Chapter 1

First Order Ordinary Differential Equations

Introduction:

A differential equation is an equation that involves one or more derivatives, or differentials. Differential equations are classified as:

a) **Type** (ordinary or partial),
b) **Order** (which is the highest order derivative that occurs in the equation),
c) **Degree** (the exponent of the highest power of the highest order derivative, after the equation has been cleared of fractions and radicals in the dependent variable and its derivatives).

For Example:

$$\left(\frac{d^3y}{dx^3}\right)^2 + \left(\frac{d^2y}{dx^2}\right)^5 + \frac{y}{x^2+1} = e^x$$

is an ordinary differential equation, of order three and degree two.

Only "ordinary" derivatives occur when the dependent variable y is a function of a single independent variable x. On the other hand, if the dependent variable y is a function of two or more independent variables, like

$$y = f(x,t),$$

where x and t are independent variables, then partial derivatives of y may occur. For example,

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$$

is a partial differential equation, of order two and degree one, (It is one-dimensional "wave equation").

Many physical problems, when formulated in mathematical terms, lead to differential equations. For example,

$$m\frac{d^2x}{dt^2} = 0, \qquad m\frac{d^2y}{dt^2} = -mg$$

describes the motion of a projectile (neglecting air resistance).

Indeed, one of the chief sources of differential equation is Newton's second law:

$$F = \frac{d}{dt}(mv),$$

where F is the resultant of the forces acting on a body of mass m and v is its velocity.

Solutions of Differential Equations (D.E):

A function

$$y = f(x)$$

is said to be a solution of a D.E if the latter is satisfied when y and its derivatives are replaced by $f(x)$ and its corresponding derivatives. For example, if C_1 and C_2 are any constants, then

$$y = C_1 \cos x + C_2 \sin x \qquad (1a)$$

is a solution of the D.E

$$\frac{d^2 y}{dx^2} + y = o \qquad (1b)$$

A physical problem that translates into a D.E usually involves additional conditions not expressed by the D.E itself. In mechanics, for example, the initial position and velocity of the moving body are usually prescribed, as well as the forces. The D.E, or equations, of motion will usually have solutions in which certain arbitrary constants occur, as shown in (Eq. 1a) above. Specific values are then assigned to these arbitrary constants to meet the prescribed initial conditions.

A D.E of order n will generally have a solution involving n arbitrary constants. This solution is called the ***general*** solution. Once the general solution is known, it is only a matter of algebra to determine specific values of the constants if initial conditions are also prescribed.

The following topics will be considered for ordinary differential equations solution.

1. **First order.**
   - **a)** Variable separable.
   - **b)** Exact differentials.
   - **c)** Homogeneous.
   - d) Linear.
2. **Special types of second order.**
3. **Linear equations with constant coefficients.**
   - **a)** Homogeneous.
   - **b)** Inhomogeneous.

1.1 First Order Ordinary Differential Equations

1.1.1 Variable Separable Differential Equations:

Any D.E that can be written in the form

$$\boxed{P(x)+Q(y).y'=0}$$

Is a separable equation, (because the dependent and independent variables are separated). We can obtain an implicit by integrating with respect to x.

$$\int P(x).dx+\int Q(y).\frac{dy}{dx}.dx=c$$

$$\boxed{\int P(x).dx+\int Q(y)dy=c}$$

Example: Consider the D.E $y'=xy^2$. We separate the dependent and independent variables and integrate to find the solution.

$$\frac{dy}{dx}=x.y^2$$
$$y^{-2}dy=x.dx$$
$$\int y^{-2}dy=\int x.dx+c$$
$$-y^{-1}=\frac{x^2}{2}+c$$
$$\left[y=\frac{-1}{x^2/2+c}\right]$$

Example: The equation $y'=y-y^2$ is separable.

$$\left(\frac{y'}{y-y^2}=1\right)$$

We expand in partial fraction and integrate.

$$\left(\frac{1}{y}-\frac{1}{y-1}\right).y'=1$$
$$\ln|y|-\ln|y-1|=x+c$$

We have an implicit function for *y(x)*. Now we solve for *y(x)*.

$$\ln\left|\frac{y}{y-1}\right| = x + c$$
$$\left|\frac{y}{y-1}\right| = e^{x+c}$$
$$\frac{y}{y-1} = \pm e^{x+c}$$
$$\frac{y}{y-1} = Ce^{x}$$

Example: Consider the D.E $(xy^2 + x)dx + (yx^2 + y)dy = 0$. We separate the dependant and independent variables and integrate to find the solution.

$$x(y^2+1).dx + y.(x^2+1)dy = 0$$
$$\frac{x.dx}{x^2+1} = -\frac{y.dy}{y^2+1}$$ Multiply by 2 and Integrate

$$\ln(x^2+1) + \ln(y^2+1) = c$$
$$\ln(x^2+1).(y^2+1) = c \Rightarrow (x^2+1).(y^2+1) = e^c = C$$

Example: Solve the following D.E ?

$$(4y - \cos y).\frac{dy}{dx} - 3x^2 = 0$$
$$(4y - \cos y).\frac{dy}{dx} = 3x^2$$
$$(4y - \cos y).dy = 3x^2.dx$$
$$\int (4y - \cos y).dy = \int 3x^2.dx$$
$$\frac{4y^2}{2} - \sin y = \frac{3x^3}{3} + c$$
$$2y^2 - \sin y = x^3 + c$$

(1.1.2) Exact differential equations:

Any first order ordinary D.E's of the first degree can be written as the total D.E,

$$P(x,y).dx + Q(x,y).dy = 0.$$

If this equation can be integrated directly, that is if there is a primitive, *u(x,y)*, such that,

$$du = P.dx + Q.dy,$$

then this equation is called *exact*. The (implicit) solution of the D.E is

$$u(x,y) = c,$$

where *c* is an arbitrary constant. Since the differential of a function, *u(x,y)*, is

$$du \equiv \frac{\partial u}{\partial x}.dx + \frac{\partial u}{\partial y}.dy,$$

P and *Q* are the partial derivatives of *u*:

$$P(x,y) = \frac{\partial u}{\partial x} \quad , \qquad Q(x,y) = \frac{\partial u}{\partial y} \ .$$

In an alternative notation, the D.E

$$\frac{du}{dx} \equiv \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}.\frac{\partial y}{\partial x} = P(x,y) + Q(x,y).\frac{dy}{dx}$$

The solution of the D.E is $u(x,y) = c$.

