} \times (0.4\mathbf{i} + 0.3\mathbf{j}) = 0.6\mathbf{i} - 0.8\mathbf{j}$ m/s *Ans.*

$[\mathbf{a}_n = \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})]$ $\quad \mathbf{a}_n = -2\mathbf{k} \times (0.6\mathbf{i} - 0.8\mathbf{j}) = -1.6\mathbf{i} - 1.2\mathbf{j}$ m/s²

$[\mathbf{a}_t = \boldsymbol{\alpha} \times \mathbf{r}]$ $\quad \mathbf{a}_t = 4\mathbf{k} \times (0.4\mathbf{i} + 0.3\mathbf{j}) = -1.2\mathbf{i} + 1.6\mathbf{j}$ m/s²

$[\mathbf{a} = \mathbf{a}_n + \mathbf{a}_t]$ $\quad \mathbf{a} = -2.8\mathbf{i} + 0.4\mathbf{j}$ m/s² *Ans.*

The magnitudes of $\mathbf{v}$ and $\mathbf{a}$ are

$$v = \sqrt{0.6^2 + 0.8^2} = 1 \text{ m/s} \quad \text{and} \quad a = \sqrt{2.8^2 + 0.4^2} = 2.83 \text{ m/s}^2$$