Example:

$$x + y.\frac{dy}{dx} = 0$$

is an exact D.E since

$$\frac{d}{dx}\left(\frac{1}{2}\left(x^2 + y^2\right)\right) = x + y.\frac{dy}{dx}$$

The solution of the D.E is

$$\frac{1}{2}(x^2+y^2)=c$$

Example: Let *f(x)* and *g(x)* be known functions.

$$g(x).y'+g'(x).y=f(x)$$

is an exact D.E since

$$\frac{d}{dx}(g(x).y(x))=g\,y'+g'\,y\ .$$

The solution of D.E is

$$g(x).y(x)=\int f(x).dx+c$$

$$y(x)=\frac{1}{g(x)}.\int f(x).dx+\frac{c}{g(x)}.$$

A necessary condition for exactness. The solution of the Exact equation $P+Q.y'=0$ is $u=c$ where u is the primitive of the equation $\frac{du}{dx}=P+Q.y'$. At present the only method we have for determining the primitive is guessing. This is fine for simple equations, but for more difficult cases we would like a method more concrete than inspiration. As a first step toward this goal we determine a criterion for determining if an equation is exact.

Consider the exact equation,

$$\boldsymbol{P+Q\,.\,y'=0\,,}$$

with primitive u, where we assume that the function P and Q are continuously differentiable. Since the mixed partial derivatives of u are equal,

$$\frac{\partial^2 u}{\partial x.\partial y}=\frac{\partial^2 u}{\partial y.\partial x},$$

a necessary condition for exactness is $\boxed{\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}.}$

Example: Prove that the following D.E is exact?

$$y^2dx+2xydy=0$$

$P=y^2$ $\qquad$ $Q=2xy$

$\frac{\partial P}{\partial y}=2y$ $\qquad$ $\frac{\partial Q}{\partial x}=2y$

$\because \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad \therefore \quad D.E \quad is \quad exact$

Example: Prove that the following D.E is exact and find the general solution?

$$(3x^2 y + 2xy).dx + (x^3 + x^2 + 2y).dy = 0$$

$$P = 3x^2 y + 2xy \qquad Q = x^3 + x^2 + 2y$$

$$\frac{\partial P}{\partial y} = 3x^2 + 2x \qquad \frac{\partial Q}{\partial x} = 3x^2 + 2x$$

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad \therefore \quad D.E \quad is \quad exact$$

$$\frac{\partial u}{\partial x} = P = 3x^2 y + 2xy \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

$$\frac{\partial u}{\partial y} = Q = x^3 + x^2 + 2y \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

by integrating eq. (1) with respect to x we get

$$u = x^3 y + x^2 y + c$$

$$u = x^3 y + x^2 y + \phi(y) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

by deriving eq. (3) with respect to y we get

$$\frac{\partial u}{\partial y} = x^3 + x^2 + \phi'(y) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

and by equalizing eq. (4) with eq. (2) we get

$$x^3 + x^2 + \phi'(y) = x^3 + x^2 + 2y$$

$$\phi'(y) = 2y$$

$$\int \phi'(y) = \int 2y$$

$$\phi(y) = y^2 + c$$

and by substituting $\phi(y)$ *in eq. (3) we get the General solution as,*

$$u(x, y) = x^3 y + x^2 y + y^2 + c$$

Example: Prove that the following D.E is exact and find the general solution?

$(\cos x + y\sin x).dx = \cos x.dy$

$(\cos x + y\sin x).dx - \cos x.dy = 0$

$P = \cos x + y\sin x \qquad Q = -\cos x$

$\frac{\partial P}{\partial y} = \sin x \qquad \frac{\partial Q}{\partial x} = -(-\sin x) = \sin x$

$\because \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad \therefore \quad D.E \quad is \quad exact$

$\frac{\partial u}{\partial x} = P = \cos x + y\sin x \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$

$\frac{\partial u}{\partial y} = Q = -\cos x \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$

$by \quad integrating \quad eq. \quad (1)$

$u = \sin x - y\cos x + \phi(y) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$

$by \quad deriving \quad eq. \quad (3) \quad partially \quad to \quad y$

$\frac{\partial u}{\partial y} = -\cos x + \phi'(y) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$

$by \quad equalizing \quad eq. \quad (4) \quad to \quad eq. \quad (2) \quad we \quad get$

$-\cos x + \phi'(y) = -\cos x \quad \Rightarrow \quad \phi'(y) = 0 \quad \Rightarrow \quad \phi(y) = c$

$by \quad substituting \quad \phi(y) \quad in \quad eq. \quad (3)$

$u = \sin x - y\cos x + c \quad general \quad solution$

Example: Prove that the following D.E is exact and find the general solution?

$(xy\cos xy + \sin xy).dx + (x^2\cos xy + e^y).dy = 0$

$P = xy\cos xy + \sin xy \qquad Q = x^2\cos xy + e^y$

$\frac{\partial P}{\partial y} = -x.y.x.\sin xy + x.\cos xy + x.\cos xy \qquad \frac{\partial Q}{\partial y} = -x^2.y.\sin xy + 2x.\cos xy$

$\because \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad \therefore \quad D.E \quad is \quad exact$

$P = \frac{\partial u}{\partial x} = xy\cos xy + \sin xy \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$

$Q = \frac{\partial u}{\partial y} = x^2\cos xy + e^y \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$

$by \quad integrating \quad eq. \ (2) \quad with \quad respect \quad to \quad y$

$$u = x.\sin xy + e^{y} + \phi(x) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \ (3)$$

$drive \quad eq. \ (3) \quad paritially \quad with \quad respect \quad to \quad x$

$$\frac{\partial u}{\partial x} = x.y.\cos xy + \sin xy + \phi'(x) \qquad \ldots\ldots\ldots\ldots\ldots\ldots \ (4)$$

$by \quad equalizing \quad eq. \ (4) \quad with \quad eq. \ (1) \quad we \quad get$

$$x.y.\cos xy + \sin xy + \phi'(x) = x y \cos xy + \sin xy \quad \Rightarrow \quad \phi'(x) = 0 \quad \Rightarrow \quad \phi(x) = c$$

$$\therefore \quad u = x.\sin xy + e^{y} + c \quad which \quad is \quad the \quad general \quad solution.$$

(1.1.3) Homogeneous differential equations:

Homogeneous coefficient, first order D.E's form another class of soluble eqs. We will find that a change in dependant variable will make such eqs. separable or we can determine an integrating factor that will make such eqs. exact. First we define homogeneous functions.

$$\boxed{\frac{dy}{dx} = f\left(\frac{y}{x}\right)}$$

Example: Solve the homogeneous D.E?

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right) \qquad \ldots\ldots\ldots \ (1)$$

$$Let \quad v = \frac{y}{x} \quad \Rightarrow \quad y = v.x$$

$$\frac{dy}{dx} = v + x.\frac{dv}{dx} = f(v)$$

$$x.\frac{dv}{dx} = f(v) - v$$

$$\frac{x}{dx} = \frac{f(v) - v}{dv}$$

$$\frac{dx}{x} = \frac{dv}{F(v) - v}$$

$$\int \frac{dx}{x} + c = \int \frac{dv}{F(v) - v}$$

Example: Solve the homogeneous D.E?

$$y' = \frac{x^2 + y^2}{x.y}$$

$$\frac{dy}{dx} = \frac{x^2}{x.y} + \frac{y^2}{x.y}$$

$$\frac{dy}{dx} = \frac{x}{y} + \frac{y}{x} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

$$Let \quad v = \frac{y}{x}$$

$$\frac{dy}{dx} = v + x.\frac{dv}{dx}$$

$$substitute \quad in \quad Eq. \quad (1)$$

$$v + x.\frac{dv}{dx} = \frac{1}{v} + v$$

$$x\frac{dv}{dx} = \frac{1}{v} \quad \Rightarrow \quad \frac{x}{dx} = \frac{1}{v.dv}$$

$$\int \frac{dx}{x} = \int v.dv$$

$$\ln x = \frac{v^2}{2} \quad \Rightarrow 2(\ln x + c_1) = v^2$$

$$2(\ln x + c_1) = \left(\frac{y}{x}\right)^2 \quad \Rightarrow 2\ln x + 2c_1 = \frac{y^2}{x^2}$$

$$\ln x^2 + C = \frac{y^2}{x^2} \quad \Rightarrow y^2 = x^2(\ln x^2 + C)$$

Example: Solve the homogeneous D.E?

$$2xyy' - y^2 + x^2 = 0$$

$$\frac{2xy}{x^2}\cdot\frac{dy}{dx} - \frac{y^2}{x^2} + \frac{x^2}{x^2} = 0$$

$$\frac{2y}{x}\cdot\frac{dy}{dx} - \left(\frac{y}{x}\right)^2 + 1 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

$$Let \quad v = \frac{y}{x} \quad \Rightarrow \quad \frac{dy}{dx} = v + x.\frac{dv}{dx} \quad \Rightarrow \quad 2v\left(v + x.\frac{dv}{dx}\right) - v^2 + 1 = 0$$

$$2.v^2 + 2.x.v.\frac{dv}{dx} - v^2 + 1 = 0 \quad \Rightarrow \quad 2.x.v.\frac{dv}{dx} + v^2 + 1 = 0$$

$$2.x.v.\frac{dv}{dx} = -(v^2+1) \quad \Rightarrow \frac{x}{dx} = \frac{-(v^2+1)}{2.v.dv}$$

$$\frac{dx}{x} = \frac{2.v.dv}{-(v^2+1)} \quad \Rightarrow \quad -\int\frac{dx}{x} = \int\frac{2.v.dv}{v^2+1}$$

$$-\ln x + c = \ln(v^2+1) \quad \Rightarrow \quad \ln x^{-1} + c = \ln(v^2+1)$$

$$e^{(\ln x^{-1} + c)} = e^{\ln(v^2+1)} \quad \Rightarrow \quad e^{\ln x^{-1}}.e^c = e^{\ln(v^2+1)}$$

$$x^{-1}.c = v^2 + 1 \quad \Rightarrow \quad \frac{c}{x} = \left(\frac{y}{x}\right)^2 + 1$$

$$\frac{c}{x} = \frac{y^2}{x^2} + 1$$

(1.1.3.1) Equations reducible to homogeneous form:

Certain equations of the form

$$\frac{dy}{dx} = \frac{ax + by + c}{Ax + By + C}$$

can be reduced to the homogeneous form by substitution of

$$x = X + h \qquad y = Y + k\,, \quad then \quad \frac{dy}{dx} = \frac{dY}{dX}$$

$$\frac{dY}{dX} = \frac{a(X+h) + b(Y+k) + c}{A(X+h) + B(Y+k) + C} = \frac{aX + bY + ah + bk + c}{AX + BY + Ah + Bk + C} \quad where,$$

$$ah + bk + c = 0$$
$$Ah + Bk + C = 0$$

Example: Solve the following D.E?

$$\frac{dy}{dx}=\frac{x+2y-3}{2x+y-3}$$

assume $x=X+h \qquad y=Y+k$

$$\frac{dY}{dX}=\frac{X+h+2(Y+k)-3}{2(X+h)+Y+k-3}=\frac{X+2Y+h+2k-3}{2X+Y+2h+k-3}$$

$$h+2k-3=0$$
$$\underline{2h+k-3=0}$$
$$3k-6+3-3=0 \Rightarrow k=1 \Rightarrow h=1$$

$$\frac{dY}{dX}=\frac{X+2Y}{2X+Y} \quad , \quad Y=VX$$

$$V+X\frac{dV}{dX}=\frac{X+2VX}{2X+VX}=\frac{1+2V}{2+V}$$

$$X\frac{dV}{dX}=\frac{1+2V}{2+V}-V=\frac{1-V^2}{2+V}$$

$$\frac{(2+V)\,dV}{1-V^2}=\frac{dX}{X} \Rightarrow \frac{2+V}{1-V^2}=\frac{A}{1-V}+\frac{B}{1+V}=\frac{A(1+V)+B(1-V)}{1-V^2}$$

$$2+V=A(1+V)+B(1-V)$$
$$2+V=V(A-B)+A+B$$
$$A-B=1$$
$$\underline{A+B=2}$$
$$2A=3 \Rightarrow A=\frac{3}{2} \quad , \quad B=\frac{1}{2}$$

$$\frac{\frac{3}{2}dV}{1-V}+\frac{\frac{1}{2}dV}{1+V}=\frac{dX}{X} \Rightarrow \frac{3\,dV}{2(1-V)}+\frac{dV}{2(1+V)}=\frac{dX}{X}$$

$$\frac{3\,dV}{(1-V)}+\frac{dV}{(1+V)}=\frac{2\,dX}{X}$$

$$-3\ln(1-V)+\ln(1+V)=2\ln X$$

$$\ln(1-V)^{-3}+\ln(1+V)-\ln X^2=C$$

$$\frac{\ln(1+V)}{\ln(1-V)^3}-\ln X^2=C \Rightarrow \ln\frac{1+V}{X^2(1-V)^3}=C$$

$$\frac{1+V}{X^2(1-V)^3}=c \Rightarrow \frac{1+\frac{Y}{X}}{X^2(1-\frac{Y}{X})^3}=c \Rightarrow \frac{\frac{X+Y}{X}}{X^2\frac{(X-Y)^3}{X^3}}=c \Rightarrow \frac{X+Y}{(X-Y)^3}=c$$

$$x=X+h=X+1 \Rightarrow X=x-1 \Rightarrow Y=y-1$$

$$\therefore \quad \frac{x-1+y+1}{[x-1-(y-1)]^3}=c \Rightarrow \frac{x+y-2}{(x-y)^3}=c$$

(1.1.4) The First Order, Linear Differential Equations:

(1.1.4.1) Homogeneous Equations:

The first order, linear, homogeneous D.E has the form

$$\boxed{\frac{dy}{dx} + p(x).y = 0}$$

We can solve any equation of this type because it is separable.

$$\frac{y'}{y} = -p(x)\,dx$$

$$\ln|y| = -\int p(x).dx + c$$

$$y = \pm e^{-\int p(x).dx + c}$$

$$\boxed{y = ce^{-\int p(x).dx}}$$

Example: Consider the equation

$$\frac{dy}{dx} + \frac{1}{x}y = 0$$

$$y = ce^{-\int p(x).dx} \quad \Rightarrow \quad y(x) = ce^{-\int \frac{1}{x}dx} \quad , \quad for \quad x \neq 0$$

$$y(x) = ce^{-\ln|x|}$$

$$y(x) = \frac{c}{|x|} \quad \Rightarrow \quad y(x) = \frac{c}{x}$$

(1.1.4.2) Inhomogeneous Equations:

The first order, linear, inhomogeneous D.E has the form

$$\frac{dy}{dx} + p(x).y = f(x)$$

There are two ways for linear inhomogeneous D.E.

1) $\frac{dy}{dx} + p(x).y = f(x)$

   the solution of this D.E is:

   $I(x) = e^{\int p(x)dx}$

   $I(x).y = \int I(x)f(x)\,dx + c$

2) $\frac{dx}{dy} + p(y).x = f(y)$

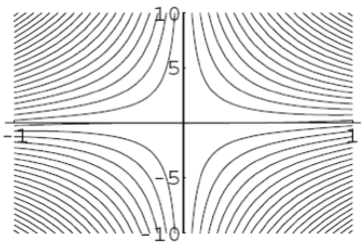
   the solution of this D.E is:

   $I(y) = e^{\int p(y)dy}$

   $I(y).x = \int I(y)f(y)\,dy + c$

Example1: consider the D.E

$$y' + \frac{1}{x}y = x^2 \quad , \quad x \succ o.$$



Solution to y' + y/x =x²

First find the integrating factor.

$$I(x) = \exp\left(\int \frac{1}{x}dx\right) = e^{\ln x} = x$$

then, multiply by the integrating factor and integrate.

$$I(x).y = \int I(x)f(x)\,dx + c$$

$$x.y = \int x.x^2\,dx + c = \int x^3 dx + c = \frac{1}{4}x^4 + c$$

$$y = \frac{1}{4}x^3 + \frac{c}{x}$$

Note that the general solution to the D.E is a one-parameter family of functions. The general solution is plotted in the figure above for various values of *c*.

Example2: Solve the following linear D.E?

$$\frac{dy}{dx}+2y=\cos x$$

$$p(x)=2 \qquad f(x)=\cos x$$

$$I(x)=e^{\int p(x).dx}=e^{\int 2.dx}=e^{2x}$$

$$I(x).y=\int I(x).f(x).dx+c$$

$$e^{2x}.y=\int e^{2x}.\cos x.dx+c \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$\int e^{2x}.\cos x.dx=u.v-\int v.du \quad integration \quad by \quad part$$

$$Let \quad u=e^{2x} \quad \Rightarrow \quad du=2e^{2x}.dx$$

$$dv=\cos x.dx \quad \Rightarrow \quad v=\int \cos x.dx=\sin x$$

$$\therefore \quad \int e^{2x}.\cos x.dx=e^{2x}.\sin x-2\int e^{2x}.\sin x.dx$$

$$Also \quad \int e^{2x}.\sin x.dx=u.v-\int v.du \quad integration \quad by \quad part$$

$$Let \quad u=e^{2x} \quad \Rightarrow \quad du=2e^{2x}.dx$$

$$dv=\sin x.dx \quad \Rightarrow \quad v=\int \sin x.dx=-\cos x$$

$$\therefore \quad \int e^{2x}.\sin x.dx=-e^{2x}.\cos x-\int -\cos x.2e^{2x}.dx$$

$$\int e^{2x}.\sin x.dx=-e^{2x}.\cos x+2\int -\cos x.e^{2x}.dx$$

$$\int e^{2x}.\cos x.dx=e^{2x}.\sin x-2\left[-e^{2x}.\cos x+2\int e^{2x}.\cos x.dx\right]$$

$$\int e^{2x}.\cos x.dx=e^{2x}.\sin x+2e^{2x}.\cos x-4\int e^{2x}.\cos x.dx$$

$$\int e^{2x}.\cos x.dx+4\int e^{2x}.\cos x.dx=e^{2x}.\sin x+2e^{2x}.\cos x$$

$$5\int e^{2x}.\cos x.dx=e^{2x}.\sin x+2e^{2x}.\cos x$$

$$\therefore \quad \int e^{2x}.\cos x.dx=\frac{e^{2x}.\sin x+2e^{2x}.\cos x}{5}$$

substitute in eq. (1)

$$e^{2x}.y=\frac{e^{2x}.\sin x+2e^{2x}.\cos x}{5}+c$$

Example3: Solve the following linear D.E?

$$\frac{dy}{dx}+2y=e^{-x}$$

Sol/

$$p(x)=2 \qquad f(x)=e^{-x}$$

$$I(x)=e^{\int p(x)dx}=e^{\int 2\,dx}=e^{2x}$$

$I(x).y = \int I(x).f(x)dx + c$

$e^{2x}.y = \int e^{2x}.e^{-x}dx = \int e^{x}dx = e^{x} + c$

$e^{2x}.y = e^{x} + c$

Example4: Solve the following linear D.E?

$$2\,\frac{dy}{dx} - y = e^{x/2}$$

Sol/

$\frac{dy}{dx} - \frac{1}{2}y = \frac{1}{2}e^{x/2}$

$p(x) = -\frac{1}{2} \qquad f(x) = \frac{1}{2}e^{x/2}$

$I(x) = e^{\int p(x)dx} = e^{\int -1/2\,dx} = e^{-x/2}$

$I(x).y = \int I(x).f(x)dx + c$

$e^{-x/2}.y = \int e^{-x/2}.\frac{1}{2}e^{x/2} = \frac{x}{2} + c$

$e^{-x/2}.y = \frac{x}{2} + c$

Example5: Solve the following linear D.E?

$$x\,dy + y\,dx = \sin x\,dx$$

Sol/

$x\,dy + y\,dx - \sin x\,dx = 0$

$x\,dy + (y - \sin x)\,dx$ \ x.dx

$\frac{dy}{dx} + \frac{y - \sin x}{x} = 0$

$\frac{dy}{dx} + \frac{y}{x} - \frac{\sin x}{x} = 0$

$\frac{dy}{dx} + \frac{1}{x}y = \frac{\sin x}{x}$

$p(x) = -\frac{1}{x} \qquad f(x) = \frac{\sin x}{x}$

$I(x) = e^{\int p(x)dx} = e^{\int -1/x\,dx} = e^{\ln(x)} = x$

$I(x).y = \int I(x).f(x)dx + c$

$x.y = \int x\frac{\sin x}{x}dx = \int \sin x\;dx = -\cos x + c$

$x.y = -\cos x + c$

Example6: Solve the following linear D.E?

$$(x-2y)\,dy + y\,dx = 0$$

Sol/

$(x-2y)\,dy + y\,dx = 0 \quad$ / *(x-2y) dx*

$\frac{dy}{dx} + \frac{y}{x-2y} = 0$

$\frac{dy}{dx} = -\frac{y}{x-2y}$

$\frac{dx}{dy} = -\frac{x-2y}{y}$

$\frac{dx}{dy} = -\frac{x-2y}{y}$

$\frac{dx}{dy} = -\frac{x}{y} + \frac{2y}{y} = -\frac{x}{y} + 2$

$\frac{dx}{dy} + \frac{1}{y}x = 2$

$p(y) = -\frac{1}{y} \qquad f(y) = 2$

$I(y) = e^{\int p(y)dy} = e^{\int 1/y\,dy} = e^{\ln(y)} = y$

$I(y).x = \int I(y).f(y)dy + c$

$y.x = \int y.2\,dy + c = y^2 + c$

$y.x = y^2 + c$

(1.1.4.3) Equation reducible to liner form (Bernoulli's equation):

The Eq. of the form $\frac{dy}{dx} + P.y = Q.y^n$ can be reduced to linear form by dividing by y^n and substituting $z = y^{1-n}$.

$$\frac{dy}{dx} + P.y = Q.y^n$$

$$y^{-n}.y' + P.y^{1-n} = Q,$$

$$\because \quad z = y^{1-n} \quad \therefore \quad \frac{dz}{dx} = (1-n).y^{-n}$$

$$y^{-n}.y' = \frac{1}{1-n}.\frac{dz}{dx} \quad \Rightarrow \quad \frac{1}{1-n}.\frac{dz}{dx} + P.z = Q$$

Example1: solve the following D.E?

$$\frac{dy}{dx} - x.y = -y^3.e^{-x^2}$$

$$y^{-3}.y' - x.y^{-2} = -e^{-x^2}$$

$$z = y^{-2} \quad \Rightarrow \quad \frac{dz}{dx} = -2y^{-3}.y'$$

$$y^{-3}.y' = -\frac{1}{2}.\frac{dz}{dx} \quad \Rightarrow \quad -\frac{1}{2}.\frac{dz}{dx} - x.z = -e^{-x^2} \qquad *\text{-}2$$

$$\frac{dz}{dx} + 2x.z = 2e^{-x^2} \quad which \quad is \quad a\, linear \quad eq.$$

Solve and re-substitute in the first eq.

Example2: solve the following D.E?

$$2\,\frac{dy}{dx} - \frac{y}{x} = -y^3 \cos x \qquad \div\, y^3 \quad \textbf{Bernoulli equation}$$

$$2y^{-3}\frac{dy}{dx} - \frac{1}{x}y^{-2} = -\cos x$$

$$let \quad z = y^{1-n} = y^{1-3} = y^{-2}$$

$$\frac{dz}{dy} = -\,y^{-3}\frac{dy}{dx}$$

$$-\frac{dz}{dy} - \frac{1}{x}\,z = -\cos x \qquad * -1$$

$$\frac{dz}{dy} + \frac{1}{x}\,z = \cos x \ \text{ linear D.E}$$

$$p(x) = \frac{1}{x} \qquad f(x) = \cos x$$

$$I(x) = e^{\int p(x)dx} = e^{\int 1/x\, dx} = e^{\ln(x)} = x$$

$$I(x).z = \int I(x)\, f(x) dx$$

By part u=x du=dx
dv=cos(x) v=sin(x)

$$x.z = \int x\,.\cos x\, dx$$

$$x.z = x.\sin x - \int \sin x\, dx$$

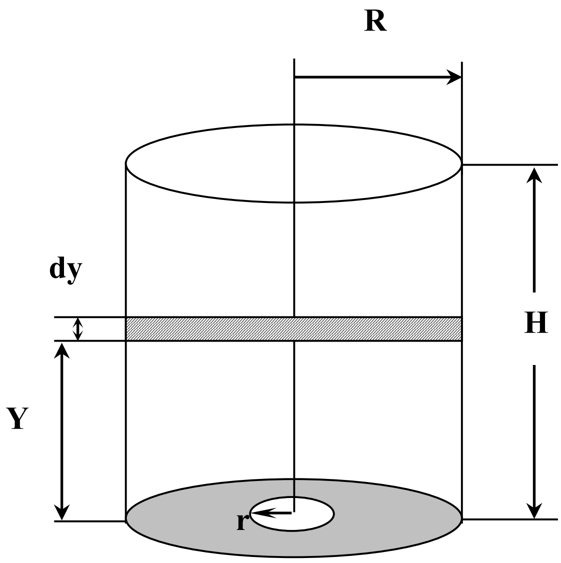
$x.z = x.\sin x + \cos x + c$
$x.y^{-2} = x.\sin x + \cos x + c$

Applications of Firs Order

Ordinary Differential Equations

Example: A cylindrical tank of radius R and height H initially filled with water. At the bottom of the tank there is a hole of radius r , through which water drains under the influence of gravity. Find the depth of water at any time t , and determine how long it takes the tank to drain off completely?

Solution: $dV = -\Pi . R^2 . dy$



$$v(velocity) = \sqrt{2\,g\,h} = \sqrt{2\,g\,y}$$

$$\frac{dV}{dt} = Q = A \times v \quad \Rightarrow \quad Q = \Pi\, r^2 \sqrt{2\,g\,y}$$

$$dV = \Pi\, r^2 \sqrt{2\,g\,y}.dt$$

$$\Pi\, r^2 \sqrt{2\,g\,y}.dt = -\Pi . R^2 . dy$$

$$r^2 \sqrt{2\,g\,y}.dt = -R^2 . dy$$

$$\frac{dy}{\sqrt{2\,g\,y}} = -\frac{r^2}{R^2}.dt \quad \Rightarrow \quad -\frac{dy}{\sqrt{2\,g\,y}} = \frac{r^2}{R^2}.dt$$

$$-(2\,g\,y)^{-1/2}.dy = \frac{r^2}{R^2}.dt$$

$$\frac{2g}{2g}\int -(2\,g\,y)^{-1/2}.dy = \int \frac{r^2}{R^2}.dt$$

$$-\frac{(2\,g\,y)^{1/2}}{1/2 \times 2\,g} = \frac{r^2}{R^2}.t + c$$

$$-\frac{\sqrt{2\,g\,y}}{g} = \frac{r^2}{R^2}.t + c$$

$$+\sqrt{\frac{2\,y}{g}} = -\frac{r^2}{R^2}.t + c \quad \textit{initial conditions}, \quad y = H \quad at \quad t = 0$$

$$\sqrt{\frac{2\,H}{g}} = 0 + c \quad \Rightarrow \quad c = \sqrt{\frac{2\,H}{g}}$$

$$\sqrt{\frac{2\,y}{g}} = -\frac{r^2}{R^2}.t + \sqrt{\frac{2\,H}{g}}$$

To find the required time for the tank to drain completely (t_o), we substitute ***y = 0***,

$$\sqrt{\frac{2y}{g}} = -\frac{r^2}{R^2}.t_o + \sqrt{\frac{2H}{g}}$$

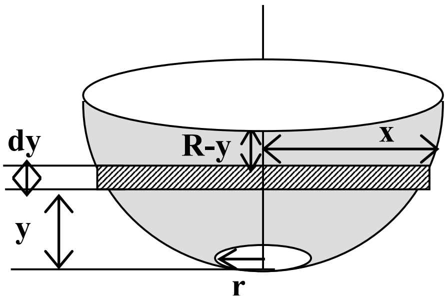
$$0 = -\frac{r^2}{R^2}.t_o + \sqrt{\frac{2H}{g}}$$

$$t_o = \sqrt{\frac{2H}{g}}.\frac{R^2}{r^2}$$

Example: A spherical (half ball) tank of radius *R* initially filled with water. At the bottom of the tank there is a hole of radius *r*, through which water drains under the influence of gravity. Find the depth of water at any time *t*, and determine how long it takes the tank to drain off completely?

Solution: $dV = -\Pi x^2\, dy$ water loose

$dV = \sqrt{2gy}.\,\Pi r^2\, dt$ water outlet



$$R^2 = x^2 + (R-y)^2$$
$$x^2 = R^2 - (R-y)^2$$
$$x^2 = R^2 - (R^2 - 2Ry + y^2)$$
$$x^2 = 2Ry - y^2$$

$$\sqrt{2gy}.\,\Pi r^2\, dt = -\Pi x^2\, dy$$

$$\sqrt{2gy}.\, r^2\, dt = -x^2\, dy$$

$$\sqrt{2gy}.\; r^2\, dt = (y^2 - 2Ry)\,dy$$

$$\sqrt{2g}\,.\; r^2\, dt = \frac{(y^2 - 2Ry)\,dy}{\sqrt{y}}$$

$$\int \sqrt{2g}\,.\; r^2\, dt = \int (y^{3/2} - 2Ry^{1/2})\,dy$$

$$\sqrt{2g}\,.\; r^2 .t = \frac{2}{3}y^{5/2} - 2Ry^{3/2} \times \frac{2}{3} + c$$

$$\frac{2}{5}y^{5/2} - \frac{4}{3}Ry^{2/3} = \sqrt{2g}\,.\; r^2 .t + c$$

$$initial\ condition: at \quad t = 0 \quad y = R$$

$$\frac{2}{5}y^{5/2} - \frac{4}{3}R\,R^{2/3} = 0 + c$$

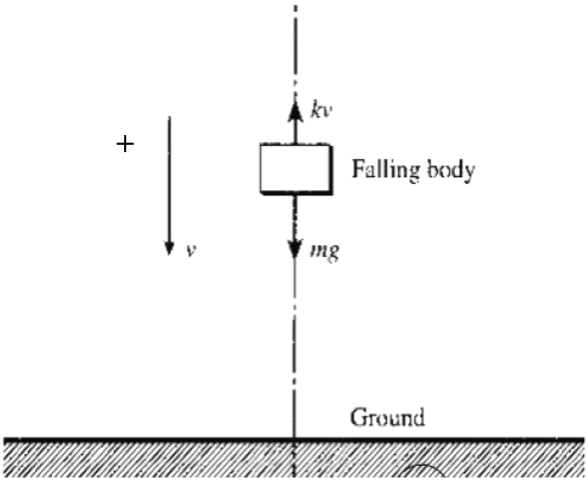
$$(\frac{2}{5} - \frac{4}{3})R^{5/2} = c$$

$$c = -\frac{14}{15}R^{5/2} = c$$

$$\frac{2}{5}y^{5/2} - \frac{4}{3}Ry^{3/2} = \sqrt{2g}\,.\; r^2 .t + \frac{14}{15}R^{5/2}$$

Example: a body falls in a medium with resistance proportional to speed at any instant. If the limiting speed is 50 *ft/sec* and the speed of the body decreases to half (25 *ft/sec*) after (1 *sec*), what was the *initial velocity*?

Solution:



$$Force = mass \times acceleration$$

$$F = m\frac{dv}{dt}, \quad Newton's\ second\ law$$

$$m\,g - k\,v = m\frac{dv}{dt}$$

$$\frac{dv}{dt} + \frac{k}{m}v = g \quad \Rightarrow \quad linear\ differential\ equation$$

$$Q = g \quad , \quad P = \frac{k}{m}$$

$$I = e^{\int p\,dt} \quad \Rightarrow \quad I = e^{\int \frac{k}{m}dt} \quad \Rightarrow \quad I = e^{\frac{k}{m}.t}$$

$$I.y = \int I.Q.dt \quad \Rightarrow \quad I.v = \int I.Q.dt$$

$$e^{\frac{k}{m}.t}.v = \int e^{\frac{k}{m}.t}.g.dt + c$$

$$e^{\frac{k}{m}.t}.v = g.\frac{m}{k}.e^{\frac{k}{m}.t} + c \qquad and\ dividing\ by\ e^{\frac{k}{m}.t}$$

$$v = \frac{g.m}{k} + \frac{c}{e^{\frac{k}{m}.t}}$$

<u>*Initial conditions*</u>:

$$at\ t = \infty \quad \Rightarrow \quad v = 50\ ft/\sec$$

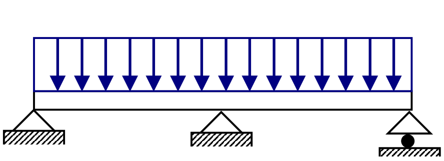
$$50 = \frac{g.m}{k} + \frac{c}{e^{\frac{k}{m}.\infty}} \quad \Rightarrow \quad 50 = \frac{g.m}{k} + \frac{c}{\infty} \quad \Rightarrow \quad k = \frac{g.m}{50}$$

$$at\ t = 1 \quad \Rightarrow \quad v = 25\ ft/\sec$$

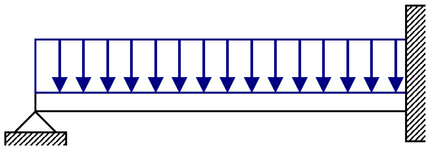
$$25 = \frac{g.m}{k} + \frac{c}{e^{\frac{k}{m}.1}} \quad \Rightarrow \quad 25 = \frac{g.m}{\frac{g.m}{50}} + \frac{c}{e^{\frac{g.m}{50.m}}}$$

$$c = -25 \times e^{\frac{g}{50}}$$

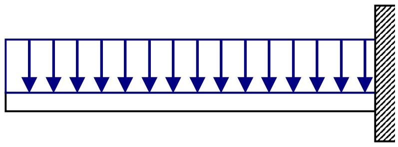
Structural Applications:



$Y = 0$	$Y = 0$
$Y' \neq 0$	$Y' \neq 0$
$M \neq 0$	$M = 0$
$V \neq 0$	$V \neq 0$



$Y = 0$	$Y = 0$
$Y' \neq 0$	$Y' = 0$
$M = 0$	$M \neq 0$
$V \neq 0$	$V \neq 0$

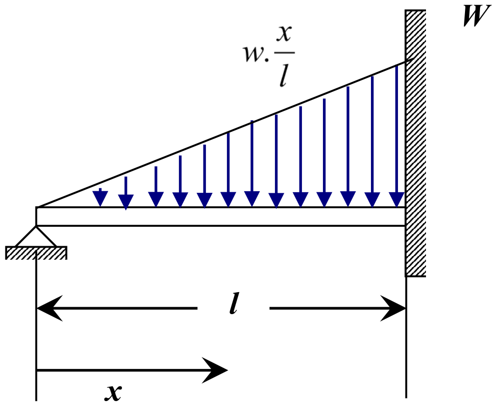


$Y \neq 0$	$Y = 0$
$Y' \neq 0$	$Y' = 0$
$M = 0$	$M \neq 0$
$V = 0$	$V \neq 0$

EI y = Deflection
EI y' = Rotation
EI y'' = Moment
EI y''' =Shear
EI y^{IV} = Load

Example: Find the equation of the deflection curve of the beam shown below?

Solution:



$$E\,I\,y^{IV} = w.\frac{x}{l} \quad \textit{Load equation}$$

$$E\,I\,y''' = \frac{w.x^2}{2l} + c_1$$

$$E\,I\,y'' = \frac{w.x^3}{6l} + c_1\,x + c_2$$

$$E\,I\,y' = \frac{w.x^4}{24l} + c_1\frac{x^2}{2} + c_2\,x + c_3$$

$$E\,I\,y = \frac{w.x5}{120l} + c_1\frac{x3}{6} + c_2\frac{x^2}{2} + c_3\,x + c_4$$

Initial Conditions:

at $x = 0$ $y = 0$

$$0 = \frac{w(0)}{120l} + \frac{c_1(0)}{6} + \frac{c_2(0)}{2} + c_3(0) + c_4 \quad \Rightarrow \quad c_4 = 0$$

at $x = 0$ $y'' = 0$

$$0 = \frac{w(0)}{6l} + c_1(0) + c_2 \quad \Rightarrow \quad c_2 = 0$$

at $x = l$ $y = 0$

$$0 = \frac{w(l^5)}{120\,l} + \frac{c_1(l^3)}{6} + \frac{0(l^2)}{2} + c_3(l) + 0 \quad \Rightarrow \quad \frac{wl^4}{120} + c_1\frac{l^3}{6} + c_3\,l = 0 \quad \ldots\ldots\ldots\ldots\ldots \quad (1)$$

at $x = l$ $y' = 0$

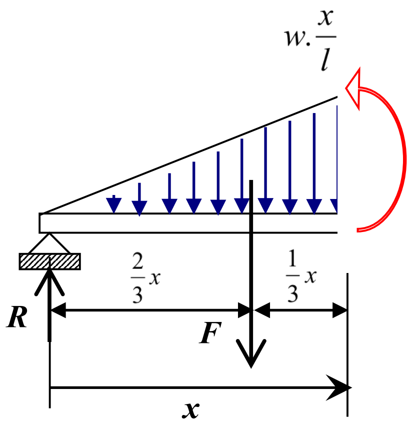
$$0 = \frac{w(l^4)}{24l} + \frac{c_1(l^2)}{2} + (0)l + c_3 \quad \Rightarrow \quad \frac{wl^3}{24} + c_1\frac{l^2}{2} + c_3 = 0 \quad \ldots\ldots\ldots\ldots\ldots \quad (2)$$

Solve equation (1) and (2) to find c_1 and c_3

$$\frac{wl^3}{120}+c_1\frac{l^2}{6}+c_3=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1') \quad \textit{dividing eq (1) by } l$$

$$\frac{wl^3}{24}+c_1\frac{l2}{2}+c_3=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

Another way to solve the previous example is by starting with the moment equation:



$$E\,I\,y''=-M$$

$$E\,I\,y''=-\left[\left(R\,x-\frac{1}{2}w\frac{x^2}{l}(\frac{1}{3}x)\right)\right]$$

$$E\,I\,y''=-R\,x+\frac{w\,x^3}{6l}$$

$$E\,I\,y'=-R\frac{x^2}{2}+\frac{w\,x^4}{24\,l}+c_1$$

$$E\,I\,y=-R\frac{x^3}{6}+\frac{w\,x^5}{120l}+c_1\,x+c_2$$

at $\quad x=0 \qquad y=0$

$$0=0+0+0+c_2 \quad \Rightarrow \quad c_2=0$$

at $\quad x=l \qquad y=0$

$$0=-\frac{Rl^3}{6}+\frac{wl^5}{120l}+c_1\,l+0 \quad \Rightarrow \quad -\frac{Rl^3}{6}+\frac{wl^4}{120}+c_1\,l=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

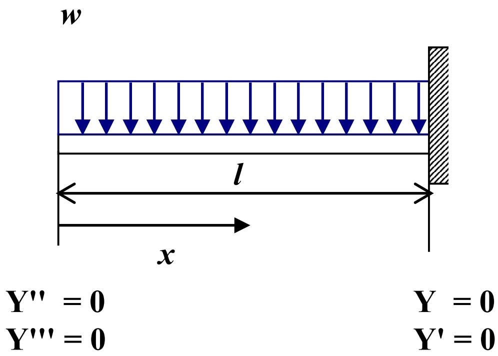
at $\quad x=l \qquad y'=o$

$$0=-R\frac{l^2}{2}+\frac{wl^4}{24\,l}+c_1 \quad \Rightarrow \quad -R\frac{l^2}{2}+\frac{wl^3}{24}+c_1=0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Solve Equation (1) and (2) to find c_1 and substitute in the deflection equation.

Example: For the cantilever beam shown below find the deflection curve?

Solution:



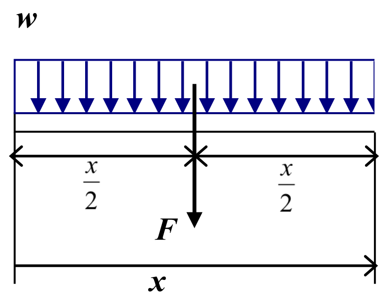
$$E\,I\,y'' = -wx\left(\frac{x}{2}\right)$$

$$E\,I\,y'' = -\frac{wx^2}{2}$$

$$y'' = -\frac{wx^2}{2EI}$$

$$y' = -\frac{wx^3}{6EI} + A$$

$$y = -\frac{wx^4}{24EI} + Ax + B$$



at $x = l \quad y = 0$

$$0 = -\frac{wl^4}{24EI} + Al + B \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

at $x = l \quad y' = 0$

$$0 = -\frac{wl^3}{6EI} + A \quad \Rightarrow \quad A = \frac{wl^3}{6EI}$$

Substitute in eq. (1)

$$0 = -\frac{wl^4}{24EI} + \frac{wl^3}{6EI}(l) + B \quad \Rightarrow \quad B = \frac{wl^4}{24EI} - \frac{wl^4}{6EI} = \frac{3}{24}.\frac{wl^4}{EI}$$

$$\therefore \quad y = -\frac{wx^4}{24EI} + \frac{wl^3}{6EI}(x) - \frac{3}{24}.\frac{wl^4}{EI} \qquad complete \quad solution$